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A-stability for multi-step method: $\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$

characteristic polynomial

$$\eta(z, w) = \sum_{m=0}^{s} (a_m - b_m z) w^m \qquad z \in \mathbb{C}$$

have roots $w_1(z), w_2(z), \ldots, w_{q}(z)$

the multistep method is A-stable iff

$$|w_i(z)| < 1 \qquad i = 1, 2, \cdots, q(z) \qquad \text{for all } z \in \mathbb{C}^-$$

⇒ Dahlquist equivalence theorem:

Suppose that the error in the starting values $y_1, y_2, \cdots y_{s-1}$ tends to zero as $h \to 0^+$. The multi step method is convergent iff ① it's of order $p \geq 1$

② the polynomial $\rho(w) = \sum_{m=0}^{s} a_m w^m$ obeys the root condition.

Example: AB ~~2nd order~~ s-step $\rho(w) = w^{s-1}(w-1)$

Dahlquist second barrier ⇒ the highest order of an A-stable multistep method is 2

Multistep method is A-stable iff $b_s > 0$ and $|w_i(it)| \leq 1 \quad t \in \mathbb{R}$

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$y' = f(x) \qquad x \geq x_0, \quad y(x_0) = y_0$

$$y(x) = y_0 + \int_{x_0}^{x} f(x')\,dx'$$

replacing the integral with a finite sum

$$\int_a^b f(x')\,\omega(x')\,dx' \approx \sum_{j=1}^{\nu} b_j f(c_j) \quad \text{—— } (*)$$

$b_1, b_2, \ldots, b_\nu \rightarrow$ quadrature weights

$c_1, c_2, \ldots c_\nu \rightarrow$ quadrature nodes

for every f with p smooth derivatives

$$\left| \int_a^b f(x)\,\omega(x)\,dx - \sum_{j=1}^{\nu} b_j f(c_j) \right| \leq C \max_{a \leq x \leq b} \left| f^{(p)}(x) \right|$$

→ such a quadrature is of order p

for every $f \in \mathbb{P}_{p-1}$ (real poly of degree $p-1$)

$(*)$ is of order p if it is exact for every $f \in \mathbb{P}_{p-1}$

for distinct quadrature nodes

→ $[b]$ can be determined uniquely

quadrature & polynomial (orthogonal) family

Thm: $c_1, c_2, \cdots c_\nu$ be zeros of P_ν and

$b_1, b_2 \dots b_\nu$ be the solution of the Vandermonde system

the quadrature method is of order 2ν

Quadrature & RK schemes:

$$\tilde{y}_\nu = y_n + h \cdot \sum_{i=1}^{\nu} \underbrace{\boxed{a_{\nu,i}}}_{A} f(x_n + c_i h, \tilde{y}_i)$$

$$y_{n+1} = y_n + h \sum_{j=1}^{\nu} b_j f(x_n + c_j h, \tilde{y}_j)$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\nu \end{bmatrix} \quad \text{is the node}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_\nu \end{bmatrix} \quad \text{is the weight}$$

$$\begin{array}{c|c} c\,[\;] & A \\ \hline & b^T [\quad\quad] \end{array}$$

1 (b) $y_{n+2} - y_n = \frac{2}{3}h\left[f(t_{n+2}, y_{n+2}) + f(t_{n+1}, y_{n+1}) + f(t_n, y_n) \right]$

$$y_{n+2} = y_n + y_n' \cdot 2h + \frac{y_n''}{2!}(2h)^2 + \frac{y_n^3}{3!}(2h)^3 + \cdots$$

$$y_{n+2}' = y_n' + y_n'' \cdot 2h + \frac{y_n^3}{2!}(2h)^2 + \frac{y_n^4}{3!}(2h)^3 + \cdots$$

$$y_{n+1}' = y_n' + y_n'' \cdot h + \frac{y_n^3}{2!}(h)^2 + \frac{y_n^4}{3!}(h)^3 + \cdots$$

$$+) \quad y_n' = y_n'$$

$$y_{n+2}' + y_{n+1}' + y_n' = 3y_n' + 3h \cdot y_n'' + \frac{5h^2}{2} \cdot y_n^3 + \frac{9h^3}{3!} y_n^4 + \cdots$$

$$y_{n+2} - y_n = 2h \cdot y_n' + 2h^2 \cdot y_n'' + \frac{8h^3}{6} y_n^3$$

$$\frac{2}{3}h(y_{n+2}' + y_{n+1}' + y_n') = 2hy_n' + 2h^2 y_n'' + \frac{5h^3}{3} y_n^3 + \cdots$$

$$\therefore \quad (y_{n+2} - y_n) - \frac{2}{3}h(f_{n+2} + f_{n+1} + f_n) = -\frac{h^3}{3} y_n^3 + O(h^4)$$

$$= O(h^3)$$

$\therefore$ consistency order is 2

(C) $y_{n+2} - y_n = \frac{2}{3}h\lambda(y_{n+2} + y_{n+1} + y_n) = \frac{2}{3}z(y_{n+2} + y_{n+1} + y_n)$

$$(-1 - \tfrac{2}{3}z)\omega^0 + (0 - \tfrac{2}{3}z)\omega + (1 - \tfrac{2}{3}z)\omega^2 = 0$$

$$\omega = \frac{\frac{2}{3}z \pm \sqrt{(\frac{2}{3}z)^2 - 4(1-\frac{2}{3}z)(1-\frac{2}{3}z)}}{2(1-\frac{2}{3}z)} = \frac{\frac{2}{3}z \pm \sqrt{(\frac{2}{3}z)^2 + 4 - 4(\frac{2}{3}z)^2}}{2(1-\frac{2}{3}z)}$$

$$w(z) = \frac{\frac{2}{3}z \pm \sqrt{4 - 3\cdot\left(\frac{2}{3}z\right)^2}}{2\left(1 - \frac{2}{3}z\right)}$$

$z = it \quad t \in R$

$$w(it) = \frac{\frac{2}{3}it \pm \sqrt{4 + \frac{4}{3}t^2}}{2\left(1 - \frac{2}{3}it\right)} = \frac{\frac{it}{3} \pm \sqrt{1 + \frac{t^2}{3}}}{1 - \frac{2}{3}it}$$

$$|w| = \frac{\sqrt{1 + \frac{t^2}{3} + \frac{t^2}{9}}}{\sqrt{1 + \frac{4}{9}t^2}} = \frac{\sqrt{1 + \frac{4}{9}t^2}}{\sqrt{1 + \frac{4}{9}t^2}} = 1 \quad \text{for all } t$$

this method is implicit and $|w(it)| \le 1$ for all $t \in \mathbb{R}$

$\Rightarrow$ it is A- stable

Lemma: The multistep is A- stable iff $b_s > 0$ and

$|w_1(it)|, \quad |w_2(it)|, \quad \dots, \quad |w_q(it)| \le 1 \quad t \in R$

where $w_1, w_2, \cdots w_q(z)$ are the zeros of the charateristic polynomial.

Lemma: $|r(z)| < 1$ for all $z \in C^{-1}$ iff all the poles of r have positive real parts and $|r(it)| \le 1$ for all $t \in \mathbb{R}$

3 (a) $y_{n+2} - y_n = 2h \cdot f(t_{n+1}, y_{n+1})$

$$y_{n+2} = y_n + y_n' \cdot 2h + \frac{y_n''}{2!}(2h)^2 + \frac{y_n^3}{3!}(2h)^3 + \cdots$$

$$y_{n+1}' = y_n' + y_n'' h + \frac{y_n^3}{2!} \cdot h^2 + \frac{y_n^4}{3!} h^3 + \cdots$$

$$y_{n+2} - y_n = y_n' \cdot 2h + \frac{y_n''}{2!}(2h)^2 + \frac{y_n^3}{3!}(2h)^3 + \cdots$$

$\therefore$ $y_{n+2} - y_n - 2h\, f(t_{n+1}, y_{n+1})$

$$= \left(\frac{8h^3}{6} - h^3\right) y_n^3 + hot$$

$$= \frac{1}{3} h^3 y_n^3 + hot = O(h^3)$$

consistency order is 2.

(b) $(-1)\,w^0 + (0 - 2z)\,w + w^2 = 0$

$$w^2 - 2zw - 1 = 0$$

$$w = \frac{2z \pm \sqrt{(2z)^2 + 4}}{2} = z \pm \sqrt{1 + z^2}$$

cannot be bound by $|w| < 1$ for all $z \in \mathbb{C}^{-1}$

$\Rightarrow$ Not A-stable

practice problem

1 (b). $u' = 2\frac{u}{x}$ $\quad \frac{u'}{u} = \frac{2}{x}$ $\quad \ln u = \ln x^2$

$u = x^2 \quad u(1) = 1$

Euler's method:

$$\frac{u_{n+1} - u_n}{h} = \frac{2u_n}{x_n} \qquad u_1 = \left(1 + \frac{2h}{x_0}\right) u_0 = (1+2h)\cdot 1$$

$$u_{n+1} = \left(\frac{2h}{x_n} + 1\right) u_n$$

$$u_{n+1} = \left(1 + \frac{2h}{x_n}\right) u_n = \left(1 + \frac{2h}{x_n}\right)\left(1 + \frac{2h}{x_{n-1}}\right) u_{n-1}$$

$$= \left(1 + \frac{2h}{x_n}\right)\left(1 + \frac{2h}{x_{n-1}}\right) \cdots \left(1 + \frac{2h}{x_0}\right) \cdot 1$$

$x_0 = 1 \quad x_1 = 1 + h \quad x_n = 1 + nh$

$$u_{n+1} = \frac{x_n + 2h}{x_n} \cdot \frac{x_{n-1} + 2h}{x_{n-1}} \cdot \cdots \cdot \frac{x_0 + 2h}{x_0}$$

$$= \frac{x_{n+2}}{x_n} \cdot \frac{x_{n+1}}{x_{n-1}} \cdot \frac{x_n}{x_{n-2}} \cdots \frac{x_2}{x_0}$$

$$= \frac{x_{n+2} \cdot x_{n+1}}{x_1} = \frac{(1+(n+2)h)(1+(n+1)h)}{1+h}$$

error: $$u_{n+1} - (x_{n+1})^2 = \frac{[1+(n+2)h][1+(n+1)h]}{1+h} - [1+(n+1)h]^2$$

$$= -(n+1) \cdot \frac{1+(n+1)h}{1+h} \cdot h^2$$

$\sim h^2$ when $h \ll 1$

3. $A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= N} + \underbrace{\begin{pmatrix} 0 & \alpha & \alpha \\ \alpha & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix}}_{= P}$

$$M = N^{-1} P = \begin{pmatrix} 0 & \alpha & \alpha \\ \alpha & 0 & \alpha \\ \alpha & \alpha & 0 \end{pmatrix}$$

$$N x^{(k+1)} = b + P x^{(k)} \qquad k = 0, 1, 2, \cdots$$

$$x^{(k+1)} = N^{-1} \cdot b + N^{-1} P x^{(k)}$$

$$x^{(k+1)} = b + P \cdot x^{(k)}$$

$$= b + P \cdot (b + P \cdot x^{(k-1)})$$

$$= b + P \cdot b + P^2 \cdot x^{(k-1)}$$

$$= b + Pb + P^2 (b + P x^{(k-2)})$$

$$= b + Pb + P^2 b + P^3 x^{(k-2)}$$

$$\therefore \boxed{x^{(m+1)} = \sum_{i=0}^{m} P^{(i)} b + P^{m+1} \cdot x^{(0)}}$$

for convergence: |eigenvalue of M| < 1

$$\begin{vmatrix} -\lambda & \alpha & \alpha \\ \alpha & -\lambda & \alpha \\ \alpha & \alpha & -\lambda \end{vmatrix} = 0 \qquad (-\lambda)(\lambda^2 - \alpha^2) - \alpha(-\alpha\lambda - \alpha^2) + \alpha(\alpha^2 + \alpha\lambda) = 0$$

$$-\lambda^3 + \lambda\alpha^2 + 2\alpha^2\lambda + 2\alpha^3 = 0$$

$$\lambda^3 - 3\alpha^2\lambda - 2\alpha^3 = 0$$

~~$\lambda^3 - 2\alpha^2\lambda$~~ $\lambda = \alpha, -\alpha, 2\alpha$

$|2\alpha| < 1 \qquad \alpha < \frac{1}{2} \qquad \boxed{\frac{1}{2} \leq \alpha}$ for non-convergence