4.4.1 $x(1-x)y'' - 2y' + 2y = 0$ $\qquad x^2(1-x)y'' - 2xy' + 2xy = 0$

$y = x^s \sum_{n=0}^{\infty} A_n x^n$ $\qquad R(x) = 1-x$

$P(x) = -2$

$Q(x) = 2x$

$s(s-1) + (-2s) + 0 = 0$

$s^2 - 3s = 0, \qquad s = 0, 3$

$s=0, \quad y_1 = \sum_{n=0}^{\infty} A_n x^n, \quad y_1' = \sum_{n=1}^{\infty} n A_n x^{n-1}, \quad y_1'' = \sum_{n=2} n(n-1) A_n x^{n-2}$

$x(1-x)\sum_{n=2}^{\infty} n(n-1) A_n x^{n-2} - 2\sum_{n=1}^{\infty} n A_n x^{n-1} + 2\cdot\sum_{n=0}^{\infty} A_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1) A_n (x^{n-1} - x^n) - 2\sum_{n=1}^{\infty} n A_n x^{n-1} + 2\sum_{n=0}^{\infty} A_n x^n = 0$

x^0: $\quad -2A_1 + 2A_0 = 0 \qquad A_1 = A_0$

x^1: $\quad 2\cdot 1\cdot A_2 - 2\cdot 2\cdot A_2 + 2A_1 = 0, \qquad +2A_2 = 2A_1, \qquad A_2 = A_1$

x^2: $\quad 3\cdot 2\cdot A_3 - 2\cdot 1\cdot A_2 - 2\cdot 3\cdot A_3 + 2A_2 = 0 \qquad 0 = 0$

x^n: $\quad (n+1)\cdot n\, A_{n+1} - n(n-1)A_n - 2(n+1)A_{n+1} + 2A_n = 0$

$(n^2 + n - 2n - 2) A_{n+1} = (n^2 - n - 2) A_n$

$(n^2 - n - 2) A_{n+1} = (n^2 - n - 2) A_n$

$A_{n+1} = A_n$

$y_1 = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$s=3 \quad y_2 = \sum_{n=0}^{\infty} B_n x^{n+3} \qquad y_2' = \sum_{n=0}^{\infty} B_n (n+3) x^{n+2} \qquad y_2'' = \sum_{n=0}^{\infty} (n+3)(n+2) B_n x^{n+1}$

$x(1-x)\sum_{n=0}^{\infty} (n+3)(n+2) B_n x^{n+1} - 2\cdot\sum_{n=0}^{\infty} B_n (n+3) x^{n+2} + 2\sum_{n=0}^{\infty} B_n x^{n+3} = 0$

$\sum_{n=0}^{\infty} (n+3)(n+2) B_n (x^{n+2} - x^{n+3}) - \sum_{n=0}^{\infty} 2B_n (n+3) x^{n+2} + \sum_{n=0}^{\infty} 2B_n x^{n+3} = 0$

x^2: $\quad 3\cdot 2\cdot B_0 - 2\cdot B_0\cdot 3 = 0 \Rightarrow 0 = 0$

x^3: $\quad 4\cdot 3\cdot B_1 - 3\cdot 2\cdot B_0 - 2B_1\cdot 4 + 2B_0 = 0 \qquad 4B_1 = 4B_0 \qquad B_1 = B_0$

x^4: $5\cdot 4\cdot B_2 - 4\cdot 3\cdot B_1 - 2\cdot B_2\cdot 5 + 2B_1 = 0.$

$10B_2 = 10B_1 \qquad B_2 = B_1$

x^{n+3}: $(n+4)(n+3)B_{n+1} - (n+3)(n+2)B_n - 2B_{n+1}(n+4) + 2B_n = 0$

$(n+4)(n+3-2)B_{n+1} = (n^2+5n+4)B_n$

$B_{n+1} = B_n$

$y_2 = x^3 \sum_{n=0}^{\infty} x^n = \frac{x^3}{1-x}$

general solution around $x=0$: $y = C_1 y_1 + C_2 y_2$

4.4.2 $\quad x^2y'' - \frac{x}{2}y' + \frac{(1+x)}{2}y = 0$

$R = 1 \qquad s(s-1) - \frac{1}{2}s + \frac{1}{2} = 0$

$P = -\frac{1}{2} \qquad s^2 - \frac{3}{2}s + \frac{1}{2} = 0 \qquad s = \frac{1}{2}, 1$

$Q = \frac{1+x}{2}$

$y_1 = x^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n x^n, \quad y_1' = \sum_{n=0}^{\infty} (n+\frac{1}{2}) A_n x^{n-\frac{1}{2}}, \quad y_1'' = \sum_{n=0}^{\infty} (n+\frac{1}{2})(n-\frac{1}{2}) A_n x^{n-\frac{3}{2}}$

$x^2y_1'' - \frac{x}{2}y_1' + \frac{1+x}{2}y_1 = 0$

$\sum_{n=0}^{\infty} (n^2-\frac{1}{4}) A_n x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty} (n+\frac{1}{2}) \frac{A_n}{2} x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{A_n}{2} (x^{n+1/2} + x^{n+3/2}) = 0$

$x^{\frac{1}{2}}$: $-\frac{1}{4}A_0 - \frac{1}{4}A_0 + \frac{A_0}{2} = 0 \qquad 0=0$

$x^{\frac{3}{2}}$: $\frac{3}{4}A_1 - \frac{3}{4}A_1 + \frac{A_1}{2} + \frac{A_0}{2} = 0 \qquad A_1 = -A_0$

$x^{n+\frac{1}{2}}$: $(n^2-\frac{1}{4})A_n - \frac{1}{2}(n+\frac{1}{2})A_n + \frac{A_n}{2} + \frac{A_{n-1}}{2} = 0.$

$(n^2 - \frac{1}{4} - \frac{1}{2}n - \frac{1}{4} + \frac{1}{2})A_n = -\frac{1}{2}A_{n-1}$

$n(n-\frac{1}{2})A_n = -\frac{1}{2}A_{n-1} \qquad A_n = \frac{-1\cdot A_{n-1}}{2n(n-\frac{1}{2})} = \frac{-1}{n(2n-1)} A_{n-1}$

$A_n = \frac{(-1)^n}{n!\,(2n-1)!!} A_0, \quad y_1 = \sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n-1)!!} x^n$

$y_2 = x\sum_{n=0}^{\infty} B_n x^n, \quad y_2' = \sum_{n=0}^{\infty}(n+1)B_n x^n, \quad y_2'' = \sum_{n=1}^{\infty} n(n+1)B_n x^{n-1}$

$x^2 y_2'' - \frac{x}{2}y_2' + \frac{1+x}{2}y_2 = 0.$

$\sum_{n=1}^{\infty} n(n+1)B_n x^{n+1} - \sum_{n=0}^{\infty}\frac{n+1}{2}B_n x^{n+1} + \frac{1+x}{2}x\sum_{n=0}^{\infty}B_n x^n = 0$

$\sum_{n=1}^{\infty} n(n+1)B_n x^{n+1} - \sum_{n=0}^{\infty}\frac{n+1}{2}B_n x^{n+1} + \sum_{n=0}^{\infty}\frac{B_n}{2}(x^{n+1}+x^{n+2}) = 0$

$x^1: \quad -\frac{1}{2}B_0 + \frac{B_0}{2} = 0, \quad 0=0$

$x^2: \quad 1\cdot 2\cdot B_1 - \frac{2}{2}B_1 + \frac{B_1}{2} + \frac{B_0}{2} = 0 \qquad \frac{3B_1}{2} = -\frac{B_0}{2} \qquad B_1 = -\frac{1}{3}B_0$

$x^{n+1}: \quad n(n+1)B_n - \frac{n+1}{2}B_n + \frac{B_n}{2} + \frac{B_{n-1}}{2} = 0.$

$\left[(n+1)(n-\frac{1}{2}) + \frac{1}{2}\right]B_n = -\frac{B_{n-1}}{2}$

$\left(n^2 + \frac{n}{2}\right)B_n = -\frac{B_{n-1}}{2}, \quad B_n = -\frac{B_{n-1}}{2n(n+\frac{1}{2})} = -\frac{B_{n-1}}{n(2n+1)}$

$B_n = \frac{(-1)^n B_0}{n!(2n+1)!!}$

$y_2 = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!(2n+1)!!}$ general soln. : $y = c_1 y_1 + c_2 y_2$ for x around zero

4.4.3 $xy'' + 2y' + xy = 0$

$x^2y'' + 2xy' + x^2y = 0.$

$R(x) = 1$ $\quad s(s-1) + 2s + 0 = 0$

$P(x) = 2$ $\quad s^2 + s = 0 \quad s(s+1) = 0 \quad s = -1, 0.$

$Q(x) = x^2$

$y_1 = x^{-1}\sum_{n=0}^{\infty}A_n x^n, \quad y_1' = \sum_{n=0}^{\infty}(n-1)A_n x^{n-2}, \quad y_1'' = \sum_{n=0}^{\infty}(n-1)(n-2)A_n x^{n-3}$

$xy_1'' + 2y_1' + xy_1 = 0$

$$\sum_{n=0}^{\infty}(n-1)(n-2)A_nX^{n-2} + \sum_{n=0}^{\infty} 2(n-1)A_nX^{n-2} + \sum_{n=0}^{\infty} A_nX^n = 0$$

X^{-2}: $(-1)(-2)A_0 + 2(-1)A_0 = 0 \qquad 0=0$

X^{-1}: $0 + 0 = 0$

X^0: $2\cdot 1\cdot A_2 + A_0 = 0 \qquad A_2 = -\frac{1}{2}A_0$

X^1: $2\cdot 1\cdot A_3 + 2\cdot 2A_3 + A_1 = 0 \qquad 6A_3 = -A_1,\quad A_3 = -\frac{1}{6}A_1$

X^{n-2}: $(n-1)(n-2)A_n + 2(n-1)A_n + A_{n-2} = 0$

$$n(n-1)A_n = -A_{n-2} \qquad A_n = -\frac{1}{n(n-1)}A_{n-2}$$

$$A_{2k} = \frac{(-1)^k}{(2k)!!(2k-1)!!}A_0$$

$$A_{2k+1} = \frac{(-1)^k}{(2k+1)!!(2k)!!}A_1$$

$y_2 = \sum_{n=0}^{\infty} B_nX^n,\quad y_2' = \sum_{n=1}^{\infty} nB_nX^{n-1},\quad y_2'' = \sum_{n=2}^{\infty} n(n-1)B_nX^{n-2}$

$$xy_2'' + 2y_2' + xy_2 = \sum_{n=2}^{\infty} n(n-1)B_nX^{n-1} + \sum_{n=1}^{\infty} 2nB_nX^{n-1} + \sum_{n=0}^{\infty} B_nX^{n+1} = 0$$

X^0: $2\, B_1 = 0 \qquad B_1 = 0$

X^1: $2\cdot 1\cdot B_2 + 2\cdot 2\cdot B_2 + B_0 = 0 \qquad B_2 = -\frac{1}{6}B_0$

X^2: $3\cdot 2\cdot B_3 + 2\cdot 3\cdot B_3 + B_1 = 0 \qquad B_3 = 0$

X^{n-1} $n(n-1)B_n + 2nB_n + B_{n-2} = 0$

$$n(n+1)B_n = -B_{n-2} \qquad B_n = -\frac{1}{n(n+1)}B_{n-2}$$

$n = 2k,\qquad B_{2k} = \frac{(-1)^k}{(2k)!!(2k+1)!!}B_0$

4.4.4 $\quad y'' - xy = 0 \qquad x^2y'' - x^3y = 0 \qquad R=1 \quad P=0 \quad Q=-X^3$

$S(S-1) = 0 \qquad s = 0, 1$

$y_1 = \sum_{n=0}^{\infty} A_nX^n,\quad y_1'' = \sum_{n=2}^{\infty} n(n-1)A_nX^{n-2}$

$$y_1'' - xy_1 = \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2} - \sum_{n=0}^{\infty} A_n x^{n+1} = 0$$

x^0: $2\cdot 1\cdot A_2 = 0$, $A_2 = 0$

x^1: $3\cdot 2\, A_3 - A_0 = 0$ $A_3 = \frac{1}{6}A_0$

x^2: $4\cdot 3\cdot A_4 - A_1 = 0$ $A_4 = \frac{1}{12}A_1$

x^3: $5\cdot 4\cdot A_5 - A_2 = 0$ $A_5 = 0$

x^{n-2}: $n(n-1)A_n - A_{n-3} = 0$ $A_n = \frac{A_{n-3}}{n(n-1)}$

$n = 3k$ $A_{3k} = \frac{1}{3k!!!\,(3k-1)!!!}A_0$

$n = 3k+1$ $A_{3k+1} = \frac{1}{(3k+1)!!!\,(3k)!!!}A_1$

$n = 3k+2$ $A_n = 0$

$$y_2 = x\sum_{n=0}^{\infty} B_n x^n,\quad y_2' = \sum_{n=0}^{\infty}(n+1)B_n x^n,\quad y_2'' = \sum_{n=1}^{\infty} n(n+1)B_n x^{n-1}$$

$$y_2'' - xy_2 = \sum_{n=1}^{\infty} n(n+1)B_n x^{n-1} - \sum_{n=0}^{\infty} B_n x^{n+2} = 0.$$

x^0: $1\cdot 2\cdot B_1 = 0$ $B_1 = 0.$

x^1: $2\cdot 3\cdot B_2 = 0.$ $B_2 = 0$

x^2: $3\cdot 4\, B_3 - B_0 = 0$ $B_3 = \frac{1}{3\cdot 4}B_0.$

x^{n-1}: $n(n+1)B_n = B_{n-3}$

$n = 3k$: $B_n = \frac{1}{n(n+1)}B_{n-3} = \frac{1}{n!!!\,(n+1)!!!}B_0.$

4.4.5 $xy'' + (6+x)y' + y = 0.$

$x^2y'' + x(6+x)y' + xy = 0.$ $Q = 1$ $P = 6+x$ $Q = x$

$s(s-1) + 6s + 0 = 0$

$s^2+5s = 0,\ s = 0, -5$

$$y_1 = x^{-5}\sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} A_n x^{n-5}$$

$$y_1' = \sum_{n=0}^{\infty}(n-5)A_n x^{n-6},\quad y_1'' = \sum_{n=0}^{\infty}(n-5)(n-6)A_n x^{n-7}$$

$x y_1'' + (6+x) y_1' + y_1 = 0$

$$\sum_{n=0}^{\infty} (n-5)(n-6) A_n x^{n-6} + (6+x) \sum_{n=0}^{\infty} (n-5) A_n x^{n-6} + x^{-5} \sum_{n=0}^{\infty} A_n x^n = 0.$$

$$\sum_{n=0}^{\infty} (n-5)(n-6) A_n x^{n-6} + \sum_{n=0}^{\infty} (n-5) A_n (6x^{n-6} + x^{n-5}) + \sum_{n=0}^{\infty} A_n x^{n-5} = 0.$$

x^{-6} : $(-5)(-6) A_0 + (-5) A_0 \cdot 6 = 0$, $0 = 0$

x^{-5} : $(-4)(-5) A_1 + (-4) A_1 \cdot 6 + (-5) A_0 + A_0 = 0$

$-4A_1 = 4A_0$, $A_1 = -A_0$

x^{-4} : $(-3)(-4) A_2 + (-3) A_2 \cdot 6 + (-4) A_1 + A_1 = 0$

$-6 A_2 = 3 A_1$ $A_2 = -\frac{1}{2} A_1 = \frac{1}{2} A_0.$

x^{-3} : $(-2)(-3) A_3 + (-2) A_3 \cdot 6 + (-3) A_2 + A_2 = 0$ $-6 A_3 = 2A_2$, $A_3 = -\frac{1}{3} A_2$

x^{n-6} : $(n-5)(n-6) A_n + 6(n-5) A_n + (n-6) A_{n-1} + A_{n-1} = 0$

$n(n-5) A_n = -(n-5) A_{n-1}$

if $n \neq 5$, $A_n = -\frac{1}{n} A_{n-1}$

x^{-2} : $(-1)(-2) A_4 + (-1) A_4 \cdot 6 + (3-5) A_3 + A_3 = 0$

$(2-6) A_4 - A_3 = 0$ $A_4 = -\frac{1}{4} A_3$

x^{-1} : $0 + 0 + (4-5) A_4 \cancel{+ A_4} = 0$ $0 = 0$

x^0 : $0 + 6 A_6 + A_5 = 0$ $A_6 = -\frac{1}{6} A_5$

$\therefore$ $y_1 = A_0 \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4\right) x^{-5}$

$+ A_5 \left(x^5 - \frac{1}{6}x^6 + \frac{1}{42}x^7 - \frac{1}{6 \cdot 7 \cdot 8} x^8 + \dots\right) x^{-5}$

$y_2 = \sum_{n=0}^{\infty} B_n x^n$ $y_2' = \sum_{n=1}^{\infty} B_n n \cdot x^{n-1}$ $y_2'' = \sum_{n=2}^{\infty} n(n-1) B_n x^{n-2}$

$x y_2'' + (6+x) y_2' + y_2 = \sum_{n=2}^{\infty} n(n-1) B_n x^{n-1} + \sum_{n=1}^{\infty} B_n n (6x^{n-1} + x^n) + \sum_{n=0}^{\infty} B_n x^n = 0$

x^0 : $B_0 = 0$

x^1 : $2 \cdot 1 \cdot B_2 + 2 \cdot B_2 \cdot 6 + B_1 + B_1 = 0$ $B_2 = -\frac{1}{7} B_1$

x^2: $3\cdot 2\cdot B_3 + B_3\cdot 3\cdot 6 + 2B_2 + B_2 = 0$

$24B_3 = -3B_2, \quad B_3 = -\frac{1}{8}B_2$

x^{n-1}: $n(n-1)B_n + 6B_n n + B_{n-1}(n-1) + B_{n-1} = 0$

$n(n+5)B_n = -nB_{n-1}$

$B_n = -\frac{1}{(n+5)}B_{n-1} = \frac{(-1)^{n-1}B_1\cdot 6!}{(n+5)!} = \frac{(-1)^{n-1}6!}{(n+5)!}B_1$

4.4.6 $2xy'' + (1-4x)y' - 2y = 0$

$x^2y'' + x(\frac{1}{2}-2x)y' - xy = 0$

$R = 1$ $\qquad S(S-1) + \frac{1}{2}S = 0$

$P = \frac{1}{2} - 2x$ $\qquad S(S-\frac{1}{2}) = 0$

$Q = -x$

$S = 0, \frac{1}{2}$

$y_1 = \sum_{n=0}^{\infty} A_n x^n, \quad y_1' = \sum_{n=1}^{\infty} nA_n x^{n-1}, \quad y_1'' = \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2}$

$\sum_{n=2}^{\infty} n(n-1)A_n x^{n-1} + \sum_{n=1}^{\infty} nA_n(\frac{1}{2}-2x)x^{n-1} - \sum_{n=0}^{\infty} A_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1)A_n x^{n-1} + \sum_{n=1}^{\infty} nA_n(\frac{1}{2}x^{n-1} - 2x^n) - \sum_{n=0}^{\infty} A_n x^n = 0$

x^0: $1\cdot A_1\cdot\frac{1}{2} - A_0 = 0 \quad A_1 = 2A_0$

x^1: $2\cdot A_2\frac{1}{2} - 2\cdot 1\cdot A_1 - A_1 = 0 \quad A_2 = 3A_1 = 3\cdot 2A_0$

x^2: $3\cdot 2\cdot A_3 + \frac{3A_3}{2} - 2\cdot 2\cdot A_2 - A_2 = 0$

$\frac{15}{2}A_3 = 5A_2 \quad A_3 = \frac{2}{3}A_2$

x^{n-1} $n(n-1)A_n + \frac{1}{2}nA_n - 2(n-1)A_{n-1} - A_{n-1} = 0$

$$n(n-\tfrac{1}{2})A_n - (2n-2+1)A_{n-1} = 0$$

$$n(n-\tfrac{1}{2})A_n = (2n-1)A_{n-1} \qquad A_n = \frac{2}{n}A_{n-1} = \frac{2\cdot 2\cdots 2}{n(n-1)\cdots 3}A_2$$

$$= \frac{2^{n-2}\cdot 2}{n!}A_2$$

$$= \frac{2^{n-1}}{n!}\, 3\cdot 2\cdot A_0$$

$$A_n = 3\cdot\frac{2^n}{n!}A_0$$

$$\boxed{y_1 = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n} = e^{2x}$$

$$8xe^{2x} + 2(1-4x)e^{2x} - 2e^{2x} = 0 \checkmark$$

$$y_2 = \sqrt{x}\sum_{n=0}^{\infty}B_n x^n,\quad y_2' = \sum_{n=0}^{\infty}(n+\tfrac{1}{2})B_n x^{n-\frac{1}{2}},\quad y_2'' = \sum_{n=0}^{\infty}(n+\tfrac{1}{2})(n-\tfrac{1}{2})B_n x^{n-\frac{3}{2}}$$

$$\sum_{n=0}^{\infty}(n^2-\tfrac{1}{4})2\cdot B_n x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty}(n+\tfrac{1}{2})B_n(1-4x)x^{n-\frac{1}{2}} - \sum_{n=0}^{\infty}2B_n x^{n+\frac{1}{2}} = 0$$

$x^{-\frac{1}{2}}$: $\quad (-\tfrac{1}{4})2\cdot B_0 + \tfrac{1}{2}\cdot B_0 = 0. \qquad 0 = 0$

$x^{\frac{1}{2}}$ $\quad (1-\tfrac{1}{4})\cdot 2\cdot B_1 + \tfrac{3}{2}\cdot B_1 - 4\cdot\tfrac{1}{2}\cdot B_0 - 2B_0 = 0.$

$$3B_1 = 4B_0 \qquad B_1 = \tfrac{4}{3}B_0$$

$x^{n-\frac{1}{2}}$ $\quad (n^2-\tfrac{1}{4})2B_n + (n+\tfrac{1}{2})B_n - 4(n-\tfrac{1}{2})B_{n-1} - 2\cdot B_{n-1} = 0.$

$$(n+\tfrac{1}{2})(2n-1+1)B_n - 4n\,B_{n-1} = 0.$$

$$(n+\tfrac{1}{2})\cdot 2n\cdot B_n = 4n\,B_{n-1}$$

$$B_n = \frac{2}{n+1/2}B_{n-1} = \frac{4}{2n+1}B_{n-1} \Rightarrow y_2 = \sqrt{x}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}x^n$$

7.5.4 $\quad xy'' - 2y' + \lambda x^2 y = 0 \quad 0<x<1, \; y(0)=0, \; y(1)=0$

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + \left(q(x) + \lambda r(x)\right) y = 0$$

$$\frac{p'}{p} = \frac{-2}{x} \qquad \ln p = -2\ln x = \ln\frac{1}{x^2}, \quad p = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}y'\right) + \frac{\lambda}{x}y = 0$$

$$y\cdot\frac{d}{dx}\left(\frac{1}{x^2}y'\right) + \lambda\cdot\frac{y^2}{x} = 0$$

$$\int_0^1 y\frac{d}{dx}\left(\frac{1}{x^2}y'\right)dx + \lambda\int_0^1 \frac{y^2}{x}dx = 0$$

$$y\frac{1}{x^2}y'\Big|_0^1 - \int_0^1 \frac{y'^2}{x^2}dx = -\lambda\int_0^1 \frac{y^2}{x}dx$$

$$\lambda \int_0^1 \frac{y^2}{x}dx = \int_0^1 \left(\frac{y'}{x}\right)^2 dx \geq 0$$

$\lambda = 0, \; y' = 0 \Rightarrow y = 0 \Rightarrow \lambda = 0$ is not an eigenvalue.

$\boxed{\lambda > 0}$

$x^2y'' - 2xy' + \lambda x^3 y = 0 \Rightarrow$ generalized Bessel's function with $\lambda > 0$

$$x^2y'' + x(a + 2bx^r)y' + \left[c + dx^{2s} - b(1-a-r)x^r + b^2x^{2r}\right]y = 0$$

$a = -2 \qquad d = \lambda > 0 \qquad \Rightarrow y(x) = x^{\frac{1-a}{2}} e^{-\frac{bx^r}{r}} Z(\xi)$

$b = 0 \qquad 2s = 3, \; s = \frac{3}{2} \qquad \xi = \frac{\sqrt{|d|}}{s}x^s$

$c = 0$

$$p = \frac{1}{s}\sqrt{\left(\frac{1-a}{2}\right)^2 - c}$$

$$y(x) = x^{\frac{3}{2}} \cdot Z(\xi) \quad \xi = \frac{\sqrt{\lambda}}{3/2} x^{3/2}$$

$$p = \frac{1}{3/2}\sqrt{\left(\frac{3}{2}\right)^2 - 0} = 1$$

$\Rightarrow$ $p=1$

$d = \lambda > 0$

$$y(x) = x^{3/2}\left[C_1 J_1\left(\frac{\sqrt{\lambda}}{3/2} x^{3/2}\right) + C_2 Y_1\left(\frac{\sqrt{\lambda}}{3/2} x^{3/2}\right)\right]$$

boundary condition $y(0)=0$ $y(1)=0$

$y(0)=0 \Rightarrow C_2 = 0$

$y(1)=0$ $\quad \frac{\sqrt{\lambda_n}}{3/2} = \xi_n$ where $J_1(\xi_n)=0$.

$$\lambda_n = \left(\frac{3}{2}\xi_n\right)^2$$

$$\frac{d}{dx}\left(\frac{1}{x^2}y'\right) + \frac{\lambda}{x}y = 0 \Rightarrow \int_0^1 y_n y_m \frac{1}{x}\,dx = 0 \text{ if } n \neq m.$$