On the PDF of the Sum of Random Vectors

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Abstract — **There are various cases in physics and engineering sciences (specially communications) where one requires the envelope PDF of the sum of several random sinusoidal signals. According to the correspondence between a random sinusoidal signal and a random vector, sum of random vectors can be considered as an abstract mathematical model for the above sum. Now it is desired to obtain the PDF of the length of the resulting vector. Considering the common and reasonable assumption of uniform distributions for the angles of vectors, many researchers have obtained the PDF of the length of the resulting vector only for special cases. However in this paper, the PDF is obtained for the most general case in which the lengths of vectors are arbitrary dependent random variables. This PDF is in the form of a definite integral, which may be inappropriate for analytic manipulations and numerical computations. So an appropriate infinite Laguerre expansion is also derived. Finally, the results are applied to solve a typical example in computing the scattering cross section of random scatterers.**

Keywords: Cochannel interference, Electromagnetic propagation in random media, Fading channels, Radar clutter, Random vectors.

I. INTRODUCTION

In various applications we usually encounter a random signal that is composed of the sum of several random sinusoidal signals, e.g. multipath fading in communication channels [1]-[5], clutter [6]-[8] and target cross section [9] in radars, interference in communication systems [10], [11], wave propagation in random media and channels [12]-[17], laser speckle patterns [18], [19], and light scattering [20], [21]. Some other examples can be found in [22].

Any random sinusoidal signal can be considered as a random vector, i.e. a vector with random length and angle. In this way the sum of random sinusoidal signals changes to the sum of random vectors. So, irrespective of the type of application, we encounter the following general mathematical problem: There are N vectors with lengths A_i's and angles Φ_i's, where N, A_i's, and Φ_i's are random variables. It is desired to obtain the probability density function (PDF) of A, length of the resulting vector:

$$A\exp(j\Phi) = \sum_{i=1}^{N} A_i \exp(j\Phi_i) = X + jY, \qquad (1)$$

In the above formula the ith vector is represented by $A_i \exp(j\Phi_i)$, where $j = \sqrt{-1}$. Note that the PDF of A also represents the univariate envelope PDF for the sum of random sinusoidal signals.

According to a comprehensive literature survey [23], the reported results on the PDF of A, obtained under various assumptions and conditions, may be summarized as follows:

1) N is a random variable: [12], [20], [21], [24]-[30],

2) Φ_i's have nonuniform PDF's on $[0,2\pi[$: [12], [13], [23], [31]-[34],

3) N is a deterministic variable and Φ_i's have uniform PDF's on $[0,2\pi[$: [10], [12], [14], [18], [21]-[23], [28], [35]-[67], [86],

4) X and Y in (1) have a joint Gaussian PDF: [12], [13], [55], [68]-[70],

5) X and Y in (1) have a joint nonGaussian PDF: [12], [50], [53], [71].

The pioneering contributions of Rayleigh, Pearson, Kluyver, and Markov to the random vector problem have been summarized in [33]. The contribution of Russian researchers, not reported in [23] since in most cases they have published their results only in Russian, is summarized in [72]. It is interesting to note that almost all of the results obtained by English speaking researchers are derived independently by Russian speaking investigators.

In most practical cases, the two conditions stated in item 3 are usually satisfied [12]. It is interesting to note that there is a close relationship between the uniform distribution of angles and stationarity concept [73], [74]. Therefore we focus on item 3. According to [23] and under various presumptions for A_i's, the following methods have been used for obtaining the PDF of A:

a. Infinite expansion in terms of the Fourier-Bessel series [18], [38], [52],

b. Infinite expansion in terms of the Laguerre series [23], [39], [40], [45], [46], [52], [67],

c. Various analytic approximations [12], [14], [21], [50], [54], [55], [59], [61], [63],

d. Recursive [10], [39],

e. Miscellaneous [40], [42], [45], [48], [49], [57], [62], [64], [65], [86].

In the subsequent sections, we derive an expression for the PDF of A in terms of an infinite series containing Laguerre polynomials. Then we discuss methods for computing the coefficients of this infinite series. Finally and as an example, we apply our results to a random vector problem describing the scattering cross-section of a small number of random scatterers, which has been solved previously via Monte-Carlo simulations [39].

II. A General Random Vector Problem and Its Associated PDF's

Consider n random vectors with lengths A_i's and angles Φ_i's, where n is a deterministic variable. For $i = 1,...,n$, Φ_i's are independent random variables with uniform PDF's on $[0, 2\pi[$, A_i's are arbitrary dependent positive random variables, and A_i's are independent of Φ_i's. Summation of these n random vectors results in a random vector with length A and angle Φ, as defined in (1).

According to (1) we have:

$$X = A\cos\Phi = \sum_{i=1}^{n} A_i \cos\Phi_i \quad Y = A\sin\Phi = \sum_{i=1}^{n} A_i \sin\Phi_i\,, \tag{2}$$

The joint characteristic function of X and Y is defined as $\Psi_{XY}(\eta,\zeta) = E_{XY}[\exp(j\eta X + j\zeta Y)]$, in which E is the mathematical expectation. It can also be written in terms of A_i's and Φ_i's, say:

$$\Psi_{XY}(\eta,\zeta) = E_{A_1 ... A_n \Phi_1 ... \Phi_n}[\exp(j\eta X + j\zeta Y)] = E_{A_1 ... A_n}[E_{\Phi_1 ... \Phi_n}[\exp(j\eta X + j\zeta Y) | A_1 ... A_n]]. \tag{3}$$

Since A_i's are independent of Φ_i's, the condition in (3) can be omitted. Thus (3) changes to:

$$\Psi_{XY}(\eta,\zeta) = E_{A_1 ... A_n}[E_{\Phi_1 ... \Phi_n}[\exp(j\eta X + j\zeta Y)]]. \tag{4}$$

Substitution of X and Y from (2) yields:

$$\Psi_{XY}(\eta,\zeta) = E_{A_1 \ldots A_n}[E_{\Phi_1 \ldots \Phi_n}[\prod_{i=1}^{n} \exp(j\eta A_i \cos\Phi_i + j\zeta A_i \sin\Phi_i)]]. \tag{5}$$

Due to the independence of Φ_i's, (5) simplifies to:

$$\Psi_{XY}(\eta,\zeta) = E_{A_1 \ldots A_n}[\prod_{i=1}^{n} E_{\Phi_i}[\exp(j\eta A_i \cos\Phi_i + j\zeta A_i \sin\Phi_i)]]. \tag{6}$$

By introducing new variables ρ and θ in terms of η and ζ as:

$$\eta = \rho\cos\theta \quad \zeta = \rho\sin\theta, \tag{7}$$

and using the trigonometric identity $\eta\cos\Phi_i + \zeta\sin\Phi_i = \rho\cos(\Phi_i - \theta)$, (6) can be written as:

$$\Psi_{XY}(\eta,\zeta) = E_{A_1 \ldots A_n}[\prod_{i=1}^{n} \int_0^{2\pi} \exp(jA_i\rho\cos(\phi_i - \theta)) f_{\Phi_i}(\phi_i) d\phi_i], \tag{8}$$

where $f_{\Phi_i}(\phi_i)$ is the PDF of Φ_i. Uniform PDF of each Φ_i on $[0,2\pi[$ means $f_{\Phi_i}(\phi_i) = 1/2\pi$. Using this fact, and also the integral form of the zero order Bessel function, i.e. $J_0(z) = (1/2\pi)\int_0^{2\pi} \exp(jz\cos\xi)d\xi$, (8) reduces to the following form:

$$\Psi_{XY}(\eta,\zeta) = E_{A_1 \ldots A_n}[\prod_{i=1}^{n} J_0(A_i\rho)] = \Lambda(\rho) = \Lambda(\sqrt{\eta^2 + \zeta^2}). \tag{9}$$

Based on the definition of jointly spherically symmetric random variables in [75], the functional form of $\Psi_{XY}(\eta,\zeta)$ in terms of $\sqrt{\eta^2 + \zeta^2}$ implies that X and Y are jointly spherically symmetric random variables. Thus it can be deduced from theorem 1 in [75] that A and Φ, defined in (2), are independent; Φ has a uniform PDF on $[0,2\pi[$, and A has the following PDF:

$$f_A(a) = a\int_0^{\infty} \rho J_0(a\rho)\Lambda(\rho)d\rho = aH_{0a}\{\Lambda(\rho)\}, \tag{10}$$

where $H_{0z}\{G(\xi)\}$ is the zero order Hankel transform of $G(.)$ defined as [76]:

$$H_{0z}\{G(\xi)\} = \int_0^{\infty} \xi J_0(z\xi)G(\xi)d\xi .$$

Some authors also call it Fourier-Bessel transform [77], [78]. Using the tables of [79], it is possible to obtain the Hankel transform of various functions.

III. Expanding $f_A(a)$ in Terms of Laguerre Polynomials

Laguerre polynomials are a set of orthogonal polynomials on the positive real axis. Thus they make a useful basis for expanding the PDF of positive random variables [55], [80], [81]. In this section we use this approach to expand $f_A(a)$ for the case which is more general than those discussed in [39], [40], [45], and [52].

Based on the properties of the Hankel transform, it can be shown that:

$$\Lambda(\rho) = H_{0\rho}\{\frac{f_A(a)}{a}\} = \int_0^\infty J_0(\rho a) f_A(a)da = E_A[J_0(A\rho)]. \tag{11}$$

The following generating function for the Laguerre polynomials is given in [82]:

$$\exp(\sigma)J_0(2\sqrt{\tau\sigma}) = \sum_{m=0}^\infty \frac{L_m(\tau)\sigma^m}{m!}, \tag{12}$$

where $L_m(.)$ is the Laguerre polynomial of order *m*. Assuming $\tau = \beta A^2$ and $\sigma = \rho^2/4\beta$, (12) gives the following parametric expansion for $J_0(A\rho)$:

$$J_0(A\rho) = \sum_{m=0}^\infty \frac{1}{m!(4\beta)^m} L_m(\beta A^2)\rho^{2m}\exp(-\frac{\rho^2}{4\beta}); \beta \neq 0, \tag{13}$$

where β is an arbitrary non-zero real number, introduced on purpose. The role of β will be discussed later.

Now by inserting (13) into (11), one obtains:

$$\Lambda(\rho) = \sum_{m=0}^\infty \frac{1}{m!(4\beta)^m} E_A[L_m(\beta A^2)]\rho^{2m}\exp(-\frac{\rho^2}{4\beta}); \ \beta \neq 0. \tag{14}$$

Substitution of $\Lambda(\rho)$ in (10) by its expansion, presented in (14), gives:

$$f_A(a) = a\sum_{m=0}^\infty \frac{1}{m!(4\beta)^m} E_A[L_m(\beta A^2)]\int_0^\infty \rho^{2m+1} J_0(a\rho)\exp(-\frac{\rho^2}{4\beta})d\rho; \beta \neq 0. \tag{15}$$

Clearly, the integral in (15) converges only for $0 < \beta < \infty$. For this range of β, and after some manipulations [83], the following result can be obtained:

$$\int_0^\infty \rho^{2m+1} J_0(a\rho)\exp(-\frac{\rho^2}{4\beta})d\rho = \frac{m!(4\beta)^{m+1}}{2}\exp(-\beta a^2)L_m(\beta a^2); 0 < \beta < \infty. \tag{16}$$

Inserting (16) into (15) gives:

$$f_A(a) = 2\beta a\exp(-\beta a^2)\sum_{m=0}^\infty C_m L_m(\beta a^2); \ 0 < \beta < \infty, \tag{17}$$

where by definition:

$$C_m = E_A[L_m(\beta A^2)]. \tag{18}$$

It should be mentioned that the infinite series for $f_A(a)$ in (17) is obtained just by employing the fact that $f_A(a)$ and $\Lambda(\rho)$ constitute a Hankel transform pair (see (10)), along with the use of Laguerre generating function in (12). In fact, not only the PDF of *A* but also the PDF of an arbitrary positive random variable *P* can be expressed similar to (17) [55]. The functional form of $\Lambda(\rho)$ in (9) only affects the value of coefficients C_m's in (17), which is the topic of the next section. However, this subject is not discussed in [55].

The advantage of a variable β, instead of a predetermined value, lies in the fact that β can be selected in such a way to minimize the truncation error of (17). This approach is used in [23], [35], [37] for a similar random vector problem. However, determination of the β which minimizes the truncation error of (17) is under study.

IV. A Closed-Form Formula for C_m

Definition of the *m*th order Laguerre polynomial implies that:

$$L_m(z) = \sum_{k=0}^{m} \frac{(-1)^k m!}{(m-k)!(k!)^2} z^k. \tag{19}$$

So C_m in (18) can be written as:

$$C_m = \sum_{k=0}^{m} \frac{(-\beta)^k m!}{(m-k)!(k!)^2} \mu_n^{(2k)}, \tag{20}$$

where $\mu_n^{(2k)}$ is the $2k$th moment of *A*, resulted from the sum of *n* random vectors:

$$\mu_n^{(2k)} = E_A[A^{2k}]. \tag{21}$$

Inspection of (20) shows that C_m is a linear combination of $\mu_n^{(2k)}$'s. Hence for computing C_m, it is useful to obtain a closed-form formula for $\mu_n^{(2k)}$.

For $\beta \to \pm\infty$, the expansion in (13) reduces to the Maclaurin series of $J_0(A\rho)$; and $\Lambda(\rho)$ simplifies to:

$$\Lambda(\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 4^k} \mu_n^{(2k)} \rho^{2k}. \tag{22}$$

Comparison of (22) with the Maclaurin series of $\Lambda(\rho)$ reveals that:

$$\mu_n^{(2k)} = (-4)^k \frac{(k!)^2}{(2k)!} \partial^{2k} \Lambda(\rho) \big/ \partial\rho^{2k} \Big|_{\rho=0}. \tag{23}$$

When calculation of $E_{A_1 \dots A_n}[.]$ in (9) is possible in a closed form, $\mu_n^{(2k)}$'s and consequently C_m's, can be computed via (23) and (20) respectively. It should be mentioned that (23) holds not only for A but also for an arbitrary positive random variable P, where $\Lambda(\rho) = E_P[J_0(P\rho)]$. However, the corresponding $\Lambda(\rho)$ for A is presented in (9).

In (23) we must compute multiple derivatives of a function. In some cases, application of Bell polynomials can simplify this task [84]. For the case in which A_i's are independent, there is a recursive relation for $\mu_n^{(2k)}$. In fact, the following useful formula is derived in [85]:

$$\mu_l^{(2k)} = \begin{cases} \nu_1^{(2k)}; l = 1 k = 0,1,\dots \\ \sum_{i=0}^{k} (\frac{k!}{i!(k-i)!})^2 \mu_{l-1}^{(2i)} \nu_l^{(2k-2i)}; l = 2,\dots,n\, k = 0,1,\dots \end{cases} \tag{24}$$

in which $\nu_l^{(2k)}$ is defined as:

$$\nu_l^{(2k)} = E_{A_l}[A_l^{2k}];\ l = 1,\dots,n\ \ k = 0,1,\dots \tag{25}$$

V. Application of The Results to a Typical Example

In [39], a random vector model is employed to investigate the statistical behavior of the scattering cross-section, when the number of scatterers is small. Specifically, they have considered the sum of n random vectors with lengths A_i's and angles Φ_i's. In their work, n is a deterministic variable, $A_1 = \dots = A_n = A_0$ where A_0 is a positive deterministic variable, and Φ_i's are independent random variables with uniform PDF's on $[0, 2\pi[$. Based on the above assumptions, an orthonormal Laguerre polynomial representation is presented for the PDF of $S_n = |\sum_{i=1}^{n} A_i \exp(j\Phi_i)|^2$ in [39].

By noting that $S_n = A^2$ and using (17) and (18) assuming $\beta = 1 \big/ nA_0^{\ 2}$, the PDF of S_n can be written as [39, (17)]:

$$f_{S_n}(s_n) = \frac{1}{nA_0^{\ 2}} \exp(-\frac{s_n}{nA_0^{\ 2}}) \sum_{m=0}^{\infty} c_m L_m(\frac{s_n}{nA_0^{\ 2}}), \tag{26}$$

where c_m is defined as [39, (21)]:

$$c_m = E_{S_n}[L_m(\frac{S_n}{nA_0^{\ 2}})]. \tag{27}$$

Note that there is the following relationship between the c_m in [39] and the C_m defined here:

$$c_m = C_m\big|_{\beta=1/nA_0^2}. \tag{28}$$

Thus, c_m can be calculated via (20), assuming $\beta = 1/nA_0^2$ and using exactly the same $\mu_n^{(2k)}$'s. By the use of either (23) or (24), $\mu_n^{(2k)}$'s can also be computed efficiently. It should be mentioned that for the case considered in [39], i. e. $A_1 = ... = A_n = A_0$, $\Lambda(\rho)$ in (9) simplifies to:

$$\Lambda(\rho) = J_0^{\,n}(A_0\rho) \tag{29}$$

A maximum likelihood estimator $\hat{c}_m$ is presented in [39, (22)] for estimating c_m, using Monte-Carlo simulations. However, based on the above discussion and formulas, it is clear that such a procedure is not necessary for computing c_m. In Table I of [39], for $A_0 = 1$ and several *m*'s and *n*'s, the estimated values of c_m's are presented; while in Table I of this paper and under the same conditions, exact values of c_m's are reported. Comparison of these two tables indicates that there is no need for time-consuming Monte-Carlo simulations, to obtain the coefficients c_m's.

VI. Conclusion

Envelope PDF of the sum of random sinusoids is of importance in various applications. Considering the most general case for the amplitudes of random sinusoids, a closed-form expression was obtained in (10) for this PDF. Since (10) is in the form of a definite integral, which may be inappropriate specially for analytic studies, an infinite Laguerre series was also derived in (17). The coefficient of this series can be obtained through the application of closed-from formulas (20) and (23). Based on these results, time-consuming Monte-Carlo simulations for determining the envelope PDF can be completely avoided. Moreover, our approach expresses the envelope PDF just in terms of polynomials, while the Fourier-Bessel series mentioned in Section I expands the envelope PDF in terms of Bessel functions, definitely more complicated than polynomials from both numerical and analytic point of views. Thus based on these results and either numerically or analytically, performance of various modulation and coding schemes in general multipath fading channels can be assessed, efficient detection procedures may be developed in radars assuming various clutter PDFs and for different target cross sections PDFs, error probability in the presence of several interferer can be determined, suitable speckle reduction techniques can be developed against different scattering conditions, etc.

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TABLE I

EXACT VALUES OF THE COEFFICIENTS c_m IN (27), OBTAINED VIA (28) AND (20), ALONG WITH THE USE OF EITHER (23) OR (24)

	No. Scatterers n				
m	4	5	6	7	8
1	0	0	0	0	0
2	-1.2500E-01	-1.0000E-01	-8.3333E-02	-7.1429E-02	
3	-4.1667E-02	-2.6667E-02	-1.8519E-02		
4	2.5391E-02	1.9000E-02			
5	3.7240E-02	2.0587E-02			
6	1.5516E-02	4.4067E-03			
7	-1.1375E-02	-8.4549E-03			
8	-2.6038E-02				
9	-2.3673E-02				
10	-7.9373E-03				
11	1.3938E-02				
12	3.4800E-02				
13	4.9586E-02				
14	5.5953E-02				
15	5.3988E-02				
16	4.5475E-02				
17	3.3042E-02				