Backward Euler's solution:

$$y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})$$

predictor formula:

$$\bar{y}_{n+1} = y_n + h\, f(x_{n+1}, y_n)$$

$$y_{n+1} = y_n + h\, f(x_{n+1}, \bar{y}_{n+1})$$

$$\Rightarrow \quad y_{n+1} = y_n + h\, f(x_{n+1}, y_n + h\, f(x_{n+1}, y_n)) \quad \text{---} (*)$$

no longer absolutely stable

$y' = \lambda y, \quad y(0) = 1$

$$y_{n+1} = y_n + h\lambda y_n + h^2\lambda^2 y_n$$

$$y_{n+1} = y_n (1 + h\lambda + h^2\lambda^2)$$

$$y_n = (1 + h\lambda + h^2\lambda^2)^n y_0$$

if $h\lambda < -1$ $\quad 1 + h\lambda + h^2\lambda^2 > 1$

$\Rightarrow$ $(*)$ is no longer absolutely stable.

Trapezoidal Method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right] \quad n \geq 0 \quad \text{---} (*)(*)$$

$$y_0 = Y_0$$

$y' = \lambda y \quad , \quad y(0) = 1$

$y_{n+1} = y_n + \frac{h}{2}(\lambda y_n + \lambda y_{n+1})$

$$y_{n+1} = \frac{2+h\lambda}{2-h\lambda} y_n$$

$$y_n = \left(\frac{2+h\lambda}{2-h\lambda}\right)^n y_0 = \left(\frac{2+h\lambda}{2-h\lambda}\right)$$

$\left|\frac{2+h\lambda}{2-h\lambda}\right| < 1$ for all $h\lambda$ with $Real(\lambda) < 0$

$\therefore$ (**) is absolutely stable

approximated Trapezoidal Method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + h f(x_n, y_n))\right] \text{ —— } (***)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[\lambda y_n + \lambda(y_n + h\lambda y_n)\right]$$

$$y_{n+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_n$$

$$y_n = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_0$$

$\left|1 + h\lambda + \frac{h^2\lambda^2}{2}\right| < 1$ iff $-2 < h\lambda < 0$ for real λ

(***) is not absolutely stable for Trapezoidal method.

Example: $Y'(x) = \alpha x^{\alpha-1} \quad , \quad Y(0) = 0$

$\alpha = 2.5, \ 1.5, \ 1.1$ with step size $h = 0.2, \ 0.1, \ 0.05$

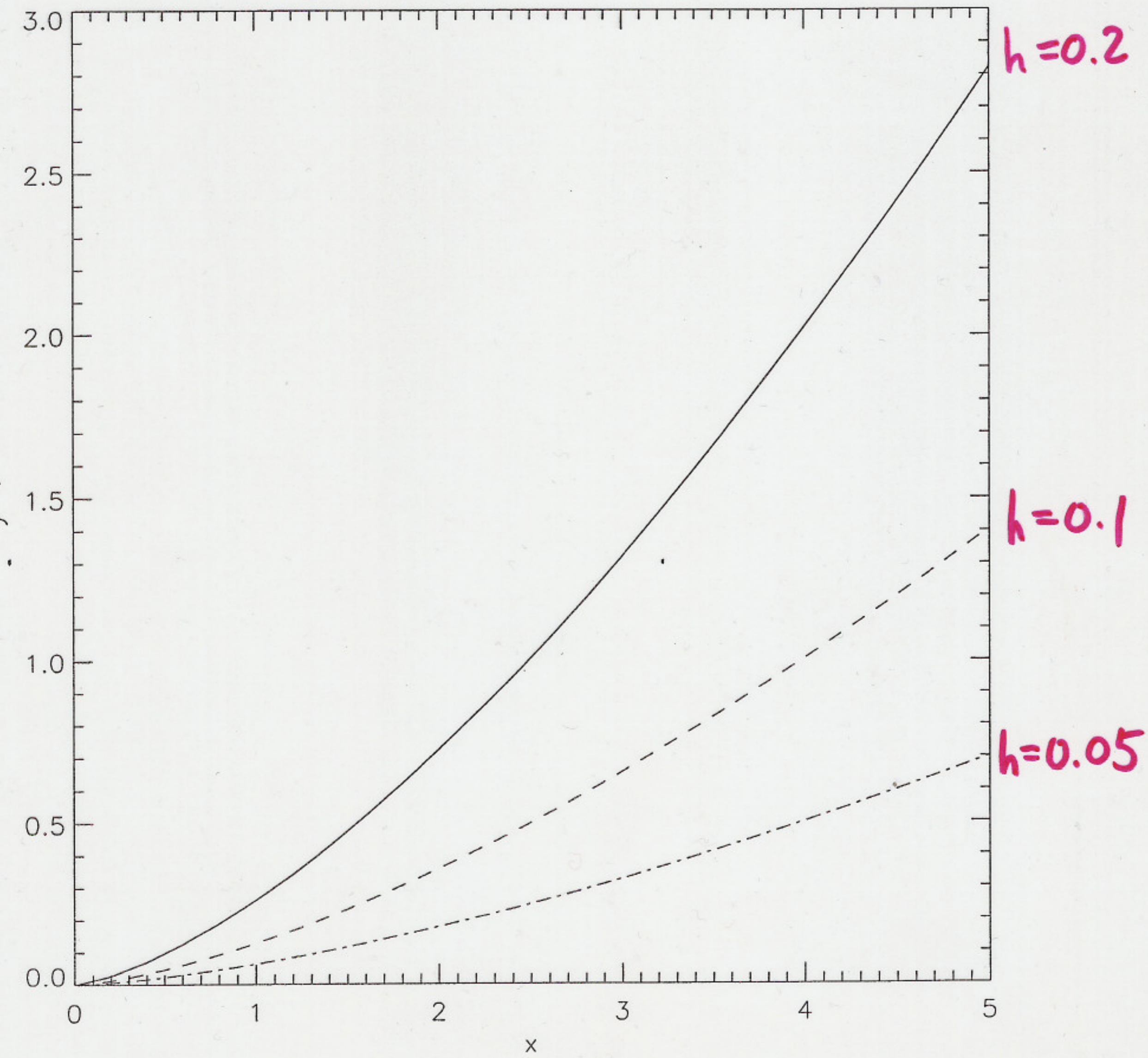
$$Y' = \alpha X^{\alpha-1} \qquad Y(0) = 0$$

$$Y(x) = x^{\alpha}$$

$$\alpha = 2.5$$

Euler's method

$$y_{n+1} = y_n + h f(x_n, y_n)$$

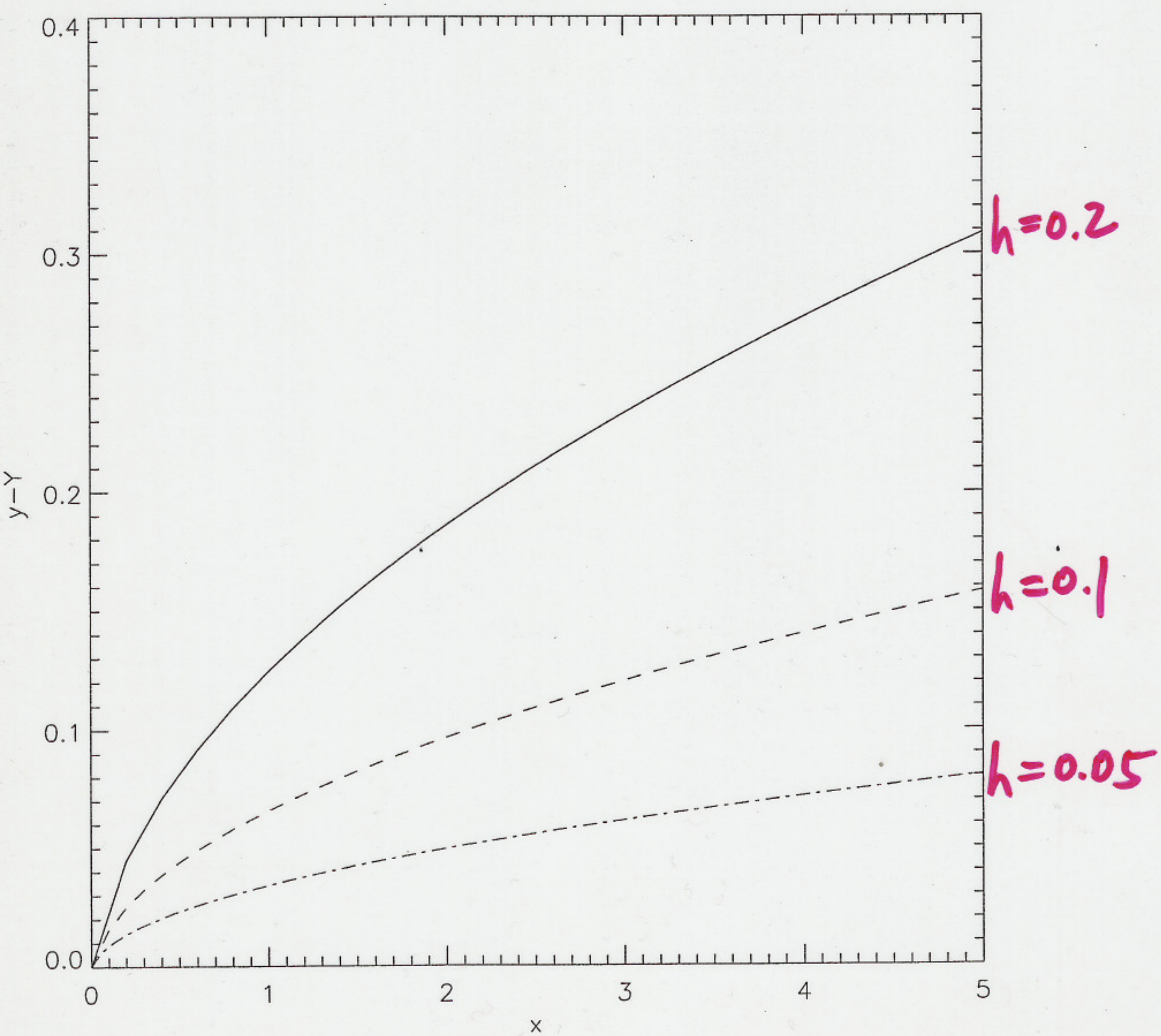


$$Y' = \alpha x^{\alpha-1} \qquad Y(0)=0$$

$$Y(x) = x^{\alpha}$$

$$\alpha = 1.5$$

$$y_{n+1} = y_n + f(x_n, y_n)\cdot h$$

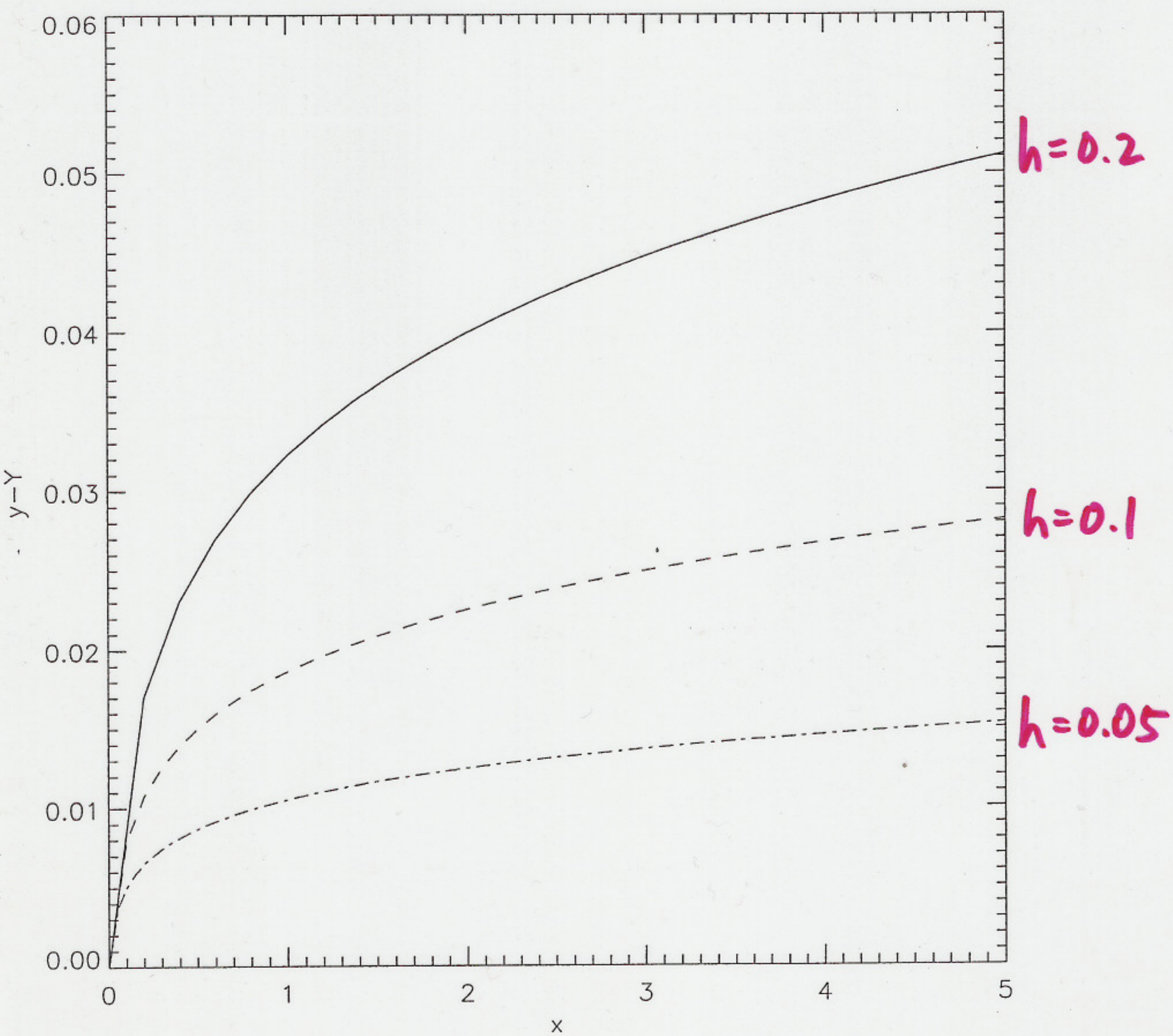


$$Y' = \alpha x^{\alpha-1} \qquad Y(0) = 0$$

$$Y(x) = x^{\alpha}$$

$$\alpha = 1.1$$

$$y_{n+1} = y_n + h f(y_n, x_n)$$



Runge-kutta methods of order 2

$$y_{n+1} = y_n + hF(x_n, y_n ; h) \qquad n \geq 0 \ , \ y_0 = Y_0$$

$$F(x, y ; h) = \gamma_1 f(x,y) + \gamma_2 f(x+\alpha h, y+\beta h f(x,y))$$

choose constants $\{\alpha, \beta, \gamma_1, \gamma_2\}$ s.t. the truncation error

$$T_{n+1} \equiv Y(x_{n+1}) - [Y(x_n) + hF(x_n, Y(x_n) ; h)] \quad \text{——— (1)}$$

will be of order $O(h^3)$.

First expand with respect to the second argument around y:

$$f(x+\alpha h, y+\beta h f(x,y))$$

$$= f(x+\alpha h, y) + f_z(x+\alpha h, y) \cdot \beta h f(x,y) + O(h^2)$$

$$f(x+\alpha h, y+\beta h f(x,y))$$

$$= f + f_x \cdot \alpha h + f_z \cdot \beta h f + O(h^2)$$

$$Y'' = f_x + f_z f$$

$$Y(x+h) = Y(x) + hY' + \frac{h^2}{2} Y'' + O(h^3)$$

$$= Y + hY' + \frac{h^2}{2}(f_x + f_z f) + O(h^3)$$

substitute into (1)

$$Y(x_{n+1}) - Y(x_n) - h F(x_n, Y(x_n); h)$$

$$= Y(x+h) - Y(x) - h F(x, Y(x); h)$$

$$= Y + hf + \frac{h^2}{2}(f_x + f_z f) - Y - h\gamma_1 f - h\gamma_2 (f + \alpha h f_x + \beta h f f_z) + O(h^3)$$

$$= h(1 - \gamma_1 - \gamma_2) f + \frac{h^2}{2}\left[(1 - 2\alpha\gamma_2) f_x + (1 - 2\beta\gamma_2) f_z f\right] + O(h^3)$$

$$\therefore \quad 1 - \gamma_1 - \gamma_2 = 0 \qquad \gamma_2 \neq 0 \qquad \gamma_1 = 1 - \gamma_2, \quad \alpha = \beta = \frac{1}{2\gamma_2}$$

$$1 - 2\alpha\gamma_2 = 0$$

$$1 - 2\beta\gamma_2 = 0$$

three favorite choices for γ_2: $\gamma_2 = \begin{cases} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{cases}$

$$\gamma_2 = \frac{1}{2} \qquad y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + h \cdot f(x_n, y_n))\right], \quad n \geq 0$$

more generally:

$$F(x, y, h; f) \equiv \gamma_1 f + \gamma_2 \{ f + h[\alpha f_x + \beta f f_y] + h^2 [\tfrac{1}{2}\alpha^2 f_{xx} + \alpha\beta f_{xy} f + \tfrac{1}{2}\beta^2 f^2 f_{yy}] \} + O(h^3)$$

$$Y' = f, \quad Y'' = f_x + f_y f, \quad Y^{(3)} = f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y f_x + f_y^2 f$$

Truncation error :

$$T_n(Y) = Y(x_{n+1}) - Y(x_n) - hF(x_n, Y(x_n), h; f)$$

$$= hY_n' + \frac{h^2}{2}Y_n'' + \frac{h^3}{6}Y_n^{(3)} + O(h^4) - hF(x_n, Y_n, h; f)$$

$$= h[1-\gamma_1-\gamma_2]f + h^2\left[\left(\frac{1}{2}-\gamma_2\alpha\right)f_x + \left(\frac{1}{2}-\gamma_2\beta\right)f_y f\right]$$

$$+h^3\left[\left(\frac{1}{6}-\frac{1}{2}\gamma_2\alpha^2\right)f_{xx} + \left(\frac{1}{3}-\gamma_2\alpha\beta\right)f_{xy}f + \left(\frac{1}{6}-\frac{1}{2}\gamma_2\beta^2\right)f_{yy}f^2 + \frac{1}{6}f_y f_x + \frac{1}{6}f_y^2 f\right] + O(h^4)$$

$$\Rightarrow \quad 1-\gamma_1-\gamma_2 = 0$$

$$\frac{1}{2}-\gamma_2\alpha = 0$$

$$\frac{1}{2}-\gamma_2\beta = 0$$

$$T_n(Y) \sim O(h^3)$$

$$= C(f,\gamma_2)h^3 + O(h^4)$$

where $|C(f, \gamma_2)| \le C_1(f) \cdot C_2(\gamma_2)$

$$C_1(f) \equiv \left[f_{xx}^2 + f_{xy}^2 f^2 + f_{yy}^2 f^4 + f_y^2 f_x^2 + f_y^4 f^2\right]^{1/2}$$

$$C_2(\gamma_2) \equiv \left[\left(\frac{1}{6}-\frac{1}{2}\gamma_2\alpha^2\right)^2 + \left(\frac{1}{3}-\gamma_2\alpha\beta\right)^2 + \left(\frac{1}{6}-\frac{1}{2}\gamma_2\beta^2\right)^2 + \frac{1}{18}\right]^{1/2}$$

minimum of $C_2(\gamma_2)$ is achieved when $\gamma_2 = \frac{3}{4}$

$$C_2\left(\frac{3}{4}\right) = \frac{1}{\sqrt{18}}$$

$$\Rightarrow \quad y_{n+1} = y_n + \frac{h}{4}\left[f(x_n,y_n) + 3f\left(x_n+\frac{2}{3}h,\ y_n+\frac{2}{3}hf(x_n,y_n)\right)\right]$$

A popular fourth-order RK Scheme :

$$v_1 = f(x_n, y_n)$$

$$v_2 = f(x_n + \frac{h}{2}, \; y_n + \frac{h}{2} v_1)$$

$$v_3 = f(x_n + \frac{h}{2}, \; y_n + \frac{h}{2} v_2)$$

$$v_4 = f(x_n + h, \; y_n + h v_3)$$

$$y_{n+1} = y_n + \frac{h}{6} [v_1 + 2v_2 + 2v_3 + v_4] \quad \text{——} \quad (*)$$

the truncation error is $O(h^5)$

when $Y'(x) = f(x)$ with no dependence of f on Y

$\Rightarrow$ equation $(*)$ reduces to Simpson's rule for numerical integration

previous example: $Y' = \alpha x^{\alpha-1} \longrightarrow Y' = \frac{\alpha Y}{x}$

which one would give more accurate solution?

$$Y' = \alpha x^{\alpha-1}$$

$$\alpha = 1.1$$

$$Y' = \frac{\alpha Y}{x}$$

$$\alpha = 1.1$$

4th order RK

$$v_1 = f(x_n, y_n)$$

$$v_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} v_1\right)$$

$$v_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} v_2\right)$$

$$v_4 = f(x_n + h, y_n + h v_3)$$

$$y_{n+1} = y_n + \frac{h}{6}\left[v_1 + 2v_2 + 2v_3 + v_4\right]$$