Newton - Cotes integration formulas

For $n \geq 1$, let $h = (b-a)/n$, $x_j = a + jh$ for $j = 0, 1, \ldots n$

$$I(f) = \int_a^b f(x)\,dx$$

$$I_n(f) = \int_a^b p_n(x)\,dx$$ where $p_n(x)$ is the interpolating polynomial on nodes $x_0, x_1, x_2, \cdots, x_n$

$$p_n(x) = \sum_{j=0}^{n} l_{j,n}(x)\, f(x_j) \qquad l_{j,n}(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \qquad i = 0, 1, 2, \cdots, n$$

$$\therefore\ I_n(f) = \int_a^b \sum_{j=0}^{n} l_{j,n}(x)\, f(x_j)\,dx$$

$$= \sum_{j=0}^{n} \left(\int_a^b l_{j,n}\,dx \right) f(x_j)$$

$$= \sum_{j=0}^{n} W_{j,n}\, f(x_j)$$

$n=1 \Rightarrow$ trapezoidal rule

$n=2 \Rightarrow$ Simpson's rule

$n=3$, $$W_{0,3} = \int_a^b l_0(x)\,dx = \int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx$$

$$W_{1,3} = \int_a^b l_1(x)\,dx = \int_{x_0}^{x_3} \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,dx$$

$\vdots$ Similarly for $W_{2,3}$ & $W_{3,3}$

$$I_3(f) = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

For error with n even (assume $f(x)$ is $n+2$ times continuously diff.)

$$I(f) - I_n(f) = C_n\, h^{n+3} \cdot f^{(n+2)}(\eta) \qquad \text{some } \eta \in [a, b]$$

$$\text{with } C_n = \frac{1}{(n+2)!} \int_0^n \mu^2(\mu-1)\cdots(\mu-n)\,d\mu$$

Proof:

$$E_n(f) = I(f) - I_n(f)$$

$$= \int_a^b f(x)\,dx - \int_a^b \sum_{j=0}^{n} l_{j,n}(x) f(x_j)\,dx$$

$$= \int_a^b (x-x_0)(x-x_1)\cdots(x-x_n)\, f[x_0, x_1, \ldots, x_n, x]\,dx$$

Define $\omega(x) = \int_a^x (t-x_0)\cdots(t-x_n)\,dt$

note that $\omega(a) = 0 = \omega(b)$, $\omega(x) > 0$ for $a < x < b$

$$E_n(f) = \int_a^b (x-x_0)\cdots(x-x_n)\, f[x_0, x_1, \cdots, x_n, x]\, dx$$

$$= \int_a^b \omega' f[x_0, x_1, \cdots x_n, x]\, dx$$

$$= \omega f[x_0, x_1, \cdots x_n, x]\Big|_a^b - \int_a^b \omega(x) \frac{d}{dx} f[x_0, x_1, \cdots, x_n, x]\, dx$$

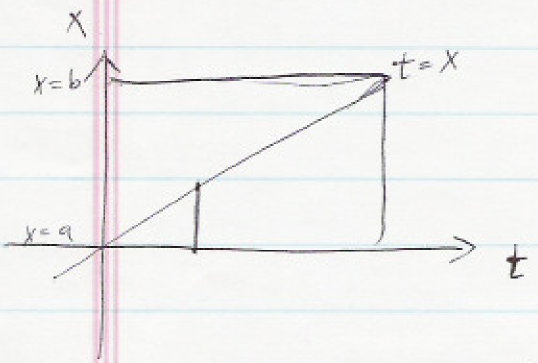
$$= 0 - \int_a^b \omega(x) f[x_0, x_1, \cdots, x_n, x, x]\, dx$$

because $\omega(x) > 0$ for $a < x < b$

$$E_n(f) = - f[x_0, x_1, \cdots x_n, \xi, \xi] \cdot \int_a^b \omega(x)\, dx \quad \text{for some } \xi$$

$$= - \frac{f^{(n+2)}(\eta)}{(n+2)!} \int_a^b \left(\int_a^x (t-x_0)(t-x_1)\cdots(t-x_n)\, dt \right) dx$$

$$= - \frac{f^{(n+2)}(\eta)}{(n+2)!} \int_a^b \left(\int_t^b (t-x_0)(t-x_1)\cdots(t-x_n)\, dx \right) dt$$



$$= - \frac{f^{(n+2)}(\eta)}{(n+2)!} \int_a^b (t-x_0)(t-x_1)\cdots(t-x_n)(x_n - t)\, dt$$

$$\Rightarrow \quad t = x_0 + \mu h, \quad 0 \le \mu \le n$$

$$E_n(f) = - \frac{f^{(n+2)}(\eta)}{(n+2)!} \int_0^n -h^{n+3} \mu(\mu-1)\cdots(\mu-n+1)(\mu-n)^2\, d\mu$$

$$= \frac{f^{(n+2)}(\eta)}{(n+2)!} h^{n+3} \int_0^n \mu(\mu-1)\cdots(\mu-n+1)(\mu-n)^2\, d\mu$$

$\nu \equiv n - \mu$

$$\int_a^b w(x)\,dx = -h^{n+3}\int_0^n (n-\nu)\cdots\cdots(1-\nu)\,\nu^2\,d\nu$$

$$\therefore E_n(f) = \frac{f^{(n+2)}(\eta)}{(n+2)!}h^{n+3}\int_0^n (n-\nu)\cdots\cdots(1-\nu)\,\nu^2\,d\nu \qquad \text{for some } \eta \in [a,b]$$

$\mathcal{F}$ be a family of continuous functions on a given interval $[a,b]$

$\mathcal{F}$ is dense in $C[a,b]$ if for every $f \in C[a,b]$ and every $\epsilon > 0$, there exists a function f_ϵ in $\mathcal{F}$ s.t.

$$\operatorname*{Max}_{a \le x \le b} |f(x) - f_\epsilon(x)| \le \epsilon$$

$\Rightarrow$ the set of all polynomials is dense in $C[a,b]$

Theorem : Let $I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n}) \qquad n \ge 1$

be a sequence of numerical integration formulas that approximate

$I(f) = \int_a^b f(x)\,dx \qquad \mathcal{F}$ is a family dense in $C[a,b]$,

$I_n(f) \to I(f)$ for all $f \in C[a,b]$

iff

I $\quad I_n(f) \to I(f) \quad \forall f \in \mathcal{F}$

II $\quad B \equiv \operatorname*{Supremum}_{n \ge 1} \sum_{j=0}^{n} |w_{j,n}| < \infty$

Gaussian Quadrature :

Newton - Cotes formulas does not give convergent results for some good integrands.

$\Rightarrow$ nodes $\{x_{jn}\}$ must be evenly spaced.

Write $\tilde{I}(f) \equiv \sum_{j=1}^{n} w_j f(x_j)$

where $\{w_j\}$ and $\{x_j\}$ are to be determined s.t.

$E_n(f) = \int_{-1}^{1} f(x)\,dx - \tilde{I}(f)$ equal zero for as high a degree polynomial $f(x)$ as possible.

$E_n(a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m)$

$= a_0 E_n(1) + a_1 E_n(x) + \cdots + a_m E_n(x^m)$

$E_n(f) = 0$ for every polynomial of degree $\leq m$ iff

$E_n(x^i) = 0, \quad i = 1, 2, \ldots, m$

Case 1 : $n = 1$

$E_n(1) = 0 \qquad E_n(x) = 0$

$\int_{-1}^{1} 1 \cdot dx - w_1 = 0 \qquad \int_{-1}^{1} x \cdot dx - w_1 x_1 = 0$

$w_1 = 2, \quad x_1 = 0 \quad \Rightarrow \int_{-1}^{1} f(x)\,dx \sim 2f(0)$

$\Rightarrow$ mid point rule

Case 2 $\quad n = 2 \Rightarrow$ four parameters w_1, w_2, x_1, x_2

$$E_n(x^i) = 0 \qquad i = 0, 1, 2, 3$$

$$\| $$

$$\int_{-1}^{1} x^i\,dx - (w_1 x_1^i + w_2 x_2^i) = 0$$

$i=0 \qquad 2 - (w_1 + w_2) = 0$

$i=1 \qquad 0 - (w_1 x_1 + w_2 x_2) = 0$

$i=2 \qquad \frac{2}{3} - (w_1 x_1^2 + w_2 x_2^2) = 0$

$i=3 \qquad 0 - (w_1 x_1^3 + w_2 x_2^3) = 0$

$$w_1 = w_2 = 1 \qquad x_1 = -\frac{\sqrt{3}}{3} \qquad x_2 = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx \sim f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

General situation: $2n$ free parameters $\{x_i\}$ & $\{w_i\}$

$\Rightarrow$ an interpolation $\sum_{j=1}^{n} w_j f(x_j)$ with a degree of precision of $2n-1$

$E_n(x^i) = 0, \quad i = 0, 1, \cdots, 2n-1$

$$\Rightarrow \sum_{j=1}^{n} w_j x_j^i = \begin{cases} 0 & i = 1, 3, \cdots, 2n-1 \\ \dfrac{2}{i+1} & i = 0, 2, \cdots, 2n-2 \end{cases}$$

$\Rightarrow$ From the theory of orthogonal polynomials, it can be shown that the nodes $\{x_1, x_2 \ldots x_n\}$ are the zeros of Legendre polynomial of degree n on the interval $[1, 1]$

Gauss-Legendre quadrature:

$$\int_{-1}^{1} f(x)\,dx \sim \sum_{j=1}^{n} w_j f(x_j)$$

with the nodes equal to the zeros of the degree n Legendre polynomial $P_n(x)$ on $[-1, 1]$.

$$w_j = \frac{-2}{(n+1) P_n'(x_i) P_{n+1}(x_i)} \qquad i = 1, 2, \cdots, n$$

and $E_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^2} \frac{f^{2n}(\eta)}{(2n)!}$

Theorem: Let $f(x)$ be continuous for $a \leq x \leq b$ and $n \geq 1$.

If we apply Gaussian numerical integration to $I = \int_a^b f(x)dx$

the error in I_n satisfies:

$$|I(f) - I_n(f)| \leq 2(b-a)\rho_{2n-1}(f)$$

where $\rho_{2n-1}(f)$ is the minimax error of degree $2n-1$ for

$f(x)$ on $[a,b]$

~~(a)~~ $\cancel{I_n(q_{2n-1}) = I(q_{2n-1})}$

prob 5

prob 12