A-stability of multi-step methods

a general s-step method:

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m}) \quad n=0,1,\cdots$$

Applied to the solution of $y' = \lambda y \quad t \geq 0, \quad y(0) = 1$

$$\sum_{m=0}^{s} a_m y_{n+m} = h\lambda \sum_{m=0}^{s} b_m y_{n+m}$$

$$\sum_{m=0}^{s} (a_m - h\lambda b_m) y_{n+m} = 0 \quad n=0,1,\cdots$$

$$\sum_{m=0}^{s} g_m \cdot x^{(m)} = 0 \quad t \geq t_0 \quad \text{linear differential equations}$$

form the characteristic polynomial

$$\eta(w) := \sum_{m=0}^{s} g_m w^m$$

if $\eta(w) = 0$ has simple roots (no multiple roots) w_i

solution to $\sum_{m=0}^{s} (a_m - h\lambda b_m) y_{n+m} = 0$

is

$$y_n = \sum_{i=1}^{s} c_i \cdot w_i^n \quad n=0,1,\cdots$$

Root condition

Lemma : suppose the roots of the characteristic polynomial

$$\eta(z = h\lambda, \omega) = \sum_{m=0}^{s} (a_m - b_m z)\, \omega^m \qquad z \in \mathbb{C}$$

are $\omega_1(z), \omega_2(z), \dots, \omega_q(z)$ with multiplicities $k_1(z), k_2(z), \dots, k_q(z)$.

The multi-step method

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m}) \quad \text{is A-stable}$$

iff $|\omega_i(z)| < 1$ for $i = 1, 2, \dots q$ for all $z \in \mathbb{C}^-$

if $|\omega_i(z)| < 1$ $y_n \to 0$ faster than any polynomial in n

if $|\omega_i(z)| \geq 1$, there exists initial values s.t. $C_i \neq 0$ and $y_n \to \infty$ as $t \to \infty$ $\Rightarrow$ impossible for $y_n \to 0$ as $n \to \infty$

Example: AB 2nd order :

$$y_{n+1} = y_n + \frac{h}{2}\left[3y_n' - y_{n-1}'\right] \qquad n \geq 1$$

$$y_{n+1} - y_n = \frac{h}{2}\left(3y_n' - y_{n-1}'\right) \qquad y_n' = \lambda y_n$$

$$\therefore \quad y_{n+1} - y_n = \frac{h\lambda}{2}(3y_n - y_{n-1})$$

$$y_{n+1} - \left(1 + \frac{3}{2}h\lambda\right) y_n + \frac{h\lambda}{2} y_{n-1} = 0$$

characteristic polynomial for 2nd order AB :

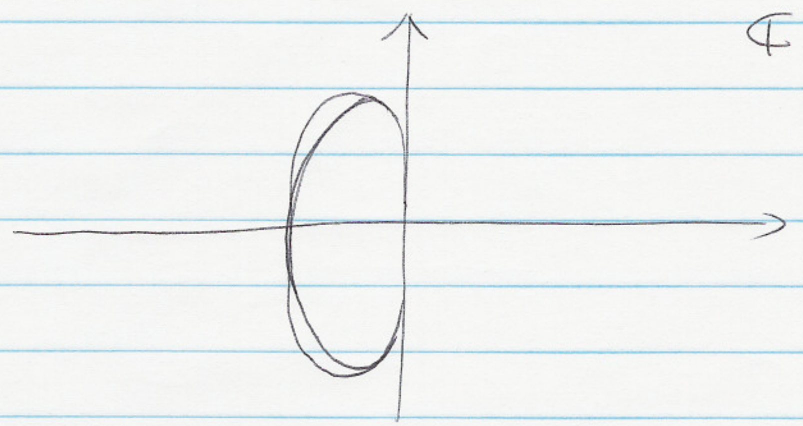
$$\omega^2 - (1 + \tfrac{3}{2}h\lambda)\,\omega + \frac{h\lambda}{2} = 0$$

$$h\lambda = z$$

roots : $\omega = \frac{1}{2}\left(1 + \frac{3}{2}z + \sqrt{1 + z + \frac{9}{4}z^2}\right)$

$$\omega = \frac{1}{2}\left(1 + \frac{3}{2}z - \sqrt{1 + z + \frac{9}{4}z^2}\right)$$

stability domain $|\omega| < 1$



Example: Adams- Moulton 2nd order

$$y_{n+1} = y_n + \frac{h}{2}\left[f'_{n+1} + f'_n\right]$$

$f'_{n+1} = \lambda y_{n+1}$

$$y_{n+1} = y_n + \frac{h\lambda}{2}\left(y_{n+1} + y_n\right)$$

$h\lambda = z$

$$\left(1 - \frac{z}{2}\right) y_{n+1} = \left(1 + \frac{z}{2}\right) y_n$$

$$y_{n+1} = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}\, y_n \longrightarrow \left|\frac{1 + z/2}{1 - z/2}\right| < 1 \quad \text{for } z \in \mathbb{C}^-$$

Example: AM 3rd order

$$y_{n+1} = y_n + \frac{h}{12}\left(5y'_{n+1} + 8y'_n - y'_{n-1}\right)$$

$$y_{n+1} = y_n + \frac{h\lambda}{12}\left(5y_{n+1} + 8y_n - y_{n-1}\right) \qquad h\lambda = z$$

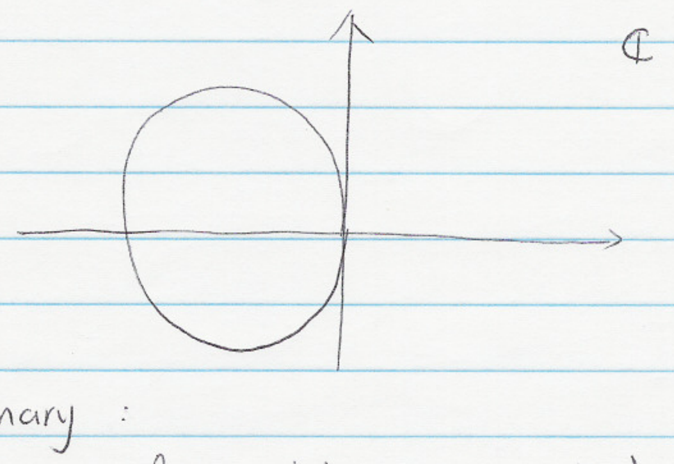
$$\left(1 - \frac{5}{12}z\right)y_{n+1} - \left(1 + \frac{2}{3}z\right)y_n + \frac{z}{12}y_{n-1} = 0$$

characteristic polynomial

$$\left(1 - \frac{5}{12}z\right)\omega^2 - \left(1 + \frac{2}{3}z\right)\omega + \frac{z}{12} = 0$$

roots: $$\omega = \frac{\left(1+\frac{2}{3}z\right) \pm \sqrt{\left(1+\frac{2}{3}z\right)^2 - 4\cdot\frac{z}{12}\left(1-\frac{5}{12}z\right)}}{2\left(1-\frac{5}{12}z\right)}$$

stability domain : $|\omega| < 1$



$\Rightarrow$ Summary :

for a general multi-step method

$$\sum_{m=0}^{s} a_m y_{n+m} = h\sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m}) \qquad n = 0, 1, \ldots$$

$b_s = 0 \rightarrow$ explicit

$b_s \neq 0 \rightarrow$ implicit

⓪ the s-step method is of order p if

$$\sum_{m=0}^{s} a_m\, y(t+mh) - h\sum_{m=0}^{s} b_m\, y'(t+mh) = \mathcal{O}(h^{p+1}) \quad h \to 0$$

① for all sufficiently smooth functions y.
Furthermore it is A-stable if the root condition is satisfied.

define $\rho(\omega) \equiv \sum_{m=0}^{s} a_m \omega^m$

$$\sigma(\omega) = \sum_{m=0}^{s} b_m \omega^m$$

NB: ⓪ amounts to

$$\rho(\omega) - \sigma(\omega)\ln\omega = c(\omega-1)^{p+1} + \mathcal{O}(|\omega-1|^{p+2}) \quad \omega \to 1$$

Thm: Dahlquist equivalence theorem:

Suppose that the error in the initial values $y_1, y_2, \cdots, y_{s-1}$ tends to zero as $h \to 0^+$, the multistep method is convergent iff

ⓐ it's of order $p \geq 1$

ⓑ $\rho(\omega)$ satisfies the root condition.

Recall what we did to derive the order of a given s-step method:

$$\psi(t,y) \equiv \sum_{m=0}^{s} a_m\, y(t+mh) - h\sum_{m=0}^{s} b_m\, y'(t+mh)$$

$$= \sum_{m=0}^{s} a_m \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k)}(t)\, m^k h^k - h\sum_{m=0}^{s} b_m \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k+1)}(t)\, m^k h^k$$

$$= \left(\sum_{m=0}^{s} a_m\right) y(t) + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{s} m^k a_m - k\sum_{m=0}^{s} m^{k-1} b_m\right) h^k y^{(k)}(t)$$

$\Rightarrow$ to obtain order p it is necessary and sufficient that

$$\left.\begin{aligned}
&\sum_{m=0}^{s} a_m = 0\\
&\sum_{m=0}^{s} m^k a_m = k\sum_{m=0}^{s} m^{k-1} b_m \qquad k=1,2,\dots,p\\
&\sum_{m=0}^{s} m^{p+1} a_m \neq (p+1)\sum_{m=0}^{s} m^p b_m
\end{aligned}\right\}\ (*)$$

$w = e^z$ $$\rho(e^z) - z\sigma(e^z) = \sum_{m=0}^{s} a_m e^{mz} - z\sum_{m=0}^{s} b_m e^{mz}$$

$$= \sum_{m=0}^{s} a_m\left(\sum_{k=0}^{\infty} \frac{1}{k!} m^k z^k\right) - z\sum_{m=0}^{s} b_m\left(\sum_{k=0}^{\infty}\frac{m^k z^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{s} m^k a_m\right) z^k - \sum_{k=1}^{\infty}\left(\frac{1}{(k-1)!}\sum_{m=0}^{s} b_m m^{k-1}\right) z^k$$

if $(*)$ are true: $\rho(e^z) - z\sigma(e^z) = c\cdot z^{p+1} + O(z^{p+2})$

Example: $y_{n+2} - 3y_{n+1} + 2y_n = h\left[\frac{13}{12} f(t_{n+2}, y_{n+2}) - \frac{5}{3} f(t_{n+1}, y_{n+1}) - \frac{5}{12} f(t_n, y_n)\right]$

$$\rho(w) = w^2 - 3w + 2 = (1+\xi)^2 - 3(1+\xi) + 2 = \xi(\xi - 1)$$

$$\sigma(w) = -\frac{5}{12} - \frac{5}{3} w + \frac{13}{12} w^2 = -\frac{5}{12} - \frac{5}{3}(1+\xi) + \frac{13}{12}(1+\xi)^2$$

$$\rho(w) - \sigma(w) \ln w$$

$$= \xi(\xi - 1) - \left(\xi - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 - \frac{1}{4}\xi^4 + \cdots\right)\left(\frac{13}{12}\xi^2 + \frac{1}{2}\xi - \frac{1}{6}\right)$$

$$= \xi^2 - \xi + \xi - \xi^2 - \frac{1}{2}\xi^3 + \cdots$$

$$= -\frac{1}{2}\xi^3 + O(\xi^4)$$

$\Rightarrow$ the method is 2nd order

but is not convergent

$\rho(w) = 0$ has roots 2 & 1 $\Rightarrow$ violate the root condition

root condition:

A polynomial obeys the root condition if all its zeros reside in the closed complex unit disc and all its zeros of unit modulus are simple.

Adams-Bashforth methods : $\rho(w) = w^{s-1}(w-1)$

for all $s \geq 1$, all AB methods satisfy the root condition

$\Rightarrow$ all AB methods are convergent

* Dahlquist first barrier

maximal order of a convergent s-step method is at most $2[(s+2)/2]$ for implicit schemes and just s for explicit schemes.

* Dahlquist 2nd barrier

The highest order of an A-stable multistep method is two.