Generalized Discrete Fourier Transform with Optimum Correlations

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Abstract— In this paper, we present design methods to optimize nonlinear phase functions of Generalized DFT (GDFT) for minimized auto and cross correlations. It is shown that GDFT offers sizable correlation improvements over DFT, Walsh, and Gold codes. We conclude the paper with a multiuser communications scenario where correlation optimized GDFT outperforms DFT and other known constant modulus codes in BER metric.

Index Terms—Generalized Discrete Fourier Transform (GDFT), OFDM, DMT, CDMA, Auto-correlation, Cross-correlation, BER.

I. GENERALIZED DFT

An N^{th} root of unity is a complex number satisfying the equation

$$z^N = 1 \quad N = 1,2,3,\ldots \tag{1}$$

If $z_p^m \neq 1$; $m = 1,2,..,N-1$, then z_p is defined as the p^{th} *primitive N^{th} root of unity* and m and N must be coprime integers. The complex number $z_1 = e^{j(2\pi/N)}$ is the primitive N^{th} root of unity with the smallest positive argument. There are N distinct N^{th} roots of unity for any primitive and expressed as

$$z_k = (z_p)^k \quad k = 1,2,3,\ \ldots\ ,N \quad \forall p \tag{2}$$

where z_p is any of the *primitive N^{th} root of unity*. As an example, $z_1 = e^{j\frac{2\pi}{4}}$ and $z_2 = e^{j\frac{3\pi}{2}}$ are the two primitive N^{th} roots of unity for N=4. All *primitive N^{th} roots of unity* satisfy the unique summation property of a geometric series expressed as follows

$$\sum_{n=0}^{N-1}(z_p)^n = \frac{(z_p)^N - 1}{z_p - 1} = \begin{Bmatrix} 1, N=1 \\ 0, N>1 \end{Bmatrix} \quad \forall p \tag{3}$$

Now, define a periodic, constant modulus, complex sequence $\{e_r(n)\}$ as the r^{th} *power* of the first *primitive N^{th} root of unity* z_1 raised to the n^{th} *power* as expressed in

$$\begin{aligned} &e_r(n) \triangleq (z_1^r)^n = e^{j(2\pi r/N)n} \\ &n = 0,1,2,\ldots,N-1 \ \text{ and } \ r = 0,1,2,\ldots,N-1 \end{aligned} \tag{4}$$

The complex sequence of (4) over a finite discrete-time interval in a geometric series is expressed according to (3) as follows [1]-[2]

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1} e_r(n) = \frac{1}{N}\sum_{n=0}^{N-1}(z_1^r)^n = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi r/N)n} \\ &= \begin{Bmatrix} 1, & r = mN \\ 0, & r \neq mN \\ & m = \text{integer} \end{Bmatrix} \end{aligned} \tag{5}$$

Let's generalize (5) by rewriting the phase as the difference of two functions $\varphi_{kl}(n) = \varphi_k(n) - \varphi_l(n) = r$ and expressing a constant modulus orthogonal set as follows,

$$\begin{aligned} &\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi r/N)n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi\varphi_{kl}(n)/N]n} \\ &= \begin{Bmatrix} 1, & \varphi_{kl}(n) = \varphi_k(n) - \varphi_l(n) = k-l = r = mN \\ 0, & \varphi_{kl}(n) = \varphi_k(n) - \varphi_l(n) = k-l = r \neq mN \\ & m = \text{integer } 0 \le k,l,n \le N-1 \end{Bmatrix} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi(\varphi_k(n)-\varphi_l(n))/N)]n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi\varphi_k(n)/N]n} e^{-j[2\pi\varphi_l(n)/N]n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} e_k(n)e_l^*(n) = \langle e_k(n), e_l^*(n) \rangle \end{aligned} \tag{6}$$

Hence, the basis functions of the new set are defined as

$$\{e_k(n)\} \triangleq e^{j(2\pi/N)\varphi_k(n)n} \qquad k,n = 0,1,\ldots,N-1 \tag{7}$$

We call this orthogonal function set as the *Generalized Discrete Fourier Transform* (GDFT) with *nonlinear phase* [1]-[2]. It is observed from (6) and (7) that there are infinitely many sets of constant modulus and *nonlinear phase functions* available. Therefore, we might methodically design such functions. As an example, one can define the discrete time function $\varphi_k(n)$ in (7) as the ratio of two polynomials,

$$\varphi_k(n)=\frac{N_k(n)}{D_k(n)}=\frac{\sum_{j=1}^{N} a_{kj} n^{b_{kj}}}{\sum_{j=1}^{M} c_{kj} n^{d_{kj}}} \quad N \le M; \; k=0,1,...,N-1 \tag{8}$$

Let's assume that the denominator polynomial $D_k(n)=1$ and the numerator polynomial is defined as follows

$$\varphi_k(n)=N_k(n)=\sum_{j=1}^{N} a_{kj} n^{b_{kj}} = a_{k1}n^{b_{k1}}+a_{k2}n^{b_{k2}}+a_{k3}n^{b_{k3}}+...+a_{kN}n^{b_{kN}} \tag{9}$$

In general, the polynomial coefficients $\{a_{kj}\}$ and $\{c_{kj}\}$ are complex, the powers $\{b_{kj}\}$ and $\{d_{kj}\}$ are real numbers.

II. CORRELATION METRICS

All the metrics used in this study depend on *aperiodic correlation functions (ACF)* of the spreading code set. The ACF metric $d_{k,l}(m)$ is defined for the complex sequences $\{e_k(n)\}$ and $\{e_l(n)\}$ [3],

$$d_{k,l}(m)=\begin{cases} \frac{1}{N}\sum_{n=0}^{N-1-m} e_k(n)e_l^*(n+m), & 0<m\le N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+m} e_k(n-m)e_l^*(n), & 1-N<m\le 0 \\ 0 \quad , & |m|\ge N \end{cases} \tag{10}$$

Out-of-phase autocorrelation sequence of the complex sequence $e_k(n)$ is defined also from (10) as the absolute sequence of $d_{k,k}(m)$ as $|d_{k,k}(m)|$. Similarly, out-of-phase cross-correlation of two complex sequences $e_k(n)$ and $e_l(n)$ is defined as the absolute function of $d_{k,l}(m)$ and expressed with $|d_{k,l}(m)|$.

In this paper, the following correlation metrics are used for optimal GDFT design and performance comparisons of various code families [4]-[6] where M is the set size and N is the length of each spreading code.

a. Maximum Value of Out-of-Phase Auto-correlation, d_{am}:

$$d_{am}=\max_{\substack{0\le k<M \\ 1\le m<N}}\left\{\left|d_{k,k}(m)\right|\right\} \tag{11}$$

b. Maximum Value of Out-of-Phase Cross-correlation, d_{cm}:

$$d_{cm}=\max_{\substack{0\le k,l<M\ k\ne l \\ 0\le m<N}}\left\{\left|d_{k,l}(m)\right|\right\} \tag{12}$$

$$d_{\max}=\max\{d_{am},d_{cm}\} \tag{13}$$

The relationship between the maximum out-of-phase auto-correlation d_{am} and the maximum out-of-phase cross-correlation d_{cm} was shown by Welch in [7],

$$d_{\max}=\max\{d_{am},d_{cm}\}=\sqrt{\frac{M-1}{M(2N-1)-1}} \tag{14}$$

c. Mean Square Value of Auto-correlation, R_{AC}:

$$R_{AC}=\frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{m=1-N \\ m\ne 0}}^{N-1}\left|d_{k,k}(m)\right|^2 \tag{15}$$

d. Mean Square Value of Cross-correlation, R_{CC}:

$$R_{CC}=\frac{1}{M(M-1)}\sum_{k=1}^{M}\sum_{\substack{l=1 \\ l\ne k}}^{M}\sum_{m=1-N}^{N-1}\left|d_{k,l}(m)\right|^2 \tag{16}$$

III. OPTIMAL GDFT AND PERFORMANCE

Now, we focus on the optimal design of the phase shaping function, $\psi(n)$, based on a performance metric. The phase function $\{\varphi_k(n)\,n\}$ of (7) is now decomposed into two functions in the time variable *n* as follows

$$\begin{aligned} &\hat{\varphi}_k(n)=\varphi_k(n)n=kn+\psi(n) \\ &k=0,1,...,N-1,\; n=1,...,N-1 \\ &\psi(n)=\hat{\varphi}_k(n)-kn=[\varphi_k(n)-k]n \\ &k=0,1,...,N-1,\; n=1,...,N-1 \\ &\psi(0)\in\mathbb{R} \quad \hat{\varphi}_k(0)=\psi(0) \end{aligned} \tag{17}$$

The linear term in (17) is highlighted due to its significance in the orthogonality requirements of (6). The GDFT framework offers us the freedom to define the phase shaping function $\psi(n)$ according to the application requirements. Note that any $\psi(n)$ function will give us an orthogonal GDFT set. Moreover, this property allows us to select $\psi(n)$ to satisfy requirement other than orthonormality.

The cross-correlation sequence of a GDFT *basis function pair (k, l)* with length N is defined as

$$R_{\hat{\varphi}_k \hat{\varphi}_l}(m) = \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})\hat{\varphi}_k(n)} e^{-j(\frac{2\pi}{N})\hat{\varphi}_l(n+m)}$$
$$= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})[-lm+(k-l)n+\psi(n)-\psi(n+m)]} \quad (18)$$

where $R_{\hat{\varphi}_k \hat{\varphi}_l}(m) = 0; \ \forall m$ for the ideal case, and $R_{\hat{\varphi}_k \hat{\varphi}_l}(0) = 0$ implies the orthogonality of the function pair. Similarly, we can define the auto-correlation function of a GDFT basis function as

$$R_{\hat{\varphi}_k \hat{\varphi}_k}(m) = \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})\hat{\varphi}_k(n)} e^{-j(\frac{2\pi}{N})\hat{\varphi}_k(n+m)}$$
$$= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})[-km+\psi(n)-\psi(n+m)]} \quad (19)$$

where $R_{\hat{\varphi}_k \hat{\varphi}_k}(m) = \delta(m)$ for the ideal case. The correlation sequence of a basis pair is incorporated in R_{AC} and R_{CC} metrics of (15) and (16), respectively, in the following design example.

Since there is no closed form solution, we used the numerical optimization software tools in Mathematica and MATLAB to obtain optimal phase shaping functions of (17) with respect to the metrics of (15) and (16). Note that the entire set uses the same phase shaping function and there are $P_M = \binom{M}{2} = \frac{M!}{(M-2)!2!}$ basis function pairs in a set of size M to be considered in the design of an optimum $\psi(n)$ yielding the best cross-correlation performance. Similarly, any basis function might have its own optimal phase shaping function optimizing the auto-correlation sequence. Therefore, one needs to pick the sequence as the optimal phase shaping function yielding the best auto-correlation for the entire set.

The resulting R_{AC} and R_{CC} values for these optimal GDFT designs are tabulated in Table I along with their DFT counterparts for comparison purposes. Note the consistency of the two numerical search tools, and the improvements in correlation metrics R_{AC} and R_{CC} are shown in this table. The optimal design method explained in this section can be generalized for any performance metric and for any size GDFT. In the following section, we will present an example of communications applications employing the proposed GDFT framework where correlations dictate the system performance.

Table I: R_{AC} and R_{CC} values for the first two functions of optimal GDFT sets with N=8 along with their DFT counterparts.

Numerical Search Tool and Optimal Phase Shaping Function	OPTIMIZATION METRIC (N=8)	
	R_{AC}	R_{CC}
GDFT (Mathematica, FindMin)	0.0877	0.4219
$\psi(n)$	{ -1.37 -2.53 -2.21 3.39 0.0 -4.21 -3.19 -0.83 }	{ 1.637 -0.79 -0.54 2.01 1.59 -0.83 1.73 2.44}
GDFT (MATLAB, fminsearch)	0.0860	0.4205
$\psi(n)$	{ -1.38 -2.56 -2.24 3.42 0.07 -4.27 -3.27 -0.80 }	{ 1.673 -0.87 -0.51 2.02 1.51 -0.86 1.70 2.46 }
DFT	4.375	0.8536

Brute Force Search for Optimal Design of GDFT:

We use the first two terms in (9)

$$\varphi_k(n) = a_{k1} n^{b_{k1}} + a_{k2} n^{b_{k2}}$$
$$a_{k1} = k, \ b_{k1} = 0 \quad (20)$$
$$\varphi_k(n) = k + a_{k2} n^{b_{k2}}$$

Therefore, the basis functions of the set are defined according to (7) as

$$e_k(n) = e^{j(2\pi/N)\varphi_k(n)n}$$
$$e_k(n) = e^{j(2\pi/N)(k+a_{k2}n^{b_{k2}})n} \quad (21)$$
$$e_k(n) = e^{j(2\pi/N)kn} e^{j(2\pi/N)[a_{k2}n^{(b_{k2}+1)}]} \quad k,n=0,1,...,N-1$$

Note that the first exponential term in (21) is the DFT kernel with linear phase while the second defines the G matrix, and $\{e_k(n)\}$ are the row sequences of A_{GDFT} as expressed in the following matrix form

$$A_{GDFT} = A_{DFT} G \quad (22)$$

In this representation, varying the values of the a_{k2} and b_{k2} coefficients generate a medley of GDFT sets along with their associated nonlinear phase functions and auto- and cross-correlation properties. This suggests that a brute force search algorithm can scan the phase space with various grid resolutions to find optimum G matrices with respect to given performance metric.

The search grid resolution is defined by the binary values of coefficients $a_2 = a_{k2}$ and $b_2 = b_{k2}$ for all *k*. Table II tabulates the optimal values of the metric $d_{\max}$ along with other performance metrics for various search grid resolutions defined as $\Delta_{a_2,b_2} = \frac{N-1}{2^b}$ where b is in bits per coefficient and $0 < a_2, b_2 \le 7$ for the code length of $N = 8$. Note that as the search grid resolution improves, the search yields better performance with respect to design metric $d_{\max}$.

Table II: Correlation Performance of Optimal GDFT Designs Based on $d_{\max}$ for *N=8*.

Search Grid Resolution (bits/c)	d_{am}	d_{cm}	$d_{\max}$ ***(OPT)***	R_{AC}	R_{CC}
4	0.301	0.442	0.442	0.526	0.925
6	0.376	0.409	0.409	0.854	0.878
8	0.341	0.387	0.387	0.576	0.918
9	0.376	0.387	0.387	1.095	0.843
Welch Bound [8x8] [7]	-----	-----	0.243	-----	-----

Table III compares the A_{GDFT} with other known constant modulus code sets based on the metrics indicated. In this case, A_{GDFT} is obtained through a brute force search based on minimization of d_{cm} for *N=8*.

Table III: Correlation Performance of Optimal GDFT along with various code families for *N=8*.

Code	d_{am}	d_{cm} ***(OPT)***	$d_{\max}$	R_{AC}	R_{CC}
Walsh [8x8], [4]	0.875	0.875	0.875	2.375	0.661
7/8 Gold, [5]	0.714	0.714	0.714	0.857	0.878
Walsh-like [8x8], [6]	0.625	0.625	0.625	0.875	0.875
DFT [8x8]	0.875	0.327	0.875	4.375	0.375
A_{GDFT} [1] [8x8] (*opt.* d_{cm})	0.762	0.309	0.762	3.341	0.522

Figure 1 displays BER performance of a 2 user asynchronous CDMA communications system with AWGN channel model [8] employing Walsh, Gold, Walsh-like, DFT code families along with d_{cm} based optimal GDFT. It is observed from this figure that optimized GDFT based on correlations outperforms other code families.

IV. CONCLUSIONS

In this paper, we proposed methodology to design the nonlinear phase functions of Generalized DFT for correlation optimization. First, we summarized the GDFT framework. Then, we introduced the correlation metrics employed for optimization of GDFT phase functions. We compared correlation performance of GDFT with popular families like Walsh, Gold and DFT. It is shown that GDFT offers significant correlation improvements over the other code families. This leads to the BER performance improvements in multiuser communications. We also displayed BER curves of various code families under the same test conditions where GDFT offers performance improvements. We expect that GDFT might be implemented in multicarrier communications systems in the future.

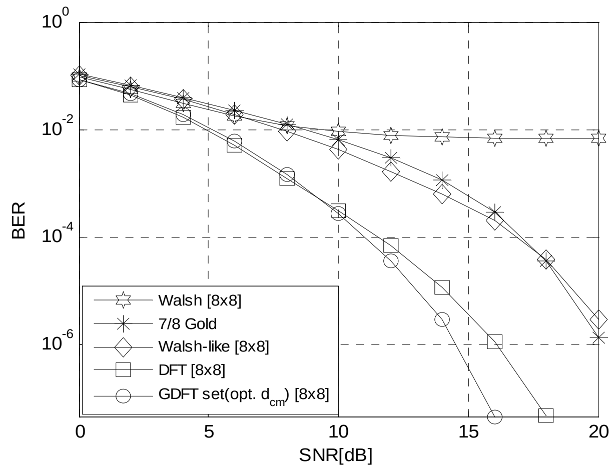

Fig. 1. BER performance of a 2-user asynchronous CDMA communications system with AWGN channel model employing various code sets of length 8 (N=7 for Gold).

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