$$\frac{1}{T}\int_0^T E'\,\mathrm{d}t = 0,$$

or

$$\frac{1}{T}\int_0^T x^2(1-\tfrac{1}{3}x^2)\,\mathrm{d}t = 0. \tag{3.18}$$

This expression is valid for all $\epsilon > 0$ and, in particular, must hold in the limit $\epsilon \to 0$. On substituting (3.17) into (3.18), we find that

$$\frac{1}{2\pi}\int_0^{2\pi} r_0^2\cos^2 t(1-\tfrac{1}{3}r_0^2\cos^2 t)\,\mathrm{d}t = O(\epsilon),$$

or

$$\tfrac{1}{2}r_0^2 - \tfrac{1}{8}r_0^4 = O(\epsilon). \tag{3.19}$$

It follows that, to an error of $O(\epsilon)$, $r_0 = 2$ and, hence, the limit cycle, to this order of approximation, is a circle of radius 2. In effect, the result (3.18) has enabled us to select from all the possible sinusoidal oscillations (3.17) precisely that one which corresponds to the limit cycle. This technique is an illustration of the method of averaging, which is the subject of chapter 9. The same result can be obtained by more straight-forward, but related, perturbation techniques which are described in chapter 7.

Next we recall from (1.9) that the stability of the limit cycle is determined by the sign of the characteristic exponent μ. Here $f_x + g_y = \epsilon(1-x^2)$, and so (1.9) becomes

$$\mu = \frac{1}{T}\int_0^T \epsilon(1-x^2)\,\mathrm{d}t. \tag{3.20}$$

Then, using the approximation (1.9), we obtain

$$\mu = -\epsilon + O(\epsilon^2). \tag{3.21}$$

This result confirms that the limit cycle is asymptotically orbitally stable, and also shows that, as $\epsilon \to 0$, the approach to the limit cycle is slow and occurs on a time scale of ϵ^{-1}. Both this result, and that for the limit cycle itself (i.e. (3.19)), are in agreement with the numerical calculations.

(ii) $\epsilon \to \infty$: In this limit it is useful to rescale the variables y and t by putting

$$y = \epsilon\eta, \qquad t = \epsilon\tau. \tag{3.22}$$

The motivation for this is that the time-scale for the limit cycle is ϵ and, since x continues to vary on a scale of unity, it follows from (3.3) that y must also scale with ϵ. In terms of η and τ, (3.3) becomes

$$\frac{dx}{d\tau} = \epsilon^2(\eta + x - \tfrac{1}{3}x^3), \qquad \frac{d\eta}{d\tau} = -x. \tag{3.23}$$

Hence

$$x\frac{dx}{d\eta} = -\epsilon^2(\eta + x - \tfrac{1}{3}x^3). \tag{3.24}$$

Here the small parameter is ϵ^{-2}, but the right-hand side of (3.23), and also the right-hand side of (3.24), is not analytic in ϵ^{-2}. Hence, a regular perturbation expansion in powers of ϵ^{-2} is not possible. Indeed, the limit $\epsilon^2 \to \infty$ is a singular perturbation. Although we shall not be developing any general techniques for singular perturbations in this text, approximate solutions for (3.24) will be obtained here by using heuristic arguments. Thus, as $\epsilon^2 \to \infty$, it is apparent from (3.24) that

$$\text{either} \quad \eta + x - \tfrac{1}{3}x^3 \approx 0, \quad \text{or} \quad \frac{d\eta}{dx} \approx 0. \tag{3.25}$$

Combining these approximations with the information obtained about the limit cycle in the course of proving theorem 6.9, we can deduce that a suitable approximation to the limit cycle as $\epsilon \to \infty$ is obtained by following the cubic $\eta = \frac{1}{3}x^3 - x$ from the point P ($x = 2$, $\eta = \frac{2}{3}$) to its turning point Q ($x = 1$, $\eta = -\frac{2}{3}$), and then following the line $\eta = -\frac{2}{3}$ until meeting the cubic again at the point R ($x = -2$, $\eta = -\frac{2}{3}$). The circuit is completed by symmetry and, hence, follows the cubic to the next turning point S ($x = -1$, $\eta = \frac{2}{3}$) and then follows the line $\eta = \frac{2}{3}$ until meeting the cubic again at the point P. The result is sketched in figure 6.10.

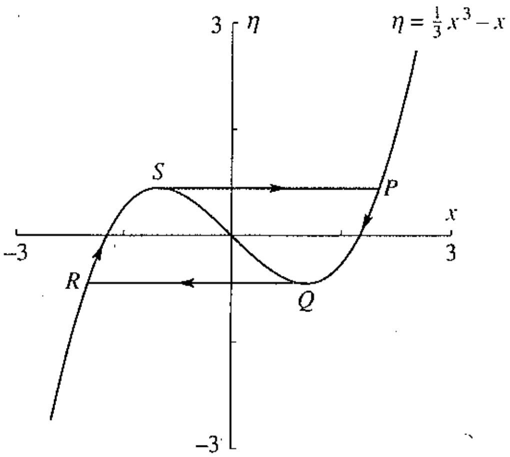

Fig. 6.10 Diagram for the Van der Pol limit cycle in the limit $\epsilon \to \infty$.

To determine an approximation for the period T, we observe that

$$T = \epsilon \oint d\tau = -\epsilon \oint \frac{1}{x}\, d\eta, \tag{3.26}$$

where the integral is around the limit cycle. Clearly, the dominant contributions come from the portions PQ and RS on the cubic and, using the symmetry, it is sufficient to consider just the portion PQ. Here $\eta = \frac{1}{3}x^3 - x$, $\mathrm{d}\eta = (x^2-1)\,\mathrm{d}x$, and so

$$T \approx 2\epsilon \int_1^2 \frac{x^2-1}{x}\,\mathrm{d}x,$$

or

$$T \approx \epsilon(3-2\ln 2). \tag{3.27}$$

On the portions PQ and RS of the limit cycle $x^2 > 1$, and the limit cycle is in a phase of positive damping, where the amplitude relaxes from $|x| \approx 2$ to $|x| \approx 1$. This is followed by a rapid transition on QR and SP, where the limit cycle passes through a region of negative damping ($x^2 < 1$) and overshoots to the points R and P. To analyze this process further, we combine the approximate expressions (3.25) with (3.23). Thus, on the portion PQ, we use the approximation $\eta \approx \frac{1}{3}x^3 - x$ together with the second equation in (3.23), to obtain

$$(x^2-1)\frac{\mathrm{d}x}{\mathrm{d}\tau} \approx -x,$$

or

$$\ln x - \tfrac{1}{2}x^2 \approx \tau + \ln 2 - 2. \tag{3.28}$$

Here we are assuming, without loss of generality, that $\tau = 0$ at the point P. Note that, as $x \to 1$, we find from (3.28) that

$$(x-1)^2 \approx (\tfrac{3}{2} - \ln 2) - \tau. \tag{3.29}$$

Thus, as $x \to 1$, $\tau \to \frac{3}{2} - \ln 2$, in agreement with our previous approximation for the period (3.27). Next, on the portion QR, $\eta \approx -\frac{2}{3}$ and, combining this approximation with the first equation in (3.23), we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} \approx -\tfrac{1}{3}\epsilon^2(x-1)^2(x+2),$$

or

$$-\frac{3}{1-x} + \ln\frac{1-x}{x+2} = 3\epsilon^2(\tau - \tau_0). \tag{3.30}$$

Here τ_0 is an arbitrary constant of integration, but, since this approximation must match with (3.28) at the point R ($x \to 1$), clearly $\tau_0 \approx \frac{3}{2} - \ln 2$. As $x \to 2$, we find from (3.30) that

$$1 - x \approx -\frac{1}{\epsilon^2(\tau-\tau_0)}, \tag{3.31}$$

while, as $x \to -2$,

$$x + 2 \approx k\exp\{-3\epsilon^2(\tau-\tau_0)\}, \tag{3.32}$$

where k is a constant. The expressions (3.18) and (3.30) can now be used to construct an approximation to the limit cycle $x(\tau)$, and the result is shown in figure 6.11, where we use the symmetry of the limit cycle to complete the portions RS and SP.

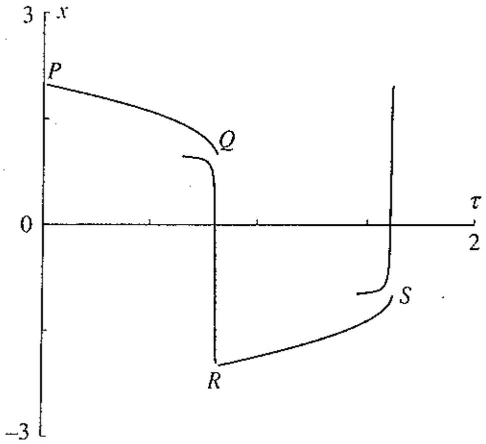

Fig. 6.11 The approximate solution $x(\tau)$, where $t = \epsilon\tau$, for the Van der Pol limit cycle in the limit $\epsilon \to \infty$.

In sketching figure 6.11, we observe from (3.32) that $x \to -2$ exponentially fast as $\epsilon^2(\tau-\tau_0) \to \infty$. Hence the approximation (3.30) joins smoothly to the approximation corresponding to (3.28) for the portion RS. However, from (3.31) we see that the approximation (3.30) is singular as $\tau \to \tau_0$ and, hence, does not join smoothly to the approximation (3.28) as $x \to 1$ (see (3.29) which shows that the approximation (3.28) develops a vertical tangent as $x \to 1$). To correct this, we need to examine the region near the point Q in more detail. Thus, we let

$$1-x = \delta u, \qquad \eta+\tfrac{2}{3} = \delta^2 v, \qquad \tau-\tau_0 = \delta^2 s, \tag{3.33}$$

where δ is a small parameter which has yet to be determined. The scaling in (3.33) is determined from the observation that Q is a turning point on the cubic and, hence, $(1-x)^2$ and $\eta+\frac{2}{3}$ must scale together, while the second equation of (3.23) implies that $\eta+\frac{2}{3}$ and $\tau-\tau_0$ must scale together. We also let $\tau_0 = \frac{3}{2}-\ln 2+\delta^2 s_0$, where $\delta^2 s_0$ is a correction term yet to be determined. Substitution of (3.33) into (3.23) now gives

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \epsilon^2\delta^3(-v+u^2-\tfrac{1}{3}\delta u^3), \qquad \frac{\mathrm{d}v}{\mathrm{d}s} = -1+\delta u. \tag{3.34}$$

Hence we must choose $\delta = \epsilon^{-2/3}$, and so obtain the approximate equations

$$\frac{\mathrm{d}u}{\mathrm{d}s} \approx u^2-v, \qquad \frac{\mathrm{d}v}{\mathrm{d}s} \approx -1. \tag{3.35}$$

It follows from the second of these equations that $v \approx -(s-s_1)$, where s_1 is a constant of integration. Then, to solve the first equation in (3.35), we put

$$u = -\frac{1}{\psi}\frac{\mathrm{d}\psi}{\mathrm{d}s},$$

and so

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + (s - s_1)\psi \approx 0. \tag{3.36}$$

This is Airy's equation, for which two linearly independent solutions are $\mathrm{Ai}(s_1 - s)$ and $\mathrm{Bi}(s_1 - s)$, which are the Airy functions of the first and second kind, respectively. To find the appropriate solution of (3.36), we first express the matching conditions (3.29) and (3.31) in the present variables (3.33). Thus (3.29) becomes

$$u \approx -\sqrt{-(s+s_0)} \quad (\text{as } s \to -\infty), \tag{3.37}$$

while (3.31) becomes

$$u \approx -\frac{1}{s} \quad (\text{as } s \to 0). \tag{3.38}$$

It follows from (3.37) that $s_1 = -s_0$ and $\psi = \mathrm{Ai}(s_1 - s)$, so that, from (3.36),

$$u = \frac{\mathrm{Ai}'(s_1 - s)}{\mathrm{Ai}(s_1 - s)}. \tag{3.39}$$

In verifying the condition (3.37), we use the result that

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\{-\tfrac{2}{3} z^{3/2}) \quad (\text{as } z \to \infty). \tag{3.40}$$

Next we observe from (3.39) that u is singular at the zeros of the Airy function. Since here s is increasing from $-\infty$, we need to consider only the smallest zero, α, such that $\mathrm{Ai}(-\alpha) = 0$, where we note that $\alpha \approx 2.338$. As $s_1 - s \to -\alpha$, (3.39) becomes

$$u \approx \frac{1}{s_1 + \alpha - s}. \tag{3.41}$$

To match with (3.38), we must now choose $s_1 = -\alpha$. This completes our examination of the structure of the limit cycle in the vicinity of the point Q. In particular, we can now improve our estimate of the period (3.27) by observing that the matching from (3.29) to (3.31) occurs over an interval $\delta^2 s_0$ (on the τ-scale), and we have shown that $s_0 = \alpha$. Hence

$$T \approx \epsilon(3 - 2\ln 2 + 2\alpha\epsilon^{-4/3} + \ldots), \tag{3.42}$$

where we can anticipate that the next term is relatively $O(\epsilon^{-2})$.

Next we turn to the approximate calculation of the characteristic exponent μ, here given by (3.20). From (3.22) and (3.23) this is equivalent to

$$\mu T = -\epsilon^2 \oint \frac{1 - x^2}{x}\, \mathrm{d}\eta, \tag{3.43}$$

where the integral is around the limit cycle. Then, using the approximations (3.25) and (3.27), we obtain

$$\mu \approx -\epsilon \frac{\frac{3}{2}+2\ln 2}{3-2\ln 2}. \tag{3.44}$$

As anticipated, this confirms that the limit cycle is asymptotically orbitally stable, and also shows that as $\epsilon \to \infty$ the approach to the limit cycle is rapid and occurs on a time scale of ϵ^{-1}. Both this result, and that for the limit cycle itself (i.e. figure 6.10), are in agreement with the numerical calculations (see figure 6.8 (b) and figure 6.9 (b)).

We have analyzed the Van der Pol limit cycle in considerable detail because it is a typical example of a relaxation oscillation. Indeed, the concepts and methodology developed for the Van der Pol equation can be adapted to study the Lienard equation:

$$u'' + f(u)\,u' + g(u) = 0. \tag{3.45}$$

To put this into the form (1.1), we let

$$x = u, \qquad y = u' + F(u),$$

where

$$F(u) = \int_0^u f(v)\,\mathrm{d}v. \tag{3.46}$$

Then (3.45) becomes

$$x' = y - F(x), \qquad y' = -g(x). \tag{3.47}$$

Suppose further that

(i) $F(x) = -F(-x)$ for all x;
(ii) $F(x) = 0$ only for $x = 0, \pm a$, and $F(x) \to \infty$ monotonically as $x \to \infty$ for $x > a$;
(iii) $g(x) = -g(-x)$ for all x;
(iv) $g(x) > 0$ for all $x > 0$. (3.48)

Note that the Van der Pol equations (3.3) satisfy these conditions. It may now be shown that there is a unique limit cycle which is asymptotically orbitally stable. The proof is similar to that for theorem 6.9, with

$$E = \tfrac{1}{2}y^2 + \int_0^x g(x')\,\mathrm{d}x'$$

replacing (3.4) as a measure of the energy. The details are left to the exercises (see also Jordan and Smith, 1977, section 11.3).