New Results on the Expected Number of Maxima in the Normal Process Envelope

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Abstract — **In this paper a new formula for the expected number of maxima in the normal process envelope is presented. In contrast to the complicated Rice's formula, our simple formula holds for an arbitrary power spectrum. The key idea is the application of characteristic functions, instead of probability density functions, for computing the expected number of level crossings of a process.**

I. INTRODUCTION

In some engineering applications we encounter a normal process. We are usually interested in $R(t)$, the envelope of normal process. But due to the complicated multivariate probability density function (PDF) and characteristic function (CF) of $R(t)$, only a limited number of its statistical properties are explored. One of these unsolved problems is N, the expected number of maxima of $R(t)$. Assuming a one-sided power spectrum $w(f)$ that has even symmetry around its center frequency f_c, Rice derived a complicated formula for N in terms of an infinite series [1]. More details can be found in [2].

II. INTRODUCING AN AUXILIARY PROCESS

It is clear that every maximum of $R(t)$ corresponds to a zero of its time derivative, $R'(t)$, with negative slope. The total number of zeros of $R'(t)$ is twice the number of its zeros with negative slope. So instead of N, we can compute $E[N_0\{R'(t)\}]/2$, half of the expected number of zeros of $R'(t)$, irrespective of their slopes. Since $R(t)>0$, we have $N_0\{R'(t)\}=N_0\{A(t)\}$, where $A(t)$ is an auxiliary process defined as $A(t)=R(t)R'(t)$. Thus we can conclude that $N=E[N_0\{A(t)\}]/2$. As becomes clear later, calculation of $E[N_0\{A(t)\}]$ is much easier than $E[N_0\{R'(t)\}]$.

III. COMPUTING $E[N_0\{A(t)\}]$

It is traditional to compute $E[N_0\{A(t)\}]$ using the joint PDF of variables A and A', $p_{AA'}(a,a')$, according to this formula [2]:

$$E[N_0\{A(t)\}]=\int_{-\infty}^{\infty}|a'|\,p_{AA'}(0,a')\,da'. \qquad (1)$$

However it is also possible to express $E[N_0\{A(t)\}]$ in terms of the joint CF of variables A and A', $\Phi_{AA'}(\omega_1,\omega_2)$ [3]:

$$E[N_0\{A(t)\}]=\frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega_2}\frac{d}{d\omega_2}\Phi_{AA'}(\omega_1,\omega_2)\,d\omega_1\,d\omega_2. \qquad (2)$$

According to [3] we have:

$$\Phi_{AA'}(\omega_1,\omega_2)=\frac{1}{1+\alpha_1\omega_1^2+\alpha_2\omega_2^2-j2B\omega_2^3}, \qquad (3)$$

where $j=\sqrt{-1}$ and:

$$\alpha_1=b_0b_2-b_1^2,$$

$$\alpha_2=b_0b_4+3b_2^2-4b_1b_3,$$

$$B=b_0b_2b_4+2b_1b_2b_3-b_2^3-b_0b_3^2-b_1^2b_4. \qquad (4)$$

In the above formulas b_n is the nth spectral moment, given by:

$$b_n=(2\pi)^n\int_0^{\infty}w(f)(f-f_c)^n\,df. \qquad (5)$$

If we put (3) in (2) and simplify the result, we finally reach at:

$$E[N_0\{A(t)\}]=\frac{K(\beta)}{\pi}\sqrt{\frac{\alpha_2}{\alpha_1}}, \qquad (6)$$

in which:

$$\beta=\frac{-B}{\alpha_2^{3/2}}=\frac{-b_0b_2b_4-2b_1b_2b_3+b_2^3+b_0b_3^2+b_1^2b_4}{\left(b_0b_4+3b_2^2-4b_1b_3\right)^{3/2}}, \qquad (7)$$

$$K(\beta)=\int_0^{\infty}\frac{\left(1+9\beta^2\zeta^2\right)^{1/2}}{\left(1+2\zeta^2+\zeta^4+4\beta^2\zeta^6\right)^{3/4}}\cos\left[\tan^{-1}(3\beta\zeta)-\frac{3}{2}\tan^{-1}\left(\frac{2\beta\zeta^3}{1+\zeta^2}\right)\right]d\zeta. \qquad (8)$$

IV. EXACT AND APPROXIMATE FORMULAS FOR N

According to the discussion in II and based on (6), we readily obtain our main result:

$$N=\frac{K(\beta)}{2\pi}\sqrt{\frac{\alpha_2}{\alpha_1}}. \qquad (9)$$

For any $w(f)$ we have $-\sqrt{3}/9\le\beta\le 1/8$, which restricts the variations of $K(\beta)$ as $1\le K(\beta)\le 1.05$ [3]. So it is reasonable to consider $2\pi N\approx\sqrt{\alpha_2/\alpha_1}$ as a good approximation. Such a behavior was previously conjectured in [4] [5], via two different approaches. Finally, it should be noted that (9) holds for spherically invariant processes, which are more general than normal processes [3].

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