TWO

The Node Method for Trusses

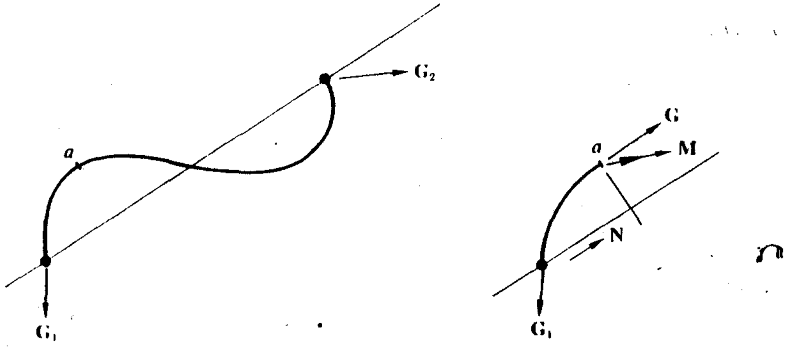
2.1 INTRODUCTION

It is the function of this chapter to provide a simple, concrete. structural example of the node method from which the reader is asked to generalize in Chapter 3. This sequence is, of course, based upon the idea that it is easier to generalize upward from a simple example (at least in an elementary course) than it is to accept a theory in terms of which all applications are special cases. Those who prefer the general case first will find it a simple matter to interchange for themselves the order of Chapters 2 and 3.

Because of its simplicity, the truss provides an ideal introduction to the node method. although it will be seen later that in the case of the mesh method, the truss is in at least one respect (due to the existence of releases). more difficult than the frame. In this chapter a direct approach to the truss problem is used. This requires first a formal description of the forces and displacements which occur, then a translation of the structural equations into matrix form. It is the form of this translation which is motivated by the network problem and which results in a generalization of the branch node incidence matrix.

2.2 THE TRUSS

In the context of this book. the truss has a very restricted defnition: It is a structure composed of pin-connected elements which is loaded only at its joints. The implication of pin-connected members which are not loaded along their length is well known from elementary mechanics, and is simply that the resultant member force lies along a straight line between the ends of the member. Since the line of action of the member force is



Curved truss bar

Free body diagram

Fig. 2. I.

known. it can be described by specifying a single scalar. the bar force. Figure 2. I shows a general curved truss bar and then a free body diagram of a part of it. The curved bar can be recognized as a truss bar because *only* forces. no moments. are shown to be acting at its ends. (By definition, a pin-ended member can have no applied end moments.) Since the sum of the forces acting on the member as a whole must be zero.

$$\mathbf{G}_1 + \mathbf{G}_2 = 0 \qquad \text{or} \qquad \mathbf{G}_1 = -\mathbf{G}_2.$$

Let **r** be the position vector of the upper end with respect to the lower end of the member. Taking moments about the lower end.

$$\mathbf{r} \times \mathbf{G}_2 = 0$$

from which it follows that since $|\mathbf{r}| \neq 0$. when $\mathbf{G}_2 \neq 0$. $\mathbf{G}_2$ is parallel to r. The vector $\mathbf{G}_2$ is then a vector whose line of action is known and which can therefore be described as a scalar multiplying either the vector **r** or a unit vector u. Figure 2.2 shows a structure composed of pin-connected bars subjected to joint loads.

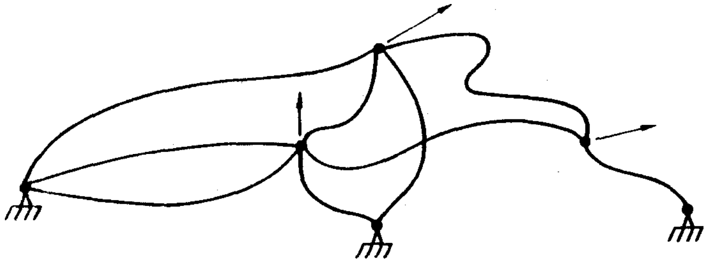


Fig. 2.2. A typical truss

2.3 A FORMAL DESCRIPTION OF THE TRUSS PROBLEM

The remainder of this chapter is concerned with a rather simple truss problem, the response of a structure composed of *straight*, pin-connected members subjected only to joint loads. While curved truss members are sometimes useful when, e.g., connections introduce eccentricity, they are not treated in the remainder of this book (but only require an appropriate modification of the "primitive stiffness matrix" described later). Such effects as thermal loads. support settlement. lack of fit are added in a later chapter, which shows them to be "secondary" with regard to the problem solution.

In truss analysis members (branches) are idealized as lines which meet at points (nodes) which are called joints. Figure 2.3 shows a typical truss joint and the quantities associated with it.

$\mathbf{P}_i$ – the applied joint load vector at joint i
$\boldsymbol{\delta}_i$ – the displacement vector at joint i

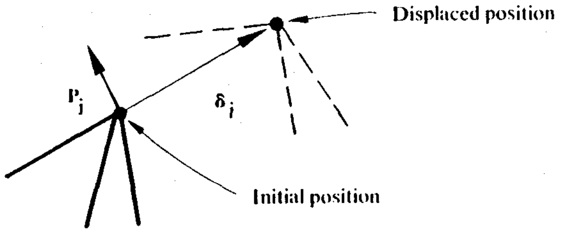


Fig. 2.3. The *i*th point

For the entire structure these quantities are assembled into the joint load matrix P and the joint displacement matrix δ.

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \delta_J \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_J \end{bmatrix}.$$

in which

$$\delta_i = \begin{bmatrix} (\boldsymbol{\delta}_i)_x \\ (\boldsymbol{\delta}_i)_y \\ (\boldsymbol{\delta}_i)_z \end{bmatrix} \qquad \text{and} \qquad P_i = \begin{bmatrix} (\mathbf{P}_i)_x \\ (\mathbf{P}_i)_y \\ (\mathbf{P}_i)_z \end{bmatrix}$$

The matrices δ_i and P_i simply contain the components of the vectors $\boldsymbol{\delta}_i$ and P, respectively; J is the number of movable joints (i.e. joints which are not supports).

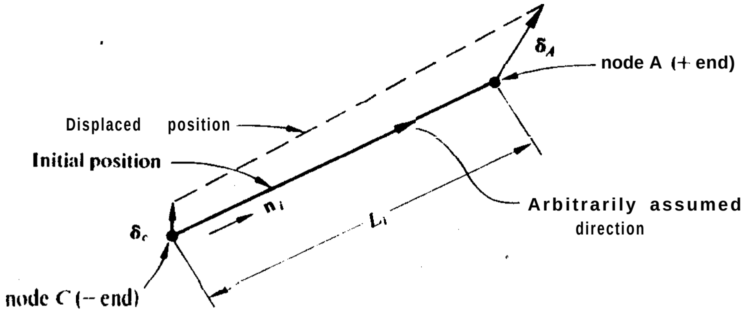


Fig. 2.4. Member *i*.

Figure 2.4 shows a typical truss bar associated with which is the bar force F_i and the bar length change Δ_i, chosen so that positive F_i and A, correspond to tension or stretching within the bar and a unit vector $\mathbf{n}_i$. Given the displacements of the ends of a bar, it is possible to compute the exact length change of the bar. In linear structural analysis it is customary to use the approximate relationship,

$$\Delta_i = \mathbf{n}_i \cdot (\boldsymbol{\delta}_A - \boldsymbol{\delta}_C) = \tilde{n}_i(\delta_A - \delta_C), \qquad (2.1)$$

which uses the notation of Fig. 2.4. This involves projecting the joint displacement vectors onto the original position of the bar to obtain the length change. It is only exact when the displaced position of the bar is parallel to its original position but is accurate when they differ by a *small rotation*. In Eq. (2. I) $\mathbf{n}_i$ is a unit vector which describes the slope of bar *i* and which has the matrix representation

$$n_i = \begin{bmatrix} (\mathbf{n}_i)_x \\ (\mathbf{n}_i)_y \\ (\mathbf{n}_i)_z \end{bmatrix}.$$

For the entire structure the F_i and Δ_i are assembled into the bar force matrix F and the bar displacement matrix A.

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_B \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_B \end{bmatrix},$$

where *B* is the number of bars in the structure.

From elementary mechanics of solids (*see* Appendix A.3) it is known that the bar forces and displacements are related through Hboke's law,

$$\cdot dd \quad |n_i| = [(n_i)_x^2 + (n_i)_y^2 + (n_i)_z^2] = 1$$

$$F_i = K_i\Delta_i,$$

in which

$$K_i = \frac{A_iE}{L_i}. \qquad (2.2)$$

(A, is the area of the ith bar and *E* is Young's modulus.)

For the entire structure Eq. (2.2) becomes

$$K = \begin{bmatrix} K_1 & & & \\ & K_2 & & 0 \\ & & \ddots & \\ & 0 & & K_B \end{bmatrix}. \qquad (2.3)$$

The matrix *K* is called the primitive stiffness matrix.

The geometrical bar-displacement joint-displacement relationship, Eq. (2.1), **can now be written for the entire structure through the aid of the** matrix N. a *B X J* (row X column) matrix whose elements N_{ij} are

$$N_{ij} = \begin{cases} \text{ii,} & \text{ifj is the + end of bar } i \\ -\tilde{n}_i & \text{ifj is the – end of bar } i \\ 0 & \text{otherwise} \end{cases} \qquad (2.4)$$

Using N. Eq. (2. I) becomes

$$\text{A} = N\delta. \qquad (2.5)$$

Note that here, as in the case of the matrix δ, it has been convenient to state the definitions in terms of *partitioned* matrices.

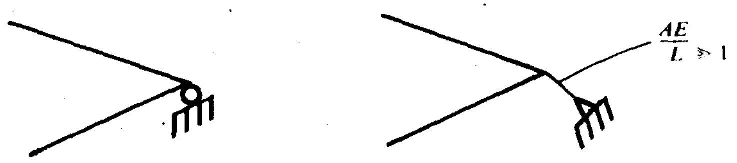
Comparing the definition of N with the definition of the matrix *A* of Chapter 1 provides the motivation for referring to N as a generalized branch node incidence matrix or simply an incidence matrix. Since N contains no columns corresponding to support joints, the implication is that all supports are "completely fixed" (i.e. if *I* is a support node, $\delta_I = 0$). It may also be noted that any partial support can be replaced by an equivalent fixed support as indicated in Fig. 2.5. but care should be taken not to ill-condition the system.

To complete the discussion of the node method for trusses, it remains only to discuss the joint equilibrium equations. It will now be shown that they can be written as

$$\tilde{N}F = I', \qquad (2.6)$$

using matrices previously defined. Fig. 2.6 shows a typical truss joint whose equilibrium equation (the vector sum of all forces on any joint

$$N = \begin{bmatrix} \tilde{n}_i & -\tilde{n}_i \end{bmatrix}$$



Partial support

Equivalent fixed support

Fig. 2.5.

free-body diagram must he zero) is

$$F_\alpha \mathbf{n}_\alpha - F_\beta \mathbf{n}_\beta - F_j \mathbf{n}_j + F_\delta \mathbf{n}_\delta = -\mathbf{P}_i . \tag{2.7}$$

In general a joint equilibrium equation must contain a term $\pm F_l \mathbf{n}_l$ for each bar incident upon the joint – the sign is determined by whether the bar is positively or negatively incident. This is precisely the form provided by Eq. (2.6).

The node method for trusses then becomes

$\tilde{N}F = P$ (node equilibrium equation)
$F = K\Delta$ (Hooke's law)
$\Delta = N\delta$ (branch-displacement joint-displacement equation)

from which it follows that

$$\tilde{N}F = P \quad \rightarrow \quad \tilde{N}K\Delta = P \quad \rightarrow \quad \tilde{N}KN\delta = P$$

or

$$\delta = (\tilde{N}KN)^{-1}P \tag{2.8}$$

Equation (2.8) is a system of simultaneous linear algebraic equations on the unknown joint displacements δ; once they have been computed it is a simple matter to compute A using Eq. (2. l), and F using Hooke's law.

The node method begins with forces which satisfy both equilibrium and Hooke's law and writes them in terms of the joint displacements. This insures that the three requirements for solutions are satisfied; in particular. the pieces fit together since their bar length changes are determined from joint displacements.

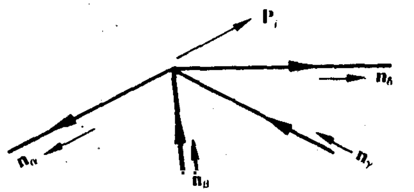


Fig. 2.6. Joint equilibrium.

Note that in the case of the equilibrium equations. the omission of support nodes from the matrix N results in not writing equilibrium equations at these nodes.

2.4 A DECOMPOSITION

At the heart of the node method is the matrix $\tilde{N}KN$, the *system matrix.* While the formal description of the node method for trusses is complete as it is given in Section 2.3, from a practical point of view it would be a mistake to program this formulation directly, largely because the matrices N and K are sparse. In this section, a method will be given by which the system matrix can be formed directly from the unit vectors $\mathbf{n}_i$, and the bar stiffnesses K_i without explicitly forming the matrices N and K.

To achieve this end it is convenient to partition the matrix N into its rows

$$N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_B \end{bmatrix}. \tag{2.9}$$

Since K is diagonal. it follows that the system matrix can be written as a sum.

$$\tilde{N}KN = \sum_{i=1}^{B} \tilde{N}_i K_i N_i . \tag{2.10}$$

The term $\tilde{N}_i K_i N_i$ can be regarded as the contribution of bar *i to* the system matrix.

Proceeding in this vein one more step, if neither end of bar i is a support. it follows from Eq. (2.4) (using the notation of Fig. 2.3) that N_i contains two non-zero terms and that $\tilde{N}_i K_i N_i$ contributes four terms

$$\tilde{N}_i K_i N_i = \begin{bmatrix} \vdots \\ n_i \\ \vdots \\ -n_i \\ \vdots \end{bmatrix} K_i [\cdots \tilde{n}_i . \cdots -\tilde{n}_i \cdots]$$

$$= \begin{array}{cc} \begin{array}{ccccc} & \text{col. } A & & \text{col. c} & \end{array} & \\ \left[\begin{array}{ccccc} - & n_i K_i \tilde{n}_i & — & -n_i K_i \tilde{n}_i & — \\ & | & & | & \\ - & -n_i K_i \tilde{n}_i & — & n_i K_i \tilde{n}_i & — \end{array}\right] & \begin{array}{l} \text{row } A \\ \\ \text{row } C \end{array} \end{array}$$

to the system matrix. When either end of bar i is a support node only one diagonal term is generated.

Figure 2.7 shows a simple 2-dimensional example which is used here to illustrate the decomposition of the system matrix into the contributions of each of the bars. It is assumed that the stiffness of each of the bars is known; the unit vectors are

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{n}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{n}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{n}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The matrix N is then

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} \tilde{n}_1 & 0 \\ \tilde{n}_2 & -\tilde{n}_2 \\ 0 & \tilde{n}_3 \\ 0 & \tilde{n}_4 \end{bmatrix},$$

and the contribution of bar 2. e.g.. to the system matrix is simply

$$\tilde{N}_2 K_2 N_2 = \begin{bmatrix} n_2 \\ -n_2 \end{bmatrix} K_2 [\tilde{n}_2 - \tilde{n}_2] = \begin{bmatrix} n_2 K_2 \tilde{n}_2 & -n_2 K_2 \tilde{n}_2 \\ -n_2 K_2 \tilde{n}_2 & n_2 K_2 \tilde{n}_2 \end{bmatrix}.$$

Since

$$n_2 K_2 \tilde{n}_2 = \frac{K_2}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

it follows that

$$\tilde{N}_2 K_2 N_2 = \frac{K_2}{2}\begin{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix},$$

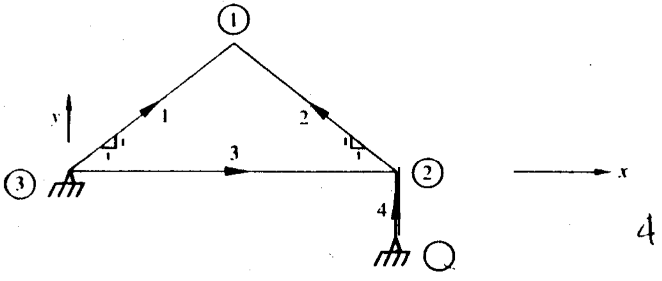


Fig. 2.7. A two dimensional example

Adding the contributions of all four bars, the system matrix for the example is finally

$$\tilde{N}KN = \begin{bmatrix} \frac{K_1}{2} + \frac{K_2}{2} & \frac{K_1}{2} - \frac{K_2}{2} & -\frac{K_2}{2} & \frac{K_2}{2} \\ \frac{K_1}{2} - \frac{K_2}{2} & \frac{K_1}{2} + \frac{K_2}{2} & \frac{K_2}{2} & -\frac{K_2}{2} \\ \frac{K_2}{2} & -\frac{K_2}{2} & \frac{K_2}{2} + K_3 & -\frac{K_2}{2} \\ \frac{K_2}{2} & -\frac{K_2}{2} & -\frac{K_2}{2} & \frac{K_2}{2} + K_4 \end{bmatrix}$$

2.5 EXERCISES

Programs P.1 and P.2 included at the end of this book illustrate the generality of the work discussed in this chapter and provide additional numerical examples. Based upon these programs. the following exercises may be useful.

1. In order to test his understanding of the material presented thus far. the reader can attempt to reconstruct Program P.2, the space truss program starting from Program P. 1, the plane truss program.
2. As an even more rigorous test of his understanding of the materiel. the reader can now anticipate Chapter 6 and modify Programs P. I or P.2 to include thermal effects.
3. Modify Programs P. I or P.2 to include “buckling” effects.

2.5.1 Discussion of Exercise 3

In linear structural analysis the equilibrium equations are written in the undeformed configuration. The classical treatment of overall truss buckling (not member buckling), on the other hand, adds to the linear formulation a nonlinearity which approximates the effect of small changes of geometry on the linear formulation in a manner similar to the method used to derive the linear response of a membrane or string. How this can be done is now indicated for a single bar one end of which is allowed to move. The general case is simply a composite in which each end of each bar is considered.

Figure 2.8 shows bar i in its undeformed and deformed configurations. Assuming that the bar force F_i remains approximately constant, the displacement δ_A generates a force normal to the undeformed position of the bar. Following the string problem this force has a magnitude of F_i x θ