1.11.1

(a) $a\sin x + b\sin^2 x + c\sin^3 x = 0$

$\sin x\,(a + b\sin x + c\sin^2 x) = 0$

when $x = n\pi$ $\sin x(a + b\sin x + c\sin^2 x) = 0$

so $\sin x, \sin^2 x, \sin^3 x$ are linearly independent.

(b) $a e^{\alpha x} + b\cdot x\cdot e^{\alpha x} = 0$

$e^{\alpha x}(a + bx) = 0$ $\quad a + bx = 0 \quad \Rightarrow$ linearly ind.

(c) $a\sin x + b\sin 2x + c\sin 3x + d\sin 4x + e\sin 5x = 0$

$\rightarrow$ linearly independent

1.11.2 $xy' + 2y = e^x$ $\qquad y' + \frac{2}{x}y = \frac{e^x}{x}$

$\mu' = \mu\frac{2}{x}$ $\qquad \frac{\mu'}{\mu} = \frac{2}{x}$ $\ln\mu = \ln x^2$ $\quad \mu = x^2$

$x^2 y = \int x e^x\,dx + C$

$x^2 y = xe^x - e^x + C$, $\quad y = \frac{1}{x^2}(xe^x - e^x + c)$

1.11.3 $y' - y\sin x = 0$

$\mu' = -\mu\sin x$

$\frac{\mu'}{\mu} = -\sin x$ $\qquad \ln\mu = \cos x$, $\quad \mu = e^{\cos x}$

$e^{\cos x} y = C$ $\qquad y = Ce^{-\cos x}$

1.11.4 $y'\sin x + y\cos x = x$ $\qquad y' + \frac{\cos x}{\sin x}y = \frac{x}{\sin x}$

$\frac{\mu'}{\mu} = \frac{\cos x}{\sin x}$ $\qquad \ln\mu = \ln\sin x$ $\quad \mu = \sin x$

$\sin x\cdot y = \frac{x^2}{2} + c$ $\qquad y = \frac{(x^2/2 + c)}{\sin x}$

1.11.5 $(1+x^2)y' + xy = x(1+x^2)$

$y' + \frac{x}{1+x^2}y = x$

$\frac{\mu'}{\mu} = \frac{x}{1+x^2}$ $\quad \ln\mu = \frac{1}{2}\ln(1+x^2)$ $\quad \mu = \sqrt{1+x^2}$

$\sqrt{1+x^2}\, y = \int x\sqrt{1+x^2}\,dx + c$

$\int x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,\frac{du}{2}$ $\quad u = 1+x^2$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{1}{3}u^{3/2}$

$y = \frac{1}{3}(1+x^2) + c(1+x^2)^{-1/2}$

1.11.6 $y'' - 4y' + 4y = 0$ $\quad r^2 - 4r + 4 = 0$ $\quad r = 2, 2$

$y = c_1 e^{2x} + c_2 x e^{2x}$

1.11.7 $y'' - 4y' + 13y = 0$ $\quad r^2 - 4r + 13 = 0$ $\quad (r-2)^2 = -9$

$r = 2 \pm 3i$

$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

1.11.8 $y'' - 4y' + 3y = 0$ $\quad r^2 - 4r + 4 - 1 = 0$ $\quad (r-2)^2 = 1$ $\quad r = 1, 3$

$y = c_1 e^x + c_2 e^{3x}$

1.11.9 $y^{(4)} - y = 0$ $\quad r^4 - 1 = 0$ $\quad (r^2+1)(r^2-1) = 0$ $\quad r = 1, -1, i, -i$

$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

1.11.10 $y^{(4)} + y = 0$ $\quad r^4 + 1 = 0$ $\quad r^4 = -1 = e^{\frac{\pi}{2}i + (2n\pi)i}$

$r = e^{\frac{\pi}{8}i + \frac{2n}{4}\pi i} = e^{\left(\frac{n\pi}{2} + \frac{\pi}{8}\right)i}$ $\quad n = 0, 1, 2, 3$

$r = e^{\frac{\pi}{8}i},\ e^{\frac{5\pi}{8}i},\ e^{\frac{9\pi}{8}i},\ e^{\frac{13\pi}{8}i}$

$y = C_1 e^{(e^{\frac{\pi}{8}i})x} + C_2 e^{(e^{\frac{5\pi}{8}i})x} + C_3 e^{(e^{\frac{9\pi}{8}i})x} + C_4 e^{(e^{\frac{13\pi}{8}i})x}$

1.11.11 $y^{(4)} - 2y'' + y = 0 \quad r^4 - 2r^2 + 1 = 0 \quad (r^2-1)^2 = 0$

$r = +1, +1, -1, -1$

$y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + C_4 x e^{-x}$

1.11.12 $x^2 y'' + 3xy' + y = 0$ equidimensional equation

$r(r-1) + 3r + 1 = 0 \quad r^2 + 2r + 1 = 0 \quad r = -1, -1$

$y = C_1 \frac{1}{x} + C_2 \ln x \frac{1}{x}$

1.11.13 $xy'' + 3y' = x^2$

$u = y' \quad xu' + 3u = x^2 \quad u' + \frac{3}{x}u = x$

$\frac{\mu'}{\mu} = \frac{3}{x} \quad \ln \mu = 3 \ln x, \quad \mu = x^3$

$x^3 u = \frac{x^2}{2} + c, \quad u = \frac{1}{2x} + \frac{c}{x^3} = y'$

$y = \frac{1}{2}\ln x + \frac{c'}{x^2} + D$

1.11.14 $y'' - 2y' + y = xe^x$

$r^2 - 2r + 1 = 0 \quad r = 1, 1$

$y_g = C_1 e^x + C_2 x e^x \quad y_p = (Ax+B)x^2 e^x$

$y_p' = (3Ax^2 + 2Bx + Ax^3 + Bx^2)e^x = [Ax^3 + (3A+B)x^2 + 2Bx]e^x$

$y_p'' = [3Ax^2 + (6A+2B)x + 2B + Ax^3 + (3A+B)x^2 + 2Bx]e^x$

$= [Ax^3 + (6A+B)x^2 + (6A+4B)x + 2B]e^x$

$$Ax^3+(6A+B)x^2+(6A+4B)x+2B-2(Ax^3+(3A+B)x^2+2Bx)+(Ax^3+Bx^2)=x$$

x^3: $A-2A+A=0$

x^2: $6A+B-2(3A+B)+B=0$

x: $6A+4B-4B=1$ $\quad A=\frac{1}{6}$

x^0: $2B=0$ $\quad B=0$

$$y=c_1e^x+c_2xe^x+\frac{1}{6}x^3e^x$$

1.11.15 $\quad y^{(4)}+5y''+4y=\sin x$

$r^4+5r^2+4=0,\ (r^2+1)(r^2+4)=0, \quad r=\pm 2i,\ \pm i$

$$y_g=c_1\cos x+c_2\sin x+c_3\cos 2x+c_4\sin 2x$$

$$y_p=x(A\sin x+B\cos x)$$

$$y_p'=A\sin x+B\cos x+x(A\cos x-B\sin x)$$

$$y_p''=A\cos x-B\sin x+A\cos x-B\sin x+x(-A\sin x-B\cos x)$$

$$y_p''=2A\cos x-2B\sin x-x(A\sin x+B\cos x)$$

$$y_p'''=-2A\sin x-2B\cos x-(A\sin x+B\cos x)-x(A\cos x-B\sin x)$$

$$y_p^{(3)}=-3A\sin x-3B\cos x-x(A\cos x-B\sin x)$$

$$y_p^{(4)}=-3A\cos x+3B\sin x-(A\cos x-B\sin x)-x(-A\sin x-B\cos x)$$

$$=-4A\cos x+4B\sin x+x(A\sin x+B\cos x)$$

$$y_p^{(4)}+5y_p''+4y_p=\sin x$$

$$-4A\cos x+4B\sin x+\cancel{x(A\sin x+B\cos x)}+5\left(2A\cos x-2B\sin x-\cancel{x(A\sin x+B\cos x)}\right)$$
$$+\cancel{4x(A\sin x+B\cos x)}=\sin x$$

$-4A+10A=0 \quad A=0$

$4B-10B=1 \quad -6B=1 \quad B=-\frac{1}{6}$

$$y_p=-\frac{1}{6}x\sin x$$

1.11.16 $y'' - y = (x+\sin x)e^x$

$r^2 - 1 = 0 \quad r = \pm 1 \,, \quad y_g = C_1 e^x + C_2 e^{-x}$

$y_p = x(Ax+B)e^x + (C\sin x + D\cos x)e^x$

$y_p' = (2Ax + B + Ax^2 + Bx)e^x + (C\cos x - D\sin x + C\sin x + D\cos x)e^x$

$y_p'' = (2Ax + 2A + B + Ax^2 + (2A+B)x + B)e^x$
$\quad + [-(C+D)\sin x + (-D+C)\cos x + (C+D)\cos x + (C-D)\sin x]e^x$

$y_p'' - y_p = (x + \sin x)e^x$

$$\begin{aligned}&(Ax^2 + (4A+B)x + 2A + 2B \\ &- Ax^2 - Bx)e^x + (2C\cos x - 2D\sin x \\ &- D\cos x - C\sin x)e^x = (x+\sin x)e^x\end{aligned}$$

$4A = 1 \quad A = \frac{1}{4}$ $\qquad 2C - D = 0$

$2A + 2B = 0 \quad B = -A = -\frac{1}{4}$ $\qquad 2D + C = 1$

$C + 2D = 1 \qquad C = \frac{1}{5}, D = \frac{2}{5}$

$$y = C_1 e^x + C_2 e^{-x} + \frac{x}{4}(x-1)e^x + \frac{1}{5}(\sin x + 2\cos x)e^x$$

1.11.17 $y'' - 2y' + 2y = e^x \sin x$

$r^2 - 2r + 2 = 0, \quad r = 1 \pm i \quad y_g = e^x(C_1\cos x + C_2 \sin x)$

$y_p = x(A\sin x + B\cos x)e^x$

$y_p' = (A\sin x + B\cos x)e^x + x(A\cos x - B\sin x)e^x + x(A\sin x + B\cos x)e^x$
$\quad = [(A + Ax - Bx)\sin x + (B + xA + xB)\cos x]\cdot e^x$

$y_p'' = [(A-B)\sin x + (A+B)\cos x]e^x$
$\quad + [(A + (A-B)x)\cos x + (B + Ax + Bx)(-\sin x)]e^x$
$\quad + [(A + Ax - Bx)\sin x + (B + Ax + Bx)\cos x]e^x$

$y_p'' - 2y_p' + y_p = e^x \sin x$

$[(A-B-B-Ax-Bx+A+Ax-Bx)\sin x + (A+B+A+Ax-Bx+B \quad)\cos x]$
$+Ax+Bx$

$-2[(A+Ax-Bx)\sin x + (B+Ax+Bx)\cos x]$

$+2 \quad (Ax\sin x + Bx\cos x) = \sin x$

$\cos x$: $A+2B+2\cancel{A}x - 2B - 2\cancel{A}x - 2\cancel{B}x + 2\cancel{B}x = 0 \qquad A=0$

$\sin x$: $A-2B-\cancel{A}x - Bx + A + \cancel{A}x - Bx - 2(A+\cancel{Ax}-Bx) + 2\cancel{A}x = 1$

$-2B=1, \quad B=-\frac{1}{2}$

$y = (C_1\cos x + C_2\sin x)e^x - \frac{1}{2}x\cos x e^x$

1.11.18 $\quad y'' + 4y' + 4y = e^{-2x}/x^2$

$r^2+4r+4=0 \qquad r=-2,-2, \qquad y_1 = e^{-2x}, \quad y_2 = xe^{-2x}$

$y_p = \left(-\int \frac{xe^{-2x}\frac{e^{-2x}}{x^2}}{e^{-4x}}dx\right)y_1 + \left(\int \frac{e^{-2x}\frac{e^{-2x}}{x^2}}{e^{-4x}}\right)y_2$

$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}-2xe^{-2x} \end{vmatrix} = e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}$

$y_p = e^{-2x}(-\ln x) + xe^{-2x}\left(-\frac{1}{x}\right)$

$y = C_1e^{-2x} + C_2xe^{-2x} + y_p$

1.11.19 $\quad y'' + y = f(x) \qquad r^2+1=0 \qquad r=\pm i$

$y = C_1\cos x + C_2\sin x + \left(\int \frac{-\sin x f(x)}{-1}dx\right)\cos x + \left(\int \frac{\cos x f(x)}{-1}dx\right)\sin x$

1.11.20 $(x-1)y'' - xy' + y = 0, \quad y_1 = e^x$

$a_0 = x-1 \qquad y = u \cdot y_1 \longrightarrow v' + \left(\frac{2y_1'}{y_1} + \frac{a_1}{a_0}\right) v = 0$

$a_1 = -x \qquad v' = u$

$a_2 = 1 \qquad v' + \left(2 \cdot 1 + \frac{-x}{x-1}\right) v = 0$

$v' + \frac{2x-2-x}{x-1} v = 0 \qquad v' + \frac{x-2}{x-1} v = 0$

$\frac{v'}{v} = -\frac{x-2}{x-1} = -\left(1 - \frac{1}{x-1}\right)$

$u = \int e^{-x}(x-1)\,dx = -xe^{-x}$

$\ln v = -x + \ln(x-1)$

$v = e^{-x} \cdot (x-1) \qquad \boxed{y = -xe^{-2x}}$

1.11 21 $y'' + y' + ye^{-2x} = 0. \qquad y_1 = \sin(e^{-x}) \qquad y_2 = uy_1 \quad v = u'$

$v' + \left(2\frac{-e^{-x}\cos(e^{-x})}{\sin(e^{-x})} + \frac{1}{1}\right) v = 0$

$v' + \left(1 - 2\cot(e^{-x})e^{-x}\right) v = 0$

$\frac{v'}{v} = 2\cot(e^{-x})\, e^{-x} - 1$

$\ln v = 2\int e^{-x} \frac{\cos e^{-x}}{\sin e^{-x}}\,dx - x = -2\ln \sin e^{-x} - x$

$u = -\cot \cdot e^{-x}$

$v = e^{-x} \frac{1}{\sin^2 e^{-x}} \qquad \boxed{y_2 = -\frac{\cos(e^{-x})}{1}}$

1.11. 22 $xy'' - 2(x+1)y' + 4y = 0, \quad y_1 = e^{2x} \quad y = v \cdot y_1$

$v' + \left(4 + \frac{-(2x+2)}{x}\right) v = 0 \qquad \ln v = \ln x^2 - 2x$

$\frac{v'}{v} = -4 + 2 + \frac{2}{x} = 2 + \frac{2}{x} \qquad v = x^2 e^{-2x} \qquad \boxed{y = x^2}$

1.11.23 $4x^2y'' = (16x^4+3)y$, $y_1 = x^{-1/2}e^{x^2}$, $y = uy_1$, $u' = v$

$v' + \left(2\frac{y_1'}{y_1} + \frac{a_1}{a_0}\right)v = 0$

$y_1' = -\frac{1}{2}x^{-3/2}e^{x^2} + 2x\,x^{-\frac{1}{2}}e^{x^2}$

$= \left(2x^{\frac{1}{2}} - \frac{1}{2}x^{-3/2}\right)e^{x^2}$

$v' + \left(2\cdot\frac{2x^{\frac{1}{2}} - \frac{1}{2x^{3/2}}}{x^{-1/2}} + \frac{0}{4x^2}\right)v = 0$

$v' + 2\left(2x - \frac{1}{2x}\right)v = 0.$

$\frac{v'}{v} = -4x + \frac{1}{x}$ $\quad \ln v = -2x^2 + \ln x$, $v = e^{-2x^2}\cdot x$

$u = \int xe^{-2x^2}dx$

$= -\frac{1}{4}e^{-2x^2}$

$\boxed{y = -\frac{1}{4}x^{-1/2}e^{-x^2}}$

1.11.24 $x^3y'' - xy' + y = 0$ $\quad y_1 = x$

$y_2 = ux$, $u' = v$

$v' + \left(2\cdot\frac{1}{x} + \frac{-x}{x^3}\right)v = 0$

$\frac{v'}{v} = -\frac{2}{x} + \frac{1}{x^2}$ $\quad \ln v = \ln\frac{1}{x^2} - \frac{1}{x}$

$u = -e^{-\frac{1}{x}} \Leftarrow v = \frac{e^{-\frac{1}{x}}}{x^2}$, $\quad \boxed{y = -xe^{-1/x}}$

1.1.25 $(1-x^2)y'' - 2xy' + 2y = 6$, $y_1 = x$, $y_2 = u\cdot x$, $u' = v$

$v' + \left(2\cdot\frac{1}{x} + \frac{-2x}{1-x^2}\right)v = 0$

$\frac{v'}{v} = -\frac{2}{x} + \frac{2x}{1-x^2}$ $\quad \ln v = \ln x^{-2} - \ln(1-x^2)$, $v = \frac{1}{x^2(1-x^2)}$

$y_2 = -1 + \frac{x}{2}\ln\frac{1+x}{1-x}$ $\quad u = \int\frac{1}{x^2(1-x^2)}dx = \int \frac{1}{x^2} + \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x}dx$

$= -\frac{1}{x} + \frac{1}{2}\ln\frac{1+x}{1-x}$

Upon inspection, $y_p = 3$ because

$(1-x^2)\, y_p'' - 2x\, y_p' + 2 \cdot y_p = 6$ is satisfied

general solution $\Rightarrow$ $y = c_1 x + c_2 \left(-1 + \frac{x}{2} \ln \frac{1+x}{1-x}\right) + 3$

1.11.26 $\quad x y'' - (2x^2+3) y' + 4xy = 8x^5 \cdot e^{2x^2}$, $y_1 = e^{x^2}$, $y_2 = u y_1$, $v' = u$

$$v' + \left(2 \cdot \frac{2x e^{x^2}}{e^{x^2}} + \frac{-2x^2-3}{x}\right) v = 0$$

$$\frac{v'}{v} = -4x + 2x + \frac{3}{x}$$

$\ln v = -x^2 + 3 \ln x \qquad v = e^{-x^2} \cdot x^3 \qquad y_2 = (x^2+1)$

$$y_p = \left(\int \frac{-x^3 \cdot 8x^4 e^{2x^2}}{W}\right) e^{x^2} + \left(\int \frac{e^{x^2}\ 8x^4 e^{2x^2}}{W}\right) x^3$$

$$W = \begin{vmatrix} e^{x^2} & x^2+1 \\ 2x e^{x^2} & 2x \end{vmatrix} = 2x e^{x^2} - 2x(x^2+1) e^{x^2} = -2x^3 \cdot e^{x^2}$$

1.11.27 $\quad y(x) = ax^3 + bx^2 + cx + d$

$y'' = 6ax + 2b$

$(x^2-1)(6ax+2b) + \alpha(ax^3 + bx^2 + cx + d) = 0$

$(6ax^4 + 2bx^2 - 6ax - 2b) + \alpha a x^3 + \alpha b x^2 + \alpha c x + \alpha d = 0$

$a = 0 \qquad 2b + \alpha b = 0 \qquad -2b + \alpha d = 0$

$+\alpha c = 0$

$(2+\alpha)b=0$ $\alpha c=0$ $2b=\alpha d$

$\alpha=-2$ or $b=0$

$\alpha=0$ or $c=0$

when $\boxed{\alpha=-2,}$ $C=0,$ $b+d=0$ $\boxed{y_1=(x^2-1)}$

$b=0$ $\alpha c=0,$ $\alpha d=0$ if $\alpha\neq 0 \Rightarrow$ trivial solution

$\boxed{\alpha=0,}$ $\boxed{y_2=cx+d}$

1.11.28 $P(x)=ax^2+bx+c$

$P''=2a$

$x(x+2)\cdot 2a+\alpha(ax^2+bx+c)=0$

$2ax^2+4ax+\alpha ax^2+\alpha bx+\alpha c=0$

$(2a+\alpha a)x^2+(4a+\alpha b)x+\alpha c=0$

$$\begin{cases}(2+\alpha)a=0 & \alpha=-2 \text{ or } a=0.\\ 4a+\alpha b=0 \\ \alpha c=0\end{cases}$$

if $\alpha=-2,$ $4a-2b=0,$ $c=0,$ $b=2a$

if $a=0,$ $\alpha b=0.$ $\alpha c=0 \Rightarrow \alpha=0.$

$y_1=x^2+2x$ $\alpha=-2$

$y_2=bx+c.$ $\alpha=0$