$$\partial_x^2 u + \partial_y^2 u = g(x,y) \qquad 0 < x, y < 1$$

$$u(x,y) = f(x,y) \qquad \text{for } x, y \text{ @ the boundary}$$

$$h = 1/N \quad , \quad (x_j, y_k) = (j, k)h \qquad 0 \leq j, k \leq N$$

$$\frac{1}{h^2}(u_{j+1,k} - 2u_{j,k} + u_{j-1,k}) + \frac{1}{h^2}(u_{jk+1} - 2u_{jk} + u_{jk-1})$$

$$= g_{jk} \quad + \quad \text{h.o.t.}$$

$$\frac{h^2}{12}\left\{\partial_x^4 u(\xi, y_k) + \partial_y^4 u(x_j, \eta)\right\}$$

rearranging terms:

$$u_{jk} = \frac{1}{4}(u_{j+1k} + u_{j-1k} + u_{jk+1} + u_{jk-1}) - \frac{h^2}{4} g_{jk}$$

$$\begin{matrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{matrix}$$

Gauss - Seidel iteration

$$u_{jk}^{(m+1)} = \frac{1}{4}\left[u_{j+1k}^{(m)} + u_{jk+1}^{(m)} + u_{j-1k}^{(m)} + u_{jk-1}^{(m)}\right] - \frac{h^2}{4} g_{jk}$$

with boundary points

$$u_{jk}^{(m)} = f_{jk} \qquad \text{for all } m \geq 0$$

$\Rightarrow$ slow convergence, $\quad r_\sigma(M) = 1 - \pi^2 h^2 + O(h^4)$

instead

$$v_{jk}^{(m+1)} = \frac{1}{4}\left[u_{j+1\,k}^{(m)} + u_{j\,k+1}^{(m)} + u_{j-1\,k}^{(m+1)} + u_{j\,k-1}^{(m+1)} \right] - \frac{h^2}{4} g_{jk}$$

$$u_{jk}^{(m+1)} = \omega v_{jk}^{(m+1)} + (1-\omega) u_{jk}^{(m)} \qquad \begin{array}{l} j = 1, \dots \; N-1 \\ k = 1, \dots \; N-1 \end{array}$$

optimal acceleration parameter

$$\omega = \frac{2}{1 + \sqrt{1 - \xi^2}} \qquad \xi = 1 - 2\sin^2\left(\frac{\pi}{2N}\right)$$

rate of convergence $\quad r_\sigma(M(\omega)) = \omega - 1$

$$= 1 - 2h\pi + O(h^2)$$

$\Rightarrow$ SOR method

successive over-relaxation method

Summary of Iterative Schemes: $A \cdot x = b$

Matrix Splitting

$$A = L + D + U$$

NOTE: NOT THE L & U in LU DECOMPOSITION

Rewrite $Ax = b$ as $D \cdot x = b - (L+U) \cdot x$

$$x = D^{-1} \cdot b - D^{-1}(L+U)x$$

$$= D^{-1} b - (D^{-1} A - I) \cdot x$$

Scheme:

$$x^{(m+1)} = D^{-1} b - D^{-1}(L+U)\, x^{(m)}$$

Componentwise: $$x_i^{(m+1)} = -\frac{1}{a_{ii}} \sum_{j \neq i} a_{ij} x_j^{(m)} + \frac{b_i}{a_{ii}}$$

$\Rightarrow$ Jacobi Method

Another natural splitting

$$(L+D)\, x = b - U x$$

$$x^{(m+1)} = (L+D)^{-1} b - (L+D)^{-1} U\, x^{(m)}$$

componentwise:

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{m} \right\}$$

$\Rightarrow$ Gauss-Seidel method

$\boxed{\text{General form}}$

$$x^{(m+1)} = G x^{(m)} + k$$

Jacobi: $\quad G = -D^{-1}(L+U) \qquad k = D^{-1} b$

Gauss-Seidel: $\quad G = -(L+D)^{-1} U \qquad k = (L+D)^{-1} b$

$\boxed{\text{Error analysis}}$ $\qquad e^{(m)} \equiv x - x^{(m)}$

$$e^{(m+1)} = G e^{(m)}, \qquad \cancel{G}$$

$$e^{(m)} = G^{m} \cdot e^{(0)}$$

If the initial error (guess) is such that $e^{(0)}$ is an eigenvalue of G, then $e^{(m)} = \lambda^{m} \cdot e^{(0)}$

then $e^{(m)} \to 0$ if $\lambda < 1$. True for any eigenvalue of G

$\Rightarrow$ must have $\boxed{r(G) < 1}$

since $r(G) \leq \|G\|$ for any matrix norm,
$\|G\| < 1$ is a sufficient ($\Rightarrow$) but not necessary cond.

what we have done before:

$$e^{(m)} = G^m \cdot e^{(0)}$$

$$\| e^{(m)} \| \leq \| G \|^m \, \| e^{(0)} \| \quad \text{as} \quad m \to \infty$$

$$\| G \| < 1 \quad \text{for convergence}$$

Remark: There are systems for which $\| G \| = 1$ & convergence occurs.

For Jacobi method:

$$\| G \|_\infty = \| - D^{-1} (L + U) \|_\infty = \max_{1 \leq i \leq n} \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| < 1$$

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}| \quad \forall i$$

$\Rightarrow$ The same is true for G-S method.

$$e_i^{(m)} = - \sum_{j < i} \frac{a_{ij}}{a_{ii}} e_j^{(m)} - \sum_{j > i} \frac{a_{ij}}{a_{ii}} e_j^{(m-1)}$$

$$i = 1 \qquad | e_1^{(m)} | \leq \sum_{j > 1} \left| \frac{a_{ij}}{a_{11}} \right| \, | e_j^{(m-1)} | \leq r \| e^{(m-1)} \|_\infty$$

$$| e_i^{(m)} | \leq \sum_{j < i} \left| \frac{a_{ij}}{a_{ii}} \right| | e_j^{(m)} | + \sum_{j > i} \left| \frac{a_{ij}}{a_{ii}} \right| \, | e_j^{(m-1)} |$$

$$\leq r \, \| e^{(m-1)} \|_\infty \sum_{j < i} \left| \frac{a_{ij}}{a_{ii}} \right| + \| e^{(m-1)} \|_\infty \sum_{j > i} \left| \frac{a_{ij}}{a_{ii}} \right|$$

$$\leq \| e^{(m-1)} \|_\infty \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right| \leq r \cdot \| e^{(m-1)} \|_\infty$$

Acceleration Methods :

$$z_i^{(m+1)} = \frac{1}{a_{ii}} \left\{ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(m+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(m)} \right\}$$

$$x_i^{(m+1)} = \omega z_i^{(m+1)} + (1-\omega) x_i^{(m)} \qquad i = 1, \cdots, n$$

$\omega = 1$, usual G-S method.

$\Rightarrow$

$$\cancel{z^{(m+1)} = D^{-1}}$$

$$z^{(m+1)} = -(L+D)^{-1} U x^{(m)} + (L+D)^{-1} b$$

$$x^{(m+1)} = -\omega (L+D)^{-1} U x^{(m)} + \omega (L+D)^{-1} b + (1-\omega) x^{(m)}$$

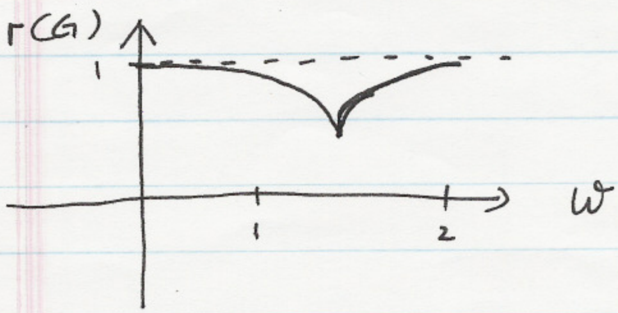
$$= (L + \omega^{-1} D)^{-1} \left[(\omega^{-1} - 1) D - U \right] x^{(m)} + (L + \omega^{-1} D)^{-1} b$$

or $[I + \omega D^{-1} L] x^{(m+1)} = \omega D^{-1} b + [(1-\omega) I - \omega D^{-1} U] x^{(m)}$

$\Rightarrow$ SOR converges when $r(G) < 1$

Thm : If A is symmetric & positive-definite

$\Rightarrow$ $r(G) < 1$ whenever $0 < \omega < 2$



Additional assumptions need to be made to show ("consistently ordered" & "property A") that

$$r(G_{GS}) = [r(G_J)]^2$$

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(G_J)^2}} = \frac{2}{1+\sqrt{1-\rho(G_{GS})}}$$

Eigenvalue Problems:

$Ax = \lambda x$ for a square matrix

x is an eigenvector
λ is an eigenvalue

$\det(A - \lambda I) = 0$ charateristic eq. for A

Characteristic polynomial of A: $f(\lambda) \equiv \det(A - \lambda I)$

Thm: (Gerschgoren's Circle thm)
Let $A = [a_{ij}]$ be an $n \times n$ square matrix & Let $r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for $i = 1, \cdots, n$. Let Z_i be a circle in the complex plane with center a_{ii} & radius r_i:
$Z_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq r_i \}$. λ be an eigenvalue of A, the λ belongs to one of the Z_i.

Example : $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix}$

$|\lambda - 4| \le 1$ since A is symmetric,

$|\lambda - 0| \le 2$ eigenvalues are real

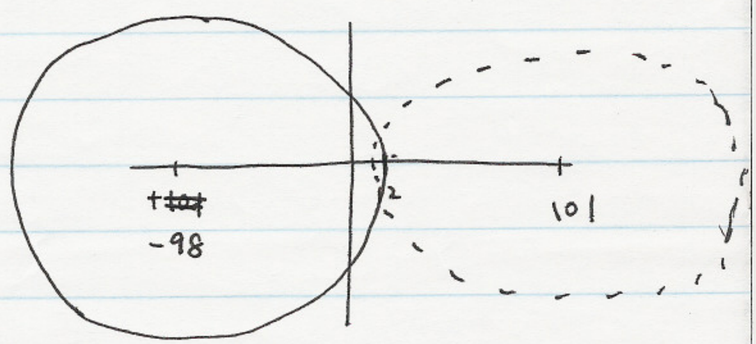
$|\lambda + 4| \le 1$ $\Rightarrow$ $\lambda \in [3,5]$

$\lambda \in [-2,2]$

$\lambda \in [-5,-3]$

Numerical evaluation of eigenvalues can be difficult for asymmetric matrixes

Example: $A = \begin{bmatrix} 101 & -90 \\ 110 & -98 \end{bmatrix}$



$\lambda = 1, 2$

$|\lambda - 101| \le 90$

$|\lambda + 98| \le 110$

Perturbed A : $\hat{A} = \begin{bmatrix} 100.999 & -90.001 \\ 110 & -98 \end{bmatrix}$

new eigenvalues : $\lambda \sim 1.298$

$\lambda \sim 1.701$

$\Rightarrow$ can be highly ill-conditioned.

$\Rightarrow$ Power method to find the largest eigenvalue

Assume that eigenvalues of A satisfy $|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_n|$