Residual Correction Method

$$r^{(k)} = b - A x^{(k)}$$

$$A e^{(k)} = r^{(k)}$$

$$x^{(k+1)} = x^{(k)} + \hat{e}^{(k)}$$

$\hat{e}^{(k)}$ is the solution from Gaussian elimination, say

$\Rightarrow$ generalized version

$$r^{(k)} = b - A x^{(k)}$$

$$N \hat{e}^{(k)} = r^{(k)}$$

$$x^{(k+1)} = x^{(k)} + \hat{e}^{(k)}$$

where N is the invertible approx. of matrix A.

$Ax = b$ $\qquad x = b + (I - A)x \equiv b + Mx$

iterative scheme $\qquad x^{(k+1)} = b + M x^{(k)} \qquad k \geq 0$

convergence analysis $\qquad e^{(k)} \leq \|M\|^k \|e^{(0)}\|$

convergence if $\|M\| < 1$

for the generalized residual correction method:

$$C = N^{-1}$$

$$X - X^{(m+1)} = X - X^{(m)} - C\, r^{(m)}$$

$$= X - X^{(m)} - C \cdot [b - A X^{(m)}]$$

$$= X - X^{(m)} - C [A X - A X^{(m)}]$$

$$\boxed{X - X^{(m+1)} = (I - CA)(X - X^{(m)})}$$

what is the rate of convergence of $X^{(m)}$ to X?

By induction

$$X - X^{(m)} = (I - CA)^m (X - X^{(0)}) \qquad m \geq 0$$

If $\|I - CA\| < 1$

$$\|X - X^{(m)}\| \leq \|I - CA\|^m \|X - X^{(0)}\|$$

and converges to zero as $m \to \infty$

$$(I - CA)^m \to 0 \qquad \text{as } m \to \infty$$

this implies that

$$r_\sigma(I - CA) < 1 \qquad \text{for convergence}$$

$$X - x^{(m+1)} = (I - CA)(x - x^{(m)})$$

$$x^{(m+1)} - x^{(m)} = e^{(m)} - e^{(m+1)}$$

$$M = (I - CA)$$

$$= Me^{(m-1)} - Me^{(m)}$$
$$= M(e^{(m-1)} - e^{(m)})$$

$$= M(x^{(m)} - x^{(m-1)})$$

$$\boxed{\therefore \; x^{(m+1)} - x^{(m)} = (I - CA)(x^{(m)} - x^{(m-1)})}$$

$$\|x - x^{(m+1)}\| \leq c\|x - x^{(m)}\|$$

$$\|x^{(m+1)} - x^{(m)}\| = \|(x - x^{(m)}) - (x - x^{(m+1)})\|$$

$$\geq \|x - x^{(m)}\| - \|x - x^{(m+1)}\|$$

$$\geq \|x - x^{(m)}\| - c\|x - x^{(m)}\|$$

$$\therefore \; \|x - x^{(m)}\| \leq \frac{1}{1-c} \|x^{(m+1)} - x^{(m)}\|$$

$$c \doteq \frac{\|x^{(m+1)} - x^{(m)}\|_\infty}{\|x^{(m)} - x^{(m-1)}\|_\infty}$$

or the maximum of several successive such ratios.

$$\boxed{\|x - x^{(m+1)}\| \leq c\|x - x^{(m)}\| \leq \frac{c}{1-c} \|x^{(m+1)} - x^{(m)}\|}$$

convergence is slow when $c \sim 1$

and

$\|x^{(m+1)} - x^{(m)}\|$ can be much smaller than the actual error.

want to find m s.t.

$$\| x - x^{(m)} \|_\infty \leq \varepsilon \| x - x^{(0)} \|_\infty$$

with ε a given factor (tolerance)

in general $\quad x - x^{(m+1)} = M(x - x^{(m)}) \qquad M = (I - cA)$

and

$$\| x - x^{(m+1)} \|_\infty \leq c \| x - x^{(m)} \|_\infty \quad \text{with } c \sim r_\sigma(M)$$

$$\Rightarrow \| x - x^{(m)} \|_\infty \leq c^m \| x - x^{(0)} \|_\infty$$

$$\therefore \quad c^m \leq \varepsilon$$

$$m \geq \frac{-\ln \varepsilon}{-\ln c}$$

say if $\varepsilon = 10^{-6}$, $\quad m \geq \dfrac{6 \ln_e 10}{-\ln c}$

the total number of operations is

$$\sim \left(\frac{6 \cdot \ln 10}{-\ln c} \right) \cdot n^2$$

for this method to be faster than Gaussian elimination

$$\frac{6 \ln 10}{-\ln c} \cdot n^2 \leq \frac{n^3}{3}$$

$$\therefore \quad \frac{6 \ln 10}{-\ln c} \leq \frac{n}{3}$$

Example: $X^{(k+1)} = b + \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} X^{(k)}$

for some real α, the method diverges.
find α s.t. the method converges

$$\Rightarrow \quad X^{(k+1)} - X^{(k)} = \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} (X^{(k)} - X^{(k-1)})$$

iteration converges if $|\lambda|$ of the matrix $\alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are smaller than 1.

eigenvalues are 3α & α $\Rightarrow$ $\boxed{-1 < \alpha < \frac{1}{3}}$
the iteration converges.

Application to solving PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y) \qquad 0 \le x, y \le 1$$

0. first discretize the domain

$$(x_j, y_k) = (jh, kh), \quad h = \frac{1}{N} \qquad 0 \le j, k \le N$$

1. $$G'' = \frac{G(x+h) - 2G(x) + G(x-h)}{h^2} - \frac{h^2}{12} G^{(4)}(\xi)$$

$$\frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h^2} + \frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{h^2}$$

$$= g_{jk} + \frac{h^2}{12}\left\{ \frac{\partial^4 u}{\partial x^4}(\xi, y_k) + \frac{\partial^4 u}{\partial y^4}(x_j, \eta) \right\}$$

② BC's $u_{jk} = f_{jk}$ on the boundary grid points.

⇒ all interior points:

$$u_{jk} = \frac{1}{4}\left\{ \underline{u_{j+1,k}} + u_{j,k+1} + \underline{u_{j-1,k}} + u_{j,k-1} \right\} - \frac{h^2}{4} g_{jk}$$

$1 \le j, k \le N-1$

⇒ $A \cdot x = b$

$$\begin{array}{c} \begin{array}{ccccc} & j & j+1 & j+2 & j+3 \end{array} \\ \begin{bmatrix} & -1 & & & \\ -1 & 4 & -1 & 0 & -1 \\ & -1 & & & \\ & 0 & & & \ddots \\ & -1 & & & \end{bmatrix} \end{array}$$