Generalized Discrete Fourier Transform*

Non-Linear Phase DFT for Improved Multicarrier Communications

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TUTORIAL

(*) Patent pending for GDFT

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Outline

I. Introduction to Orthogonal Block Transforms:
A Time-Frequency Perspective

II. Discrete Fourier Transform with Linear Phase

III. Orthogonal Transmultiplexer for Multicarrier Communications: OFDMA, TDMA, CDMA

IV. Correlation Performance Metrics

V. GDFT with Nonlinear Phase for Auto- and Cross-Correlation Improvements

Outline

VI. BER Performance Comparisons of DFT and
GDFT CDMA for Rayleigh Fading Channels

VII. Variations of CDMA Communications:
From DS-CDMA to MC-DC-CDMA

VIII. Emerging 3GPP LTE Mobile Phone Standard

IX. Discussions and Future Research Directions

I. Introduction to Orthogonal Block Transforms: A Time-Frequency Perspective

Function/Signal

Shape of a function/signal

Function/Signal = Energy Shape

Energy of a function/signal

Duality (Time-Frequency)

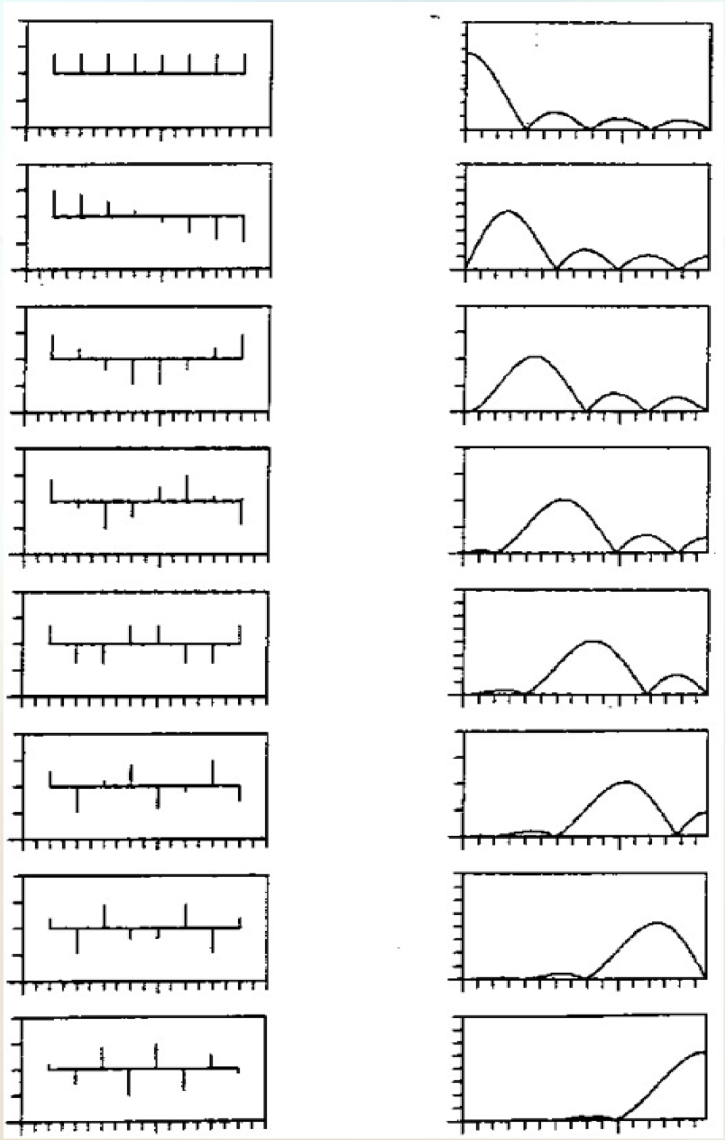
Parseval (Time-Frequency)

Function set (Time-Frequency)

Orthogonality (Time-Frequency)

DFT, DCT,WHT

Functions (Energy Shape) in Time and Frequency Domains: Duality



Signal Energy in Time-Frequency: Parseval Theorem

$$Let \;\; \phi_k(n) \xleftrightarrow[DTFT]{} \Phi_k(e^{j\omega})$$

$$\Phi_k(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \phi_k(n) e^{-j\omega n}$$

$$\phi_k(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_k(e^{j\omega}) e^{j\omega n} d\omega$$

$$E_{\phi_k} = \sum_{n=-\infty}^{n=\infty} \phi_k(n) \phi_k^*(n) = \sum_{n=-\infty}^{n=\infty} \left|\phi_k(n)\right|^2$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_k(e^{j\omega}) \Phi_k^*(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\Phi_k(e^{j\omega})\right|^2 d\omega$$

Orthogonality in Time-Frequency: Parseval Theorem

$$Let \;\; \phi_k(n) \xleftrightarrow[DTFT]{} \Phi_k(e^{j\omega})$$

$$\phi_l(n) \xleftrightarrow[DTFT]{} \Phi_l(e^{j\omega})$$

Consider a function set $\{\phi_k(n)\}; k = 0, 1, \ldots N-1$

$$\sum_{n=-\infty}^{n=\infty} \phi_k(n)\phi_l^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_k(e^{j\omega})\Phi_l^*(e^{j\omega})d\omega = \delta(k-l)$$

Correlations in Time-Frequency: Multiuser Communications (Transmultiplexer for OFDM/CDMA/TDMA)

$$Let\ \phi_k(n) \xleftrightarrow[DTFT]{} \Phi_k(e^{j\omega})$$

$$\phi_l(n) \xleftrightarrow[DTFT]{} \Phi_l(e^{j\omega})$$

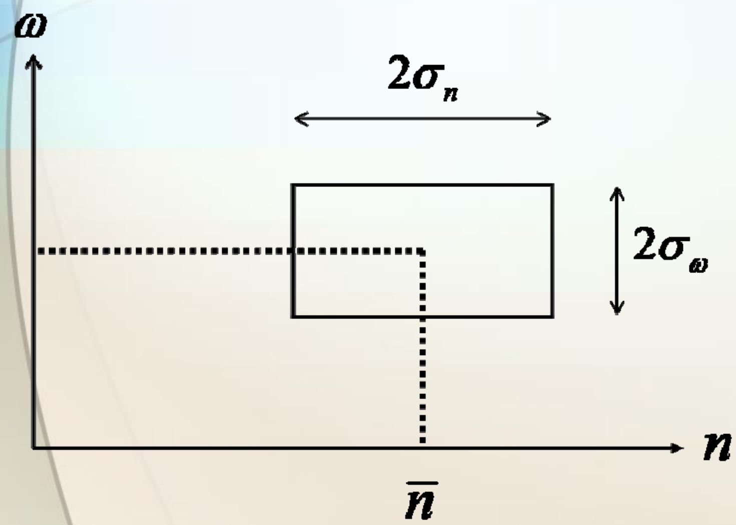
Consider a function set $\{\phi_k(n)\}; k = 0,1,...N-1$

and pairwise cross-correlations defined

$$R_{\phi_k \phi_l}(m) = \sum_{n=-\infty}^{n=\infty} \phi_k(n)\phi_l^*(n-m) \ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_k(e^{j\omega})\Phi_l^*(e^{j\omega})e^{j\omega m}d\omega$$

Correlations result in Inter Carrier Interference (ICI), Inter Symbol Imterference (ISI) and MultiUserInterference (MUI) in a Multiuser / Multicarrier Communications System (Transmultiplexer for OFDM/CDMA/TDMA or T-FMA?)

Time Frequency Localization of A Discrete Time Function



$$Let\ \ \phi_k(n) \xleftrightarrow[DTFT]{} \Phi_k(e^{j\omega})$$

$$\Phi_k(e^{j\omega}) = \sum_{n=-\infty}^{n=\infty} \phi_k(n) e^{-j\omega n}$$

$$\phi_k(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_k(e^{j\omega}) e^{j\omega n} d\omega$$

$$E_{\phi_k} = \sum_{n=-\infty}^{n=\infty} |\phi_k(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Phi_k(e^{j\omega})|^2 d\omega$$

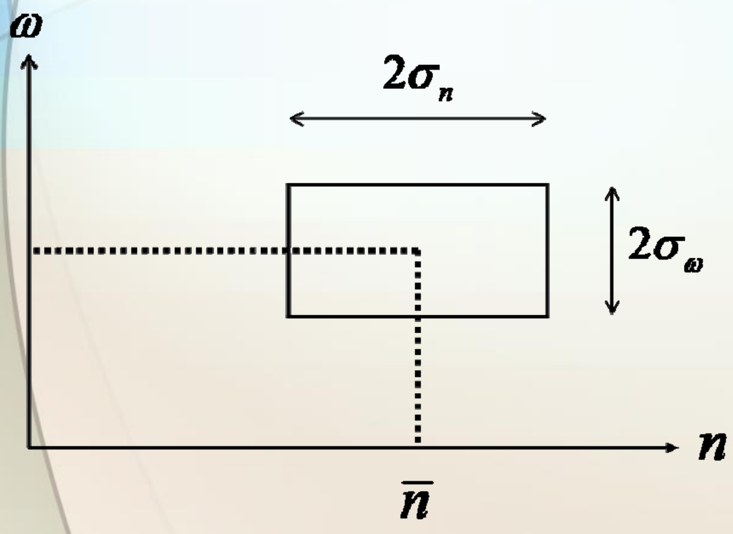
$$\bar{n} = \frac{\sum_{n=-\infty}^{n=\infty} n|(n)|^2}{E}$$

$$\bar{\omega} = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \omega |H(e^{j\omega})|^2 d\omega}{E}$$

$$\sigma_n^2 = \frac{\sum_{n=-\infty}^{\infty} (n-\bar{n})^2 |h(n)|^2}{E}$$

$$\sigma_\omega^2 = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} (\omega-\bar{\omega})^2 |H(e^{j\omega})|^2 d\omega}{E}$$

Discrete-Time Uncertainty



$$\sigma_n \sigma_\omega \geq \frac{|1-\mu|}{2}$$

$$\mu \triangleq \frac{|H(e^{j\pi})|^2}{E}$$

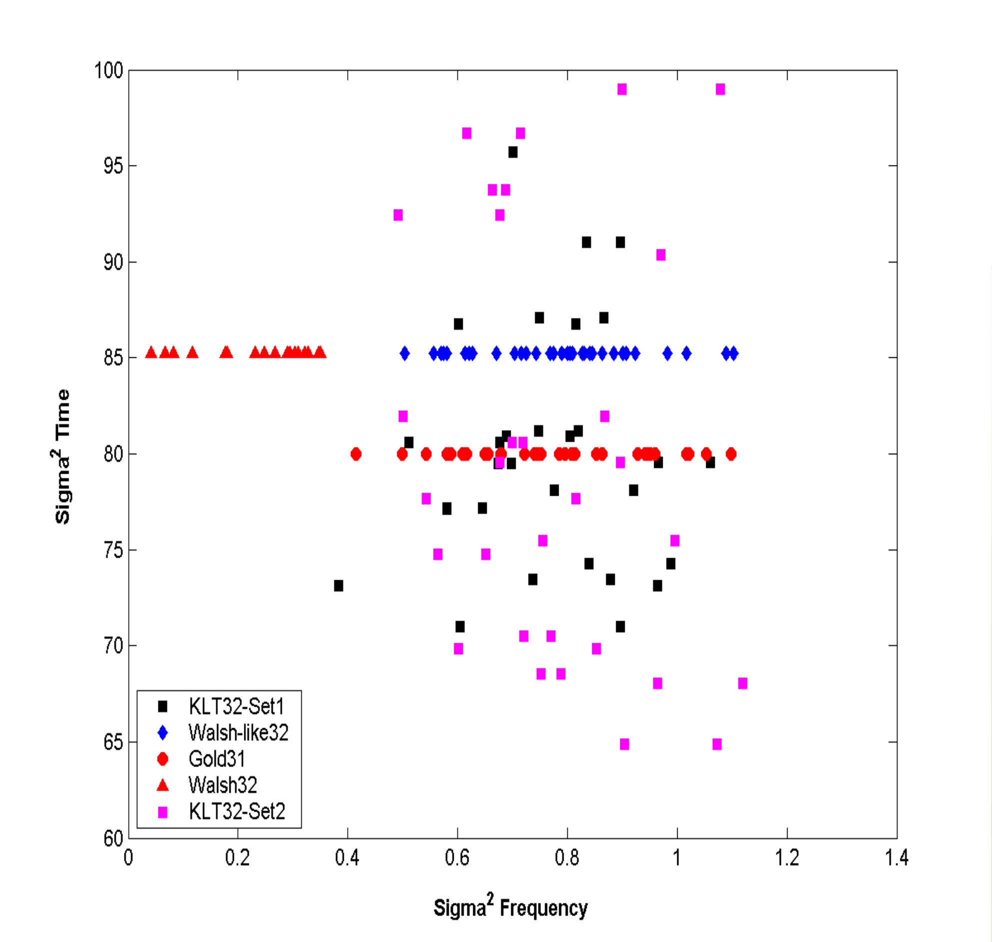
Class I: $H(e^{j\pi}) = 0 \rightarrow \sigma_n \sigma_\omega \geq \frac{1}{2}$

Class II: $H(e^{j\pi}) \neq 0 \rightarrow \sigma_n \sigma_\omega \geq \frac{|1-\mu|}{2}$

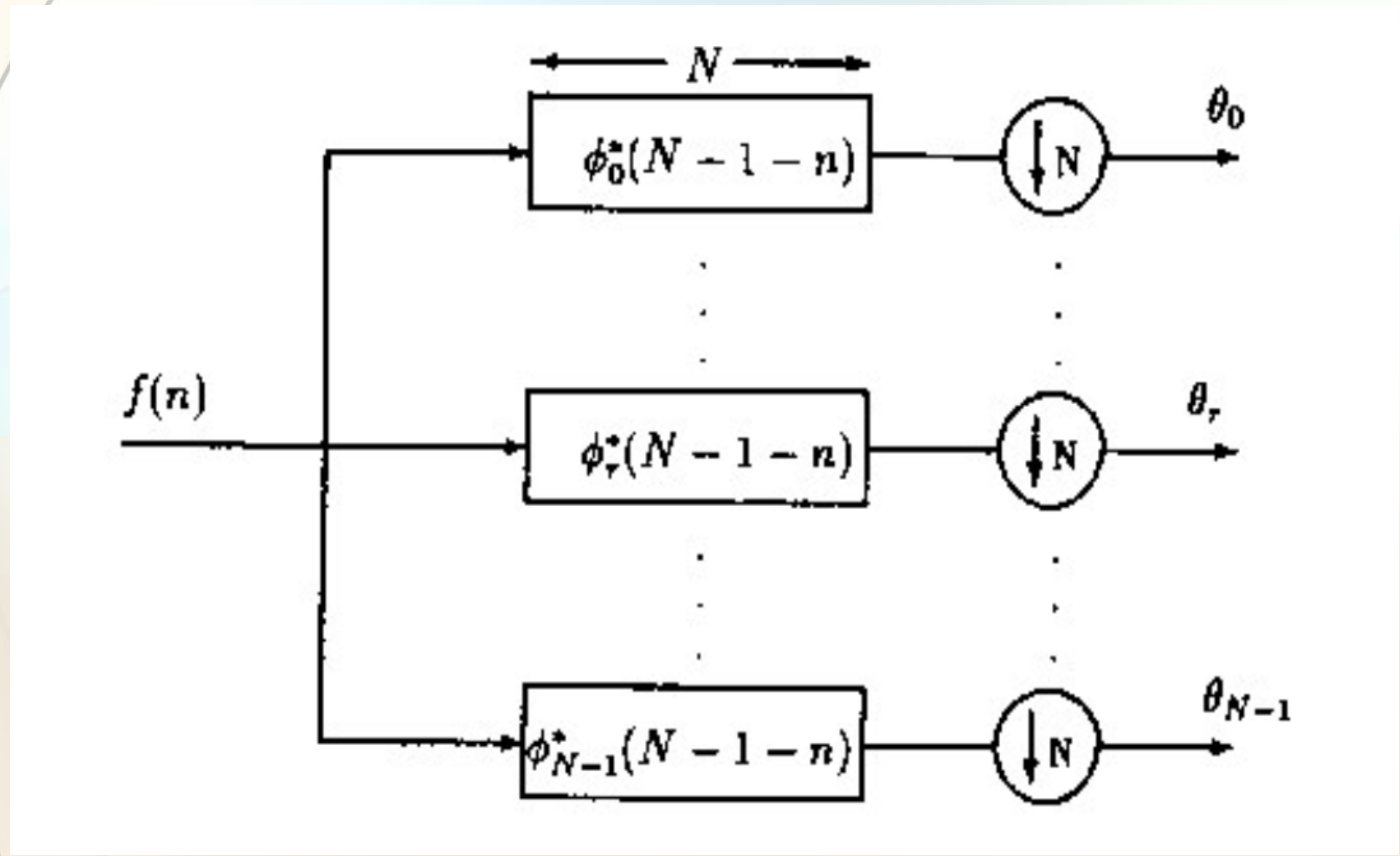
Time Frequency Localization of 31-Length Spread Spectrum KLT Codes

•Binary valued codes have constant spread in time where as multiple valued KLT codes are more spread in time

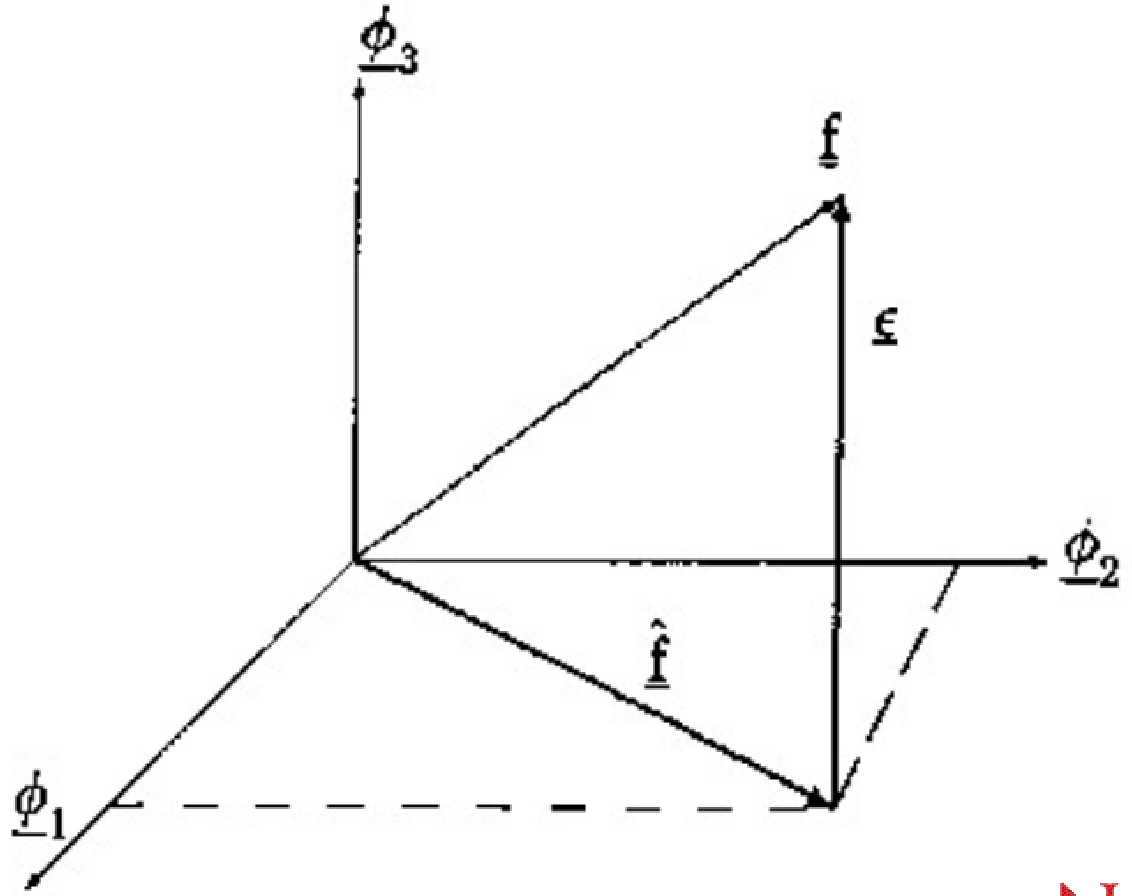
•Frequency spread for Walsh-like, Gold and KLT code sets is similar where as for Walsh codes spread is lower



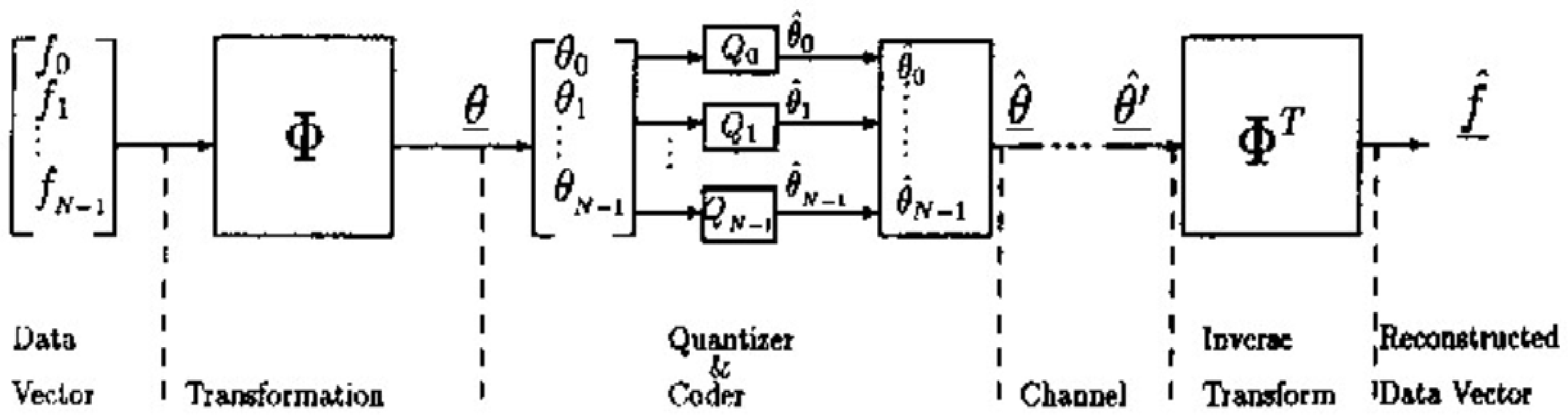
Orthonormal Spectral Analyzer as a Filter Bank



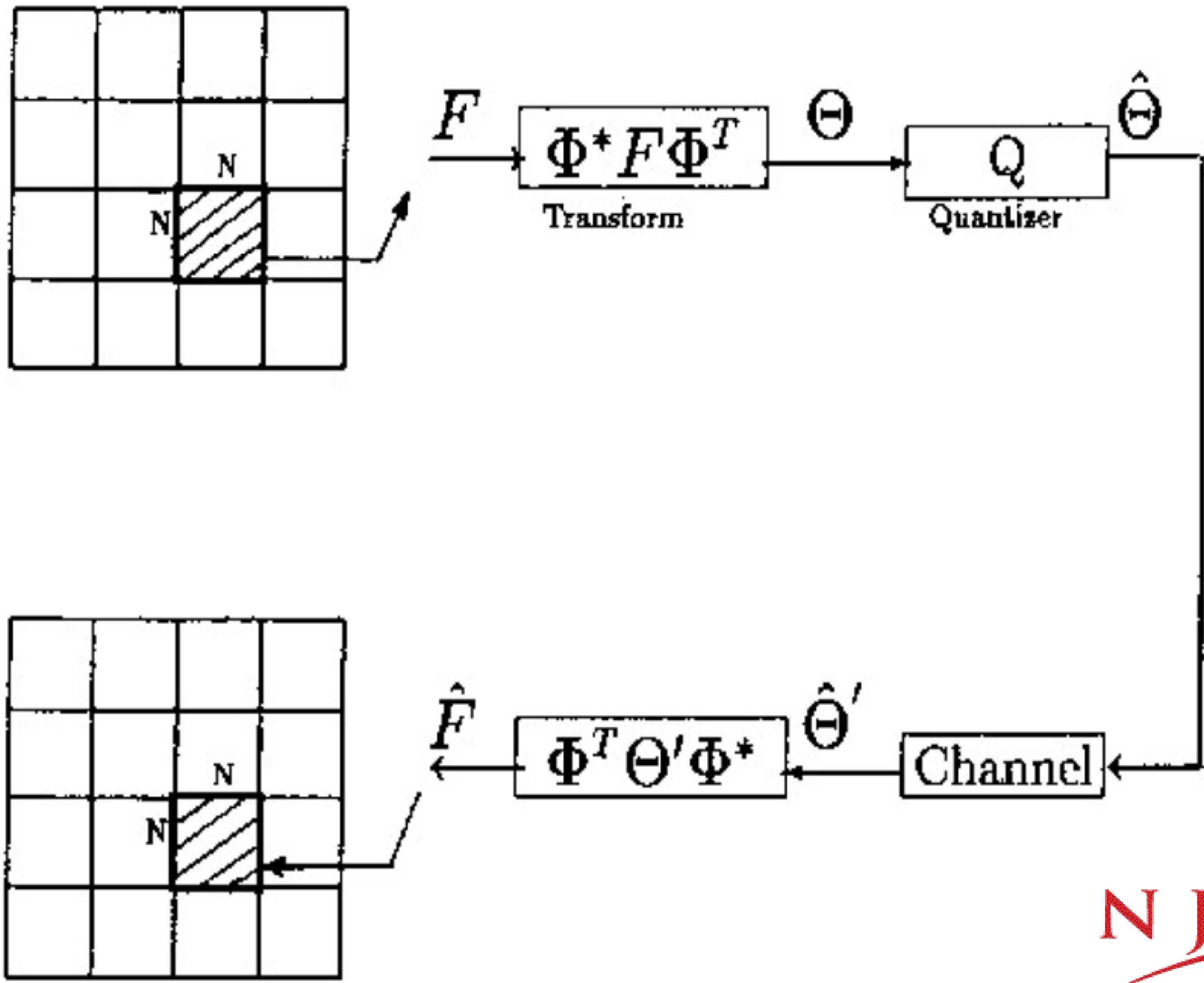
Orthogonality Principle Demonstration



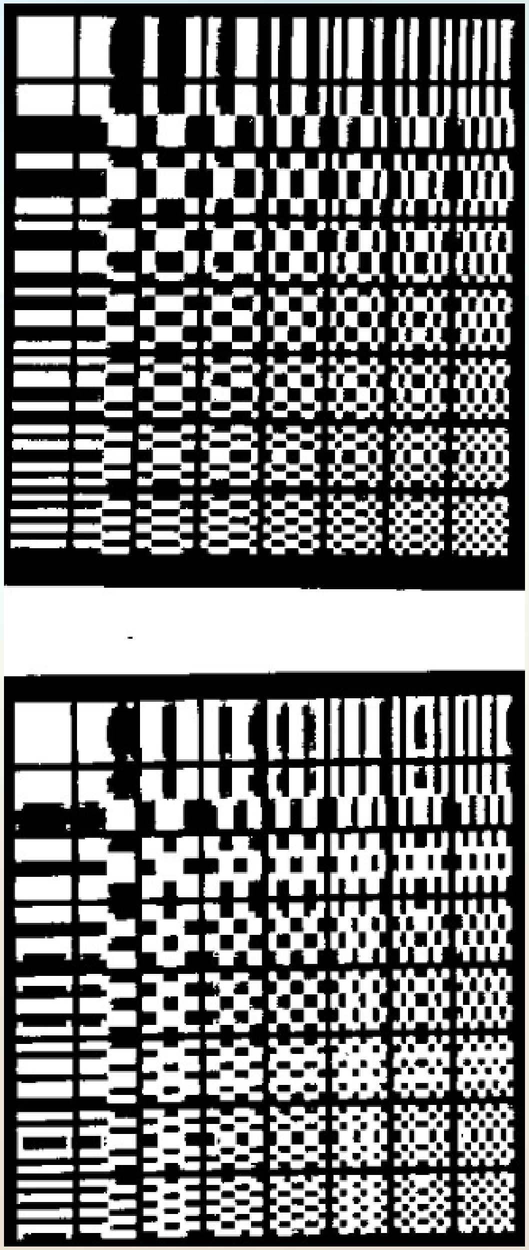
Transform Encoder / Decoder



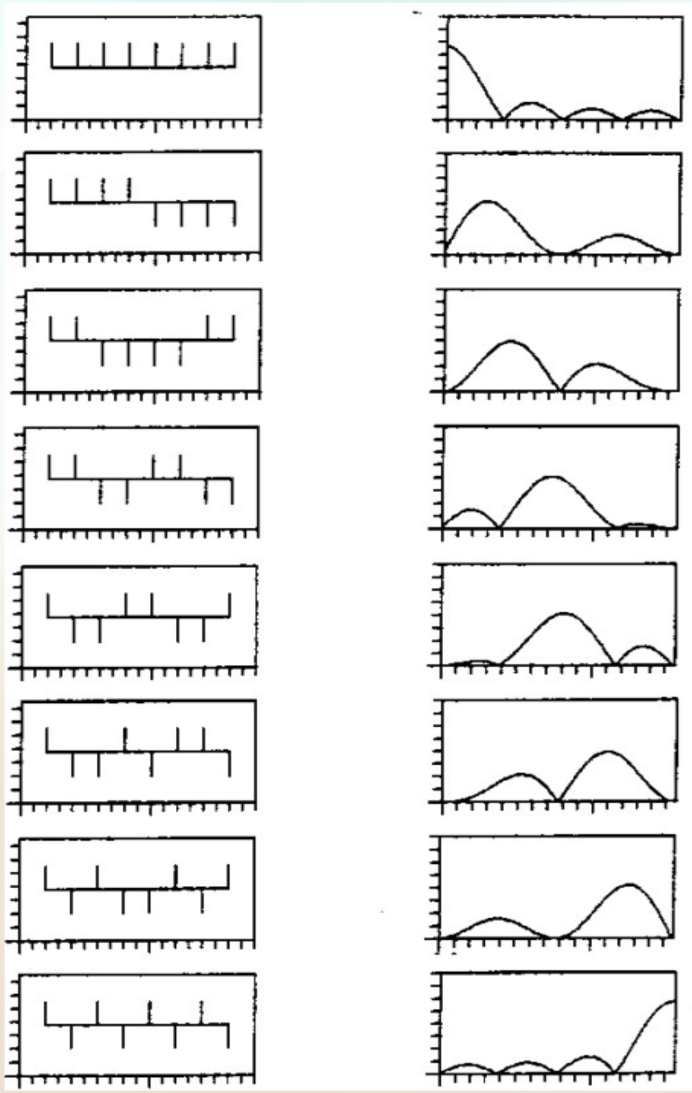
2D Transform Coding



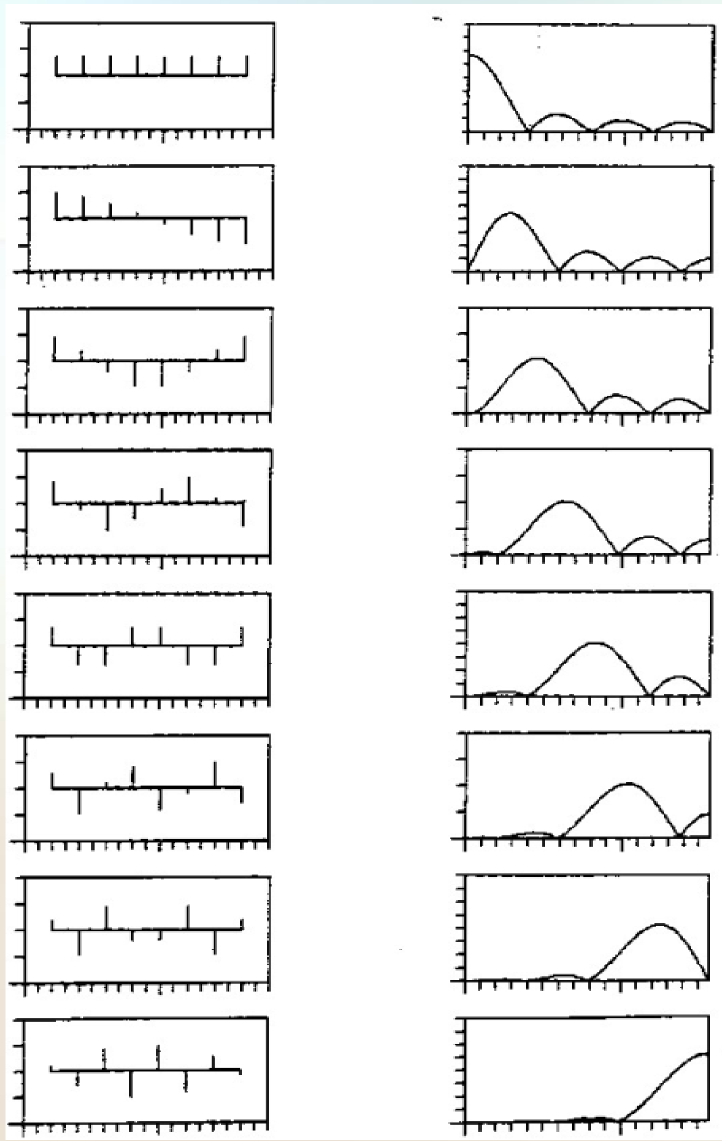
2D KLT and DCT Bases (8x8)



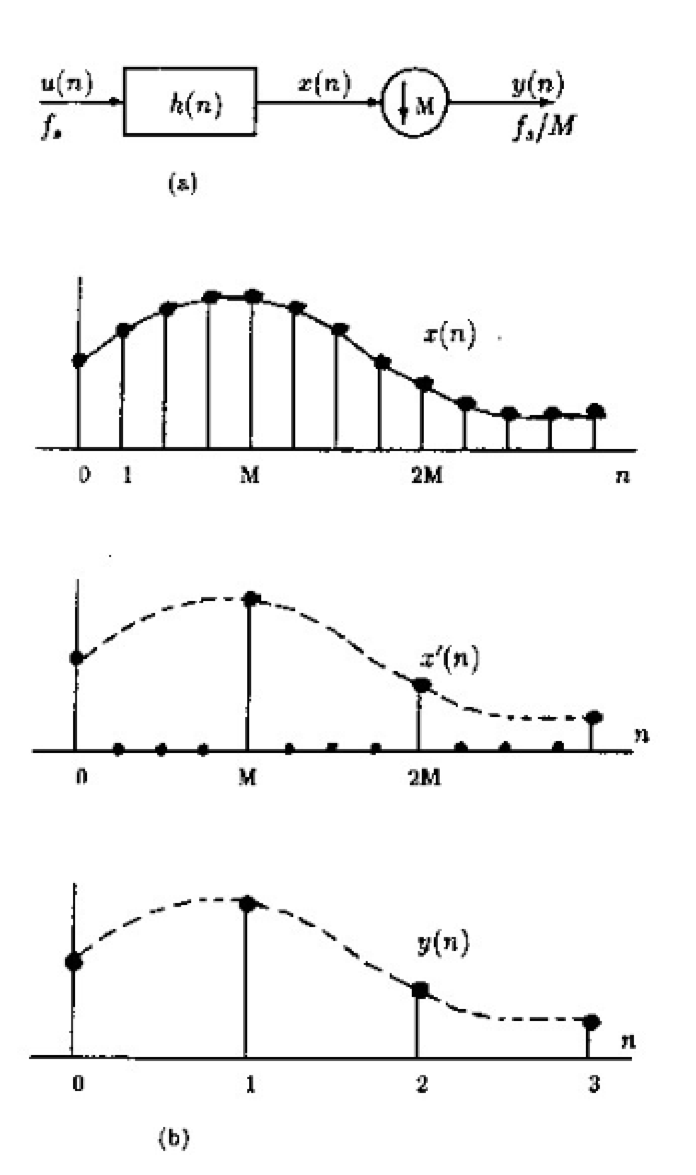
KLT Basis [AR(1), rho=0.95] in Time & Frequency (N=8)



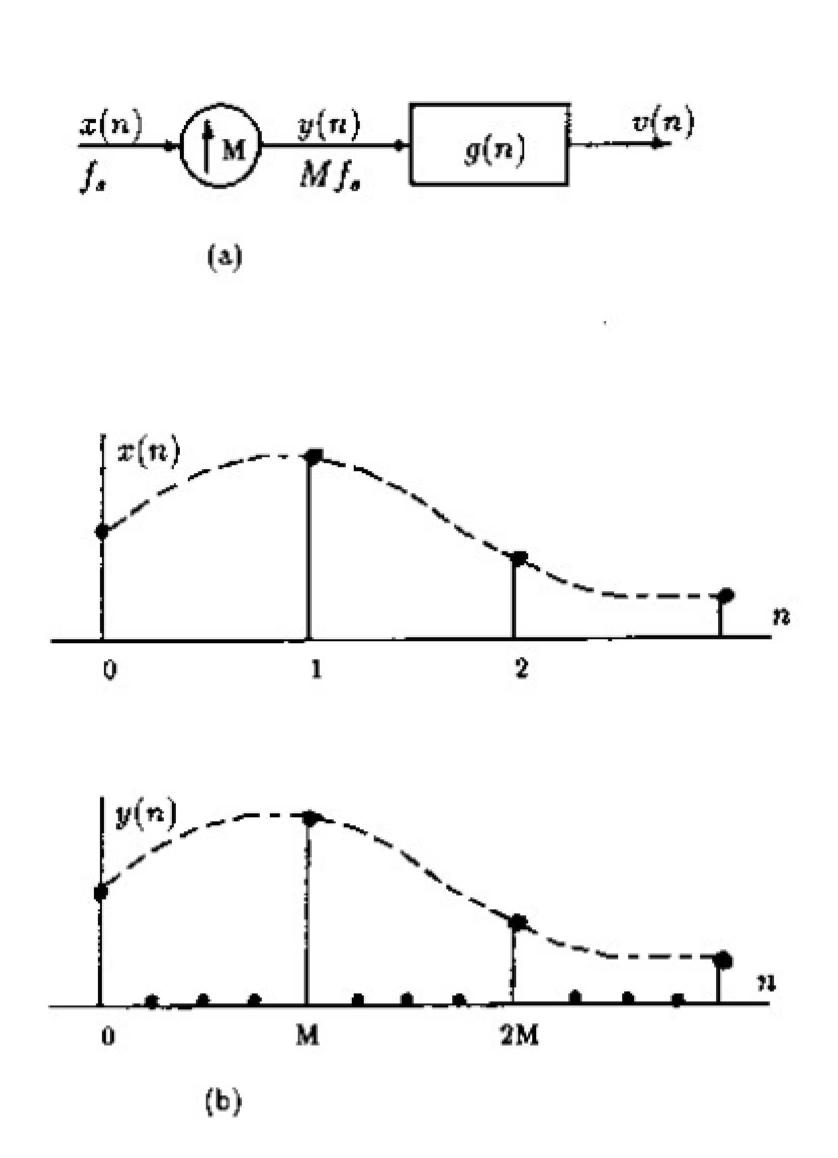
Walsh Basis (N=8)



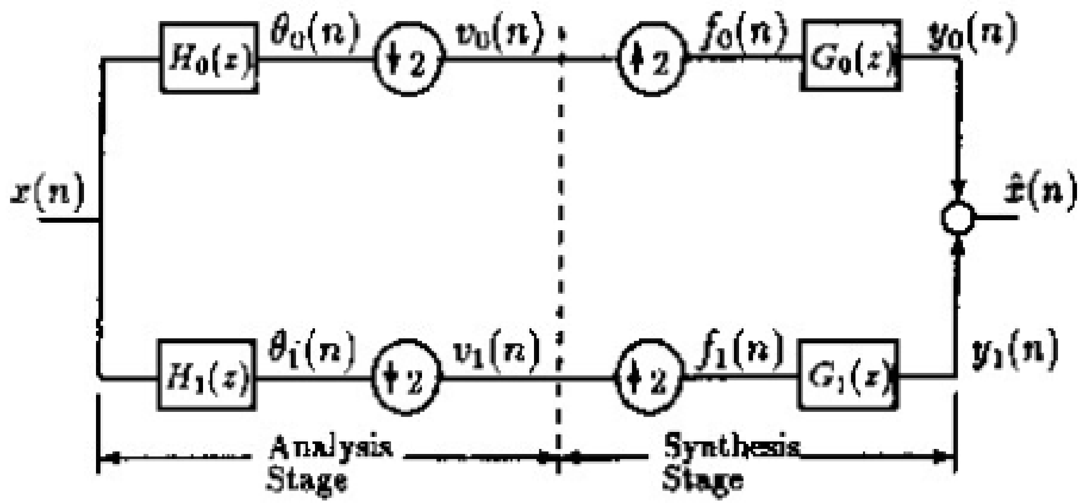
Decimation Operation



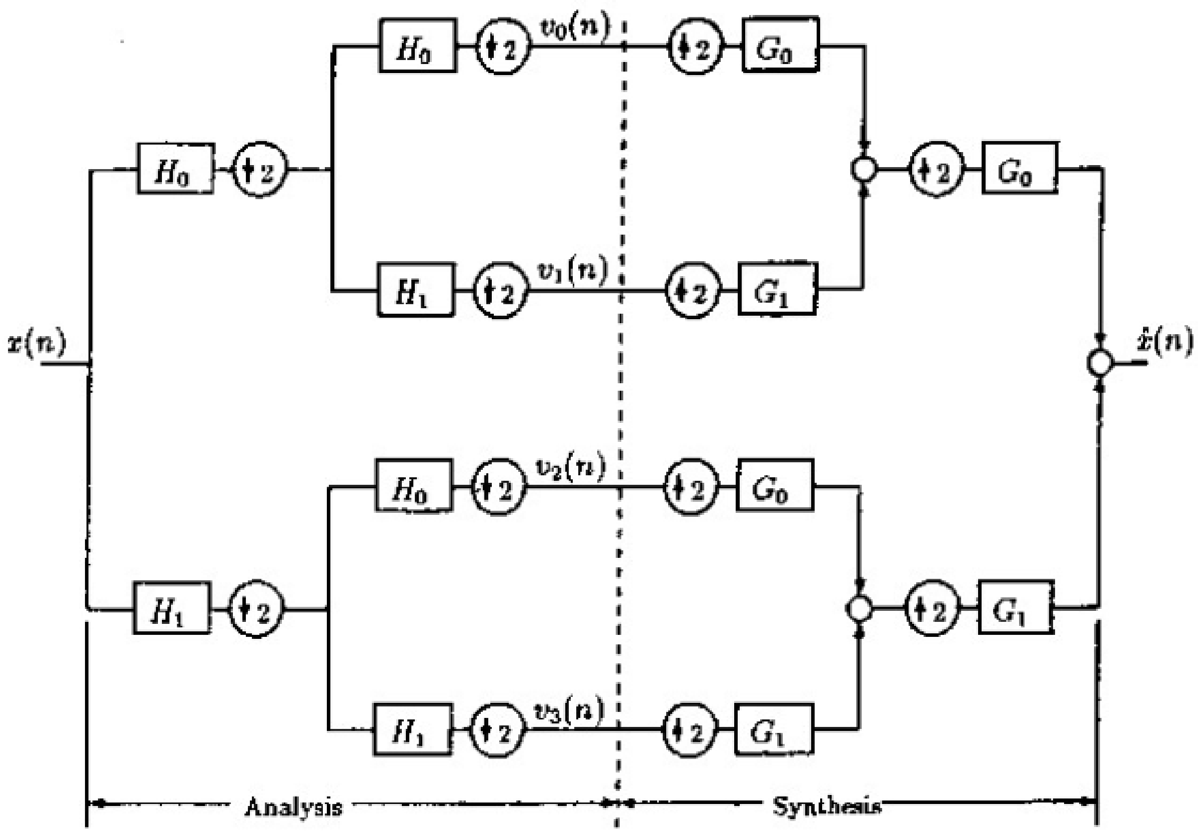
Interpolation Operation



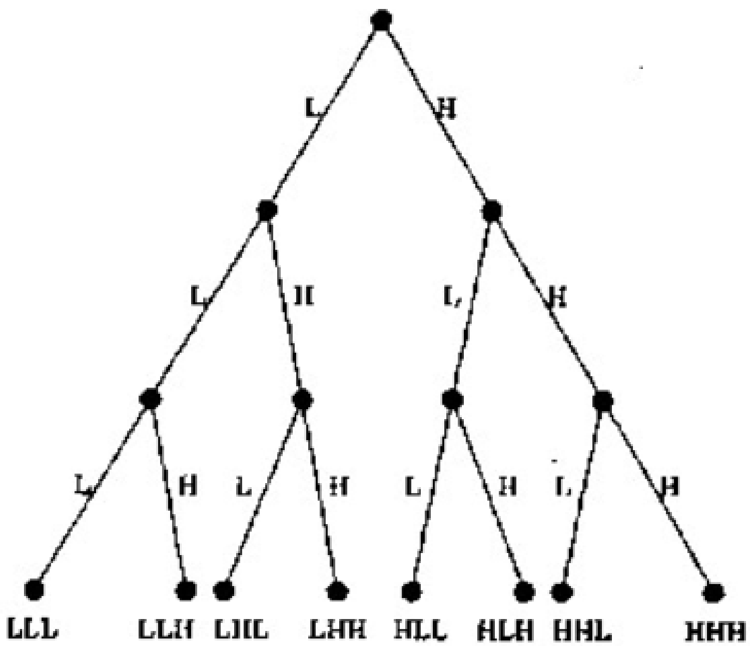
Two-Band Filter Bank

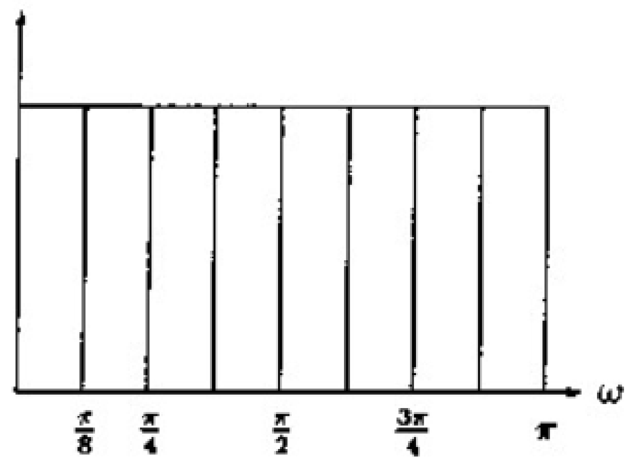


4-Band / Two-Levels Filter Bank (Subband) Tree

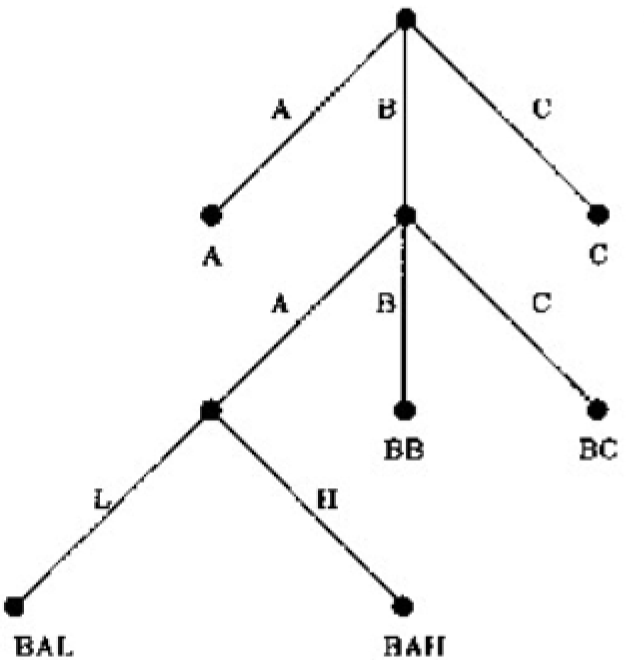


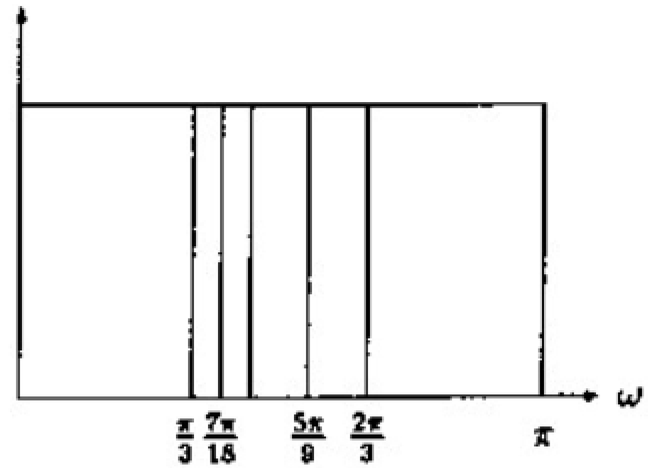
Regular Subband (Filter Bank) Tree



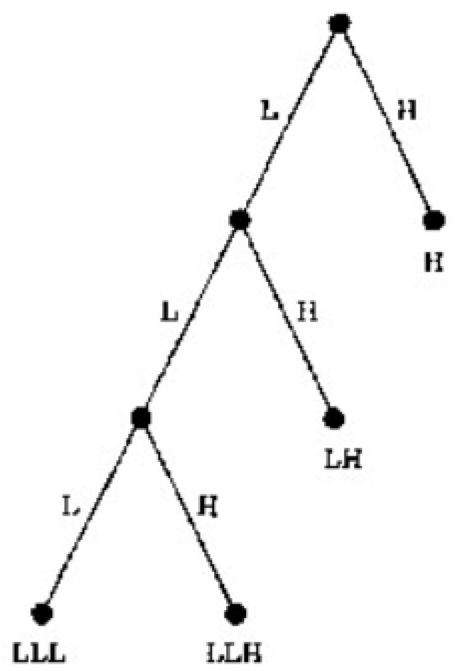


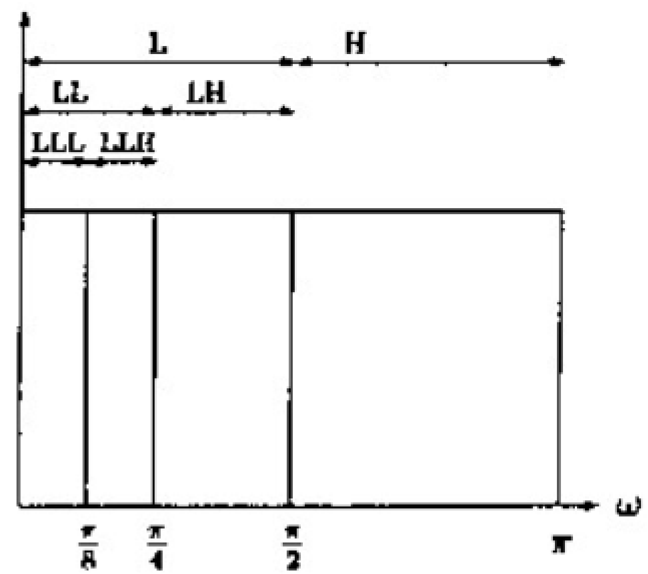
Irregular Subband Tree



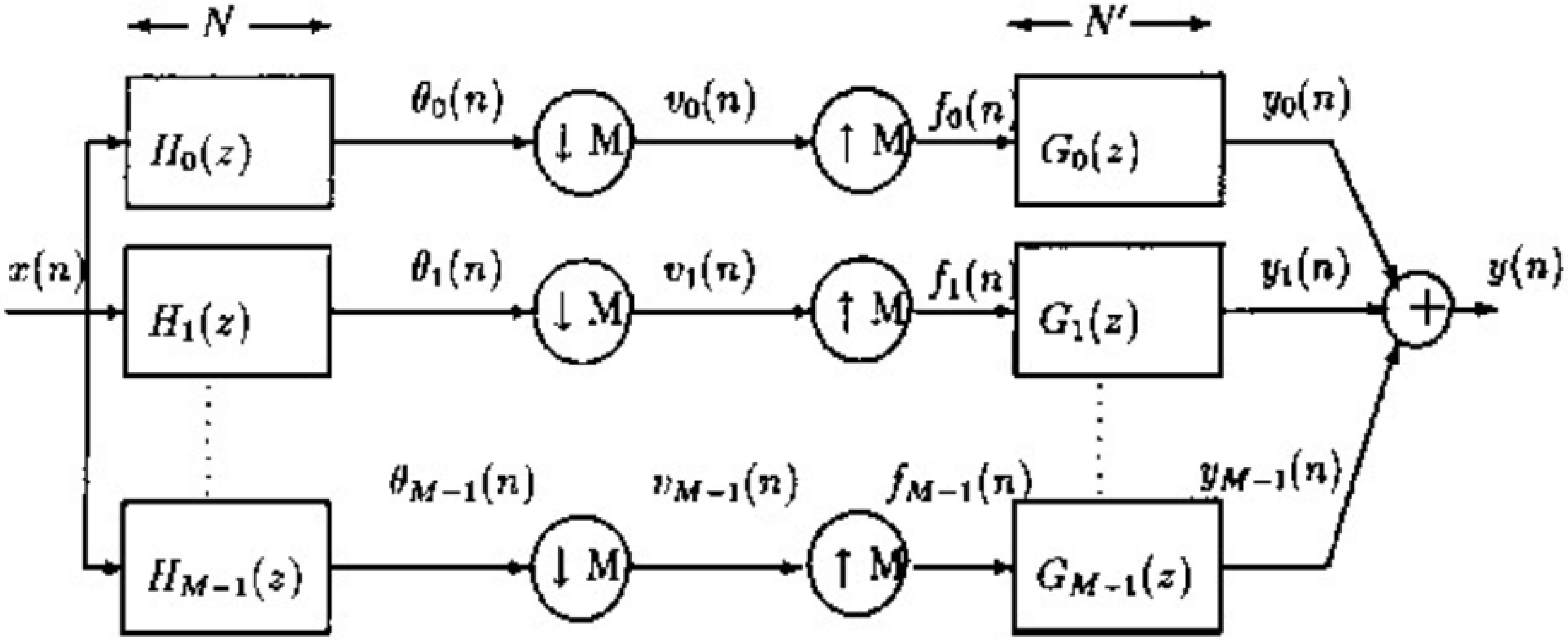


A Dyadic (Octave Band) Subband Tree

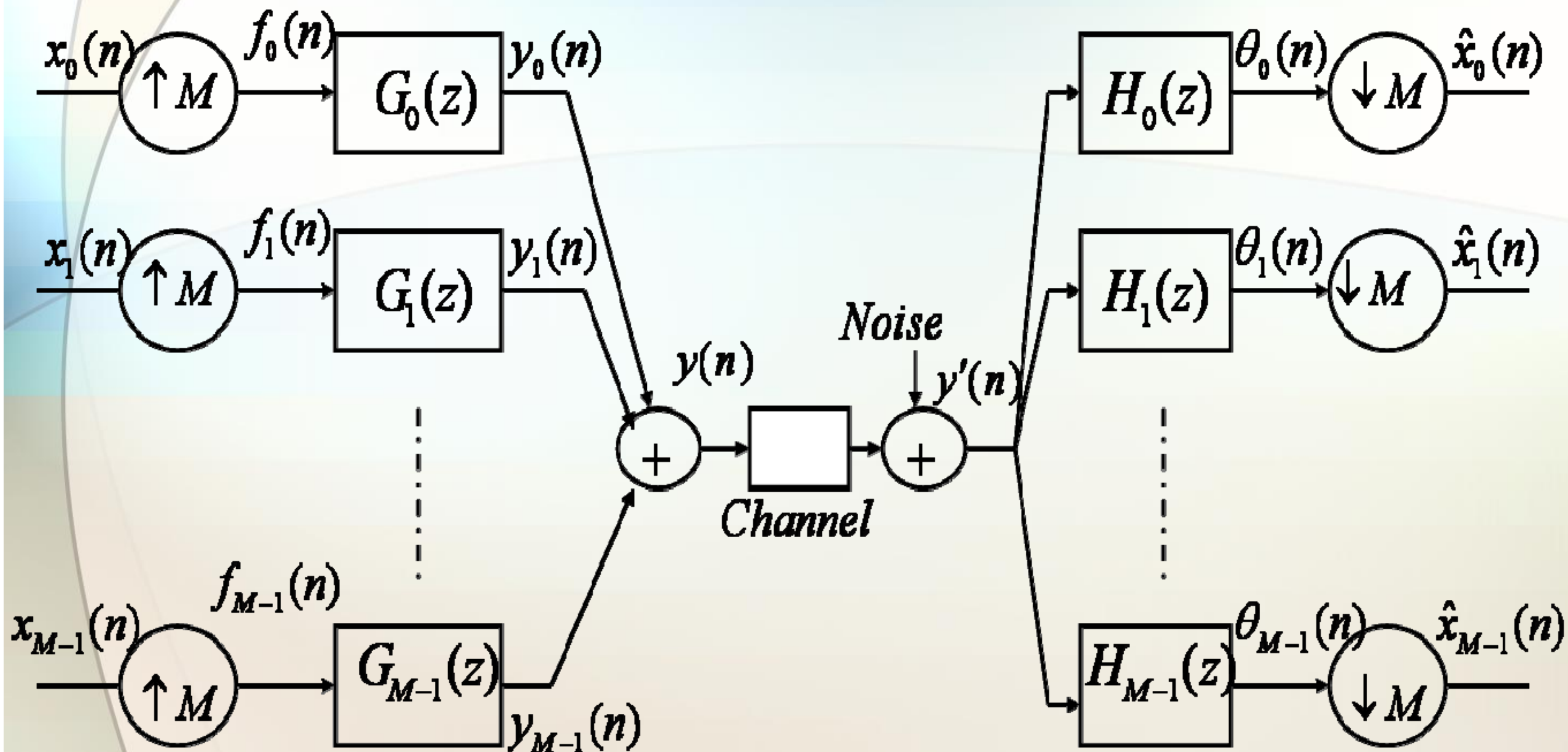




Maximally Decimated M-Band Filter Bank (Analysis/Synthesis)
Single Input Single Output (SISO)



M-Band Transmultiplexer (Synthesis/Analysis FB Configuration)
Multiple Input Multiple Output (MIMO)

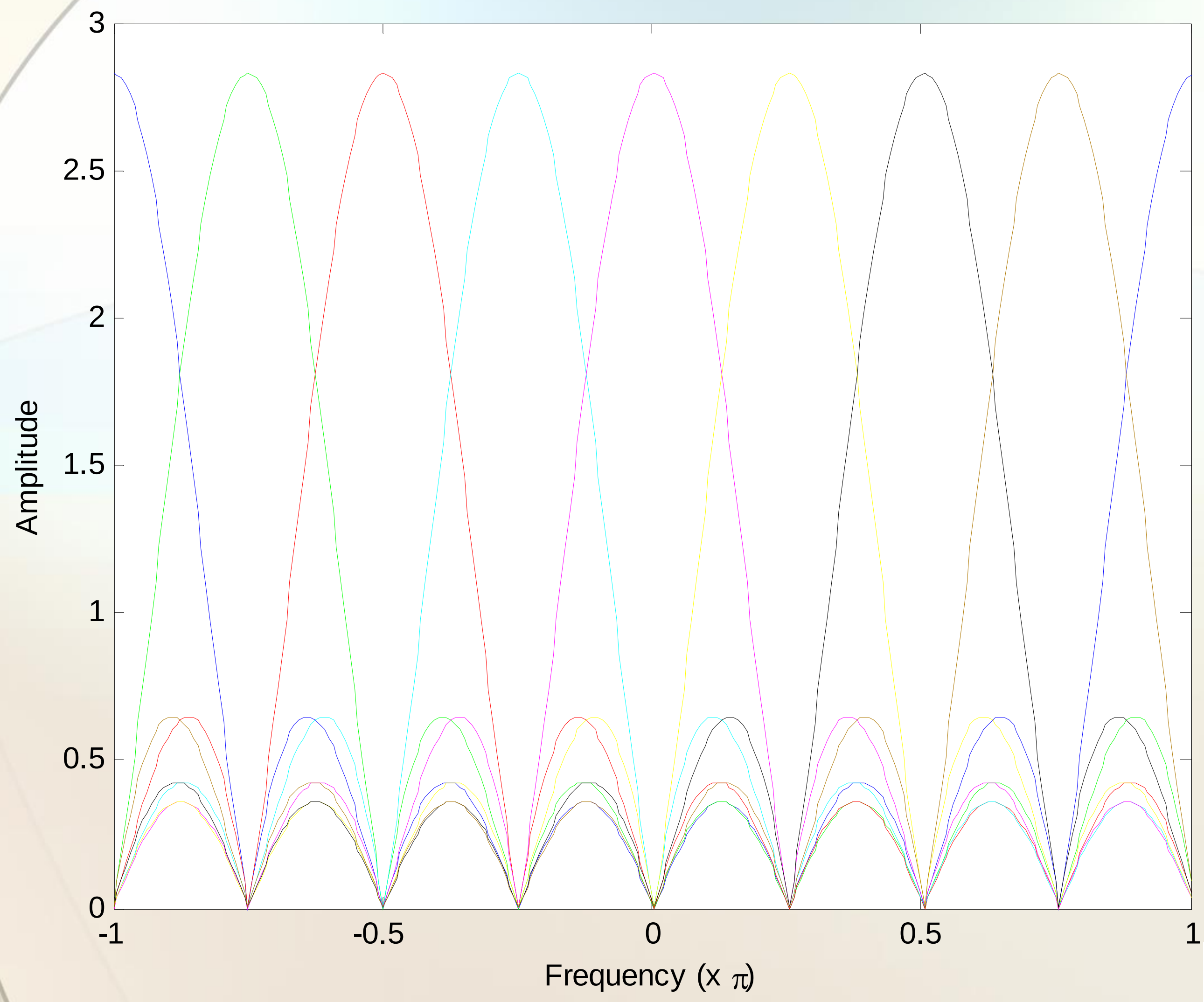


II. Discrete Fourier Transform with Linear Phase

$$e_k(n) = e^{j(2\pi/N)kn} \qquad k, n = 0, 1, ..., N-1$$

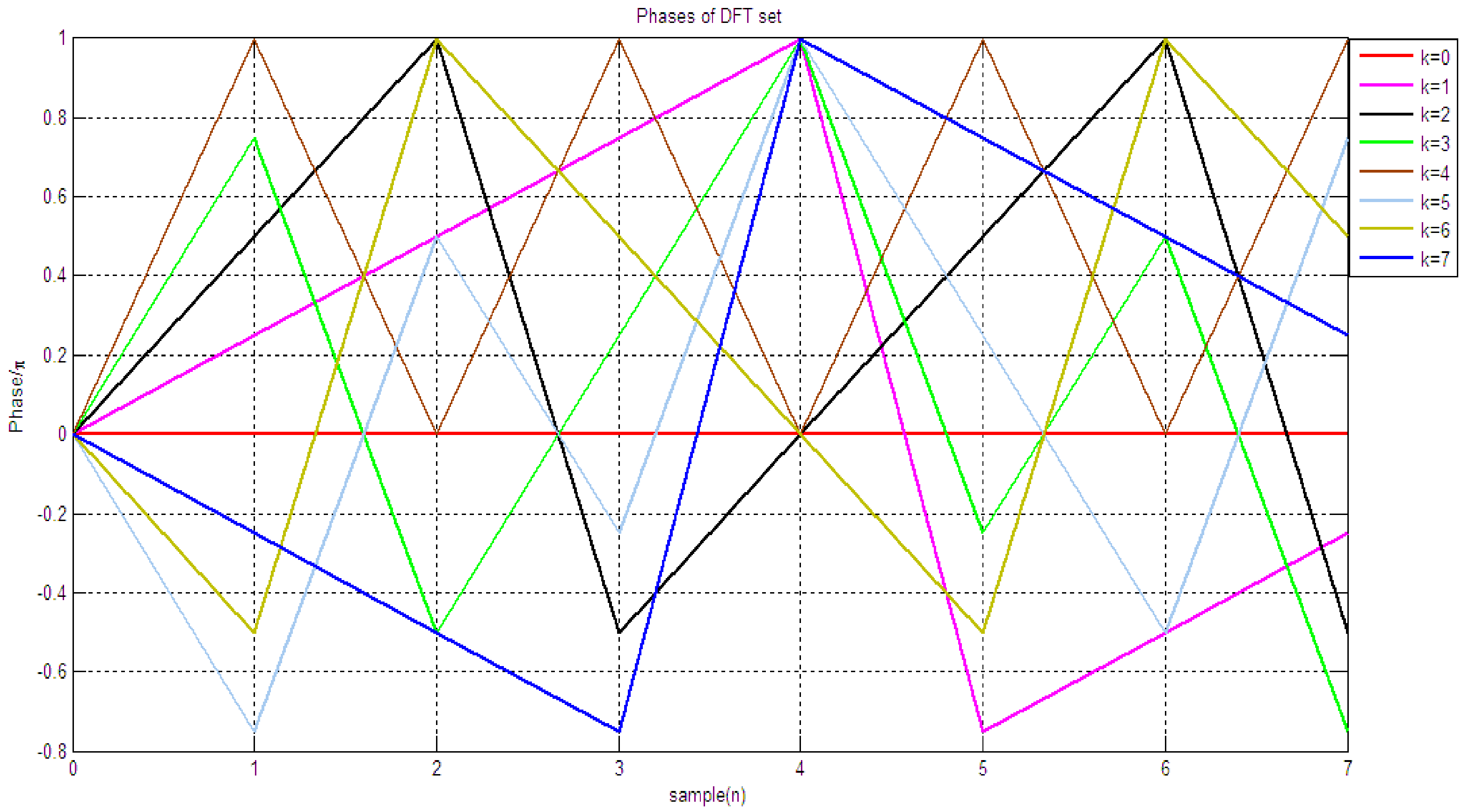
$$\frac{1}{N}\sum_{n=0}^{N-1} e_k(n) e_l^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-l)n} = \begin{cases} 1, & k-l=r=mN \\ 0, & k-l=r\neq mN \\ & m,n = \text{int}\,eger \end{cases}$$

DFT Amplitude Functions

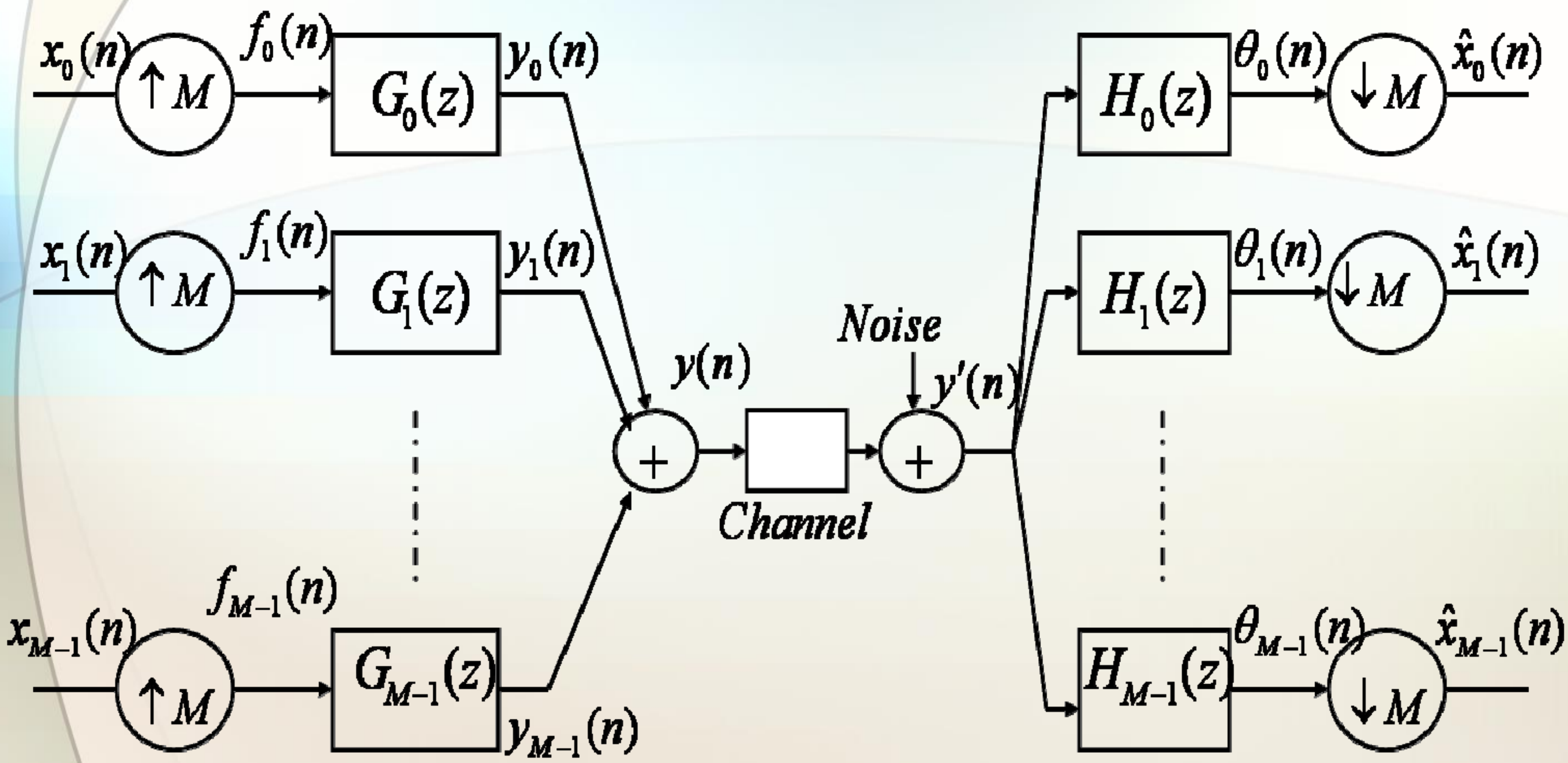


DFT Phase Functions (Linear)

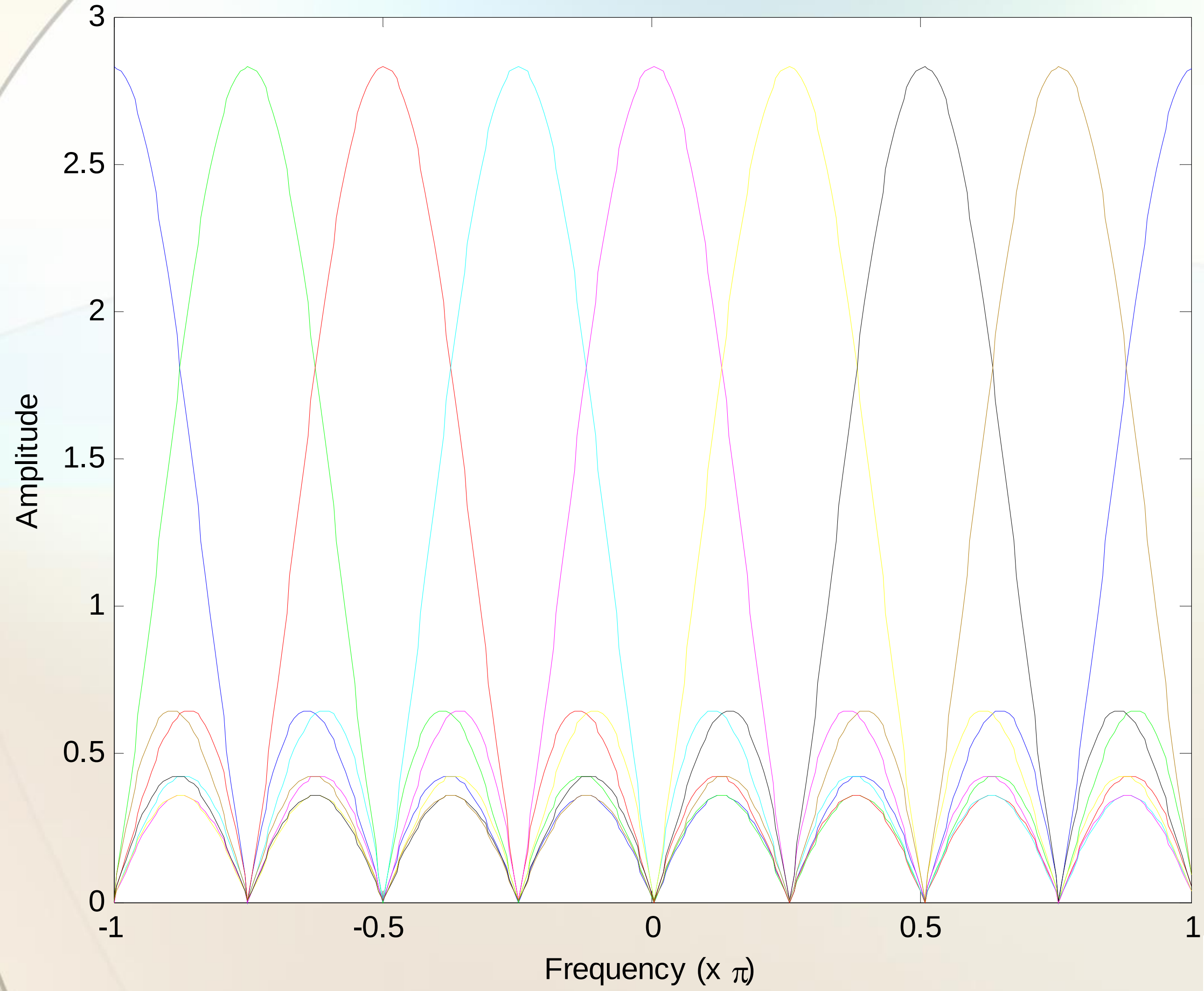
Modulo 2π



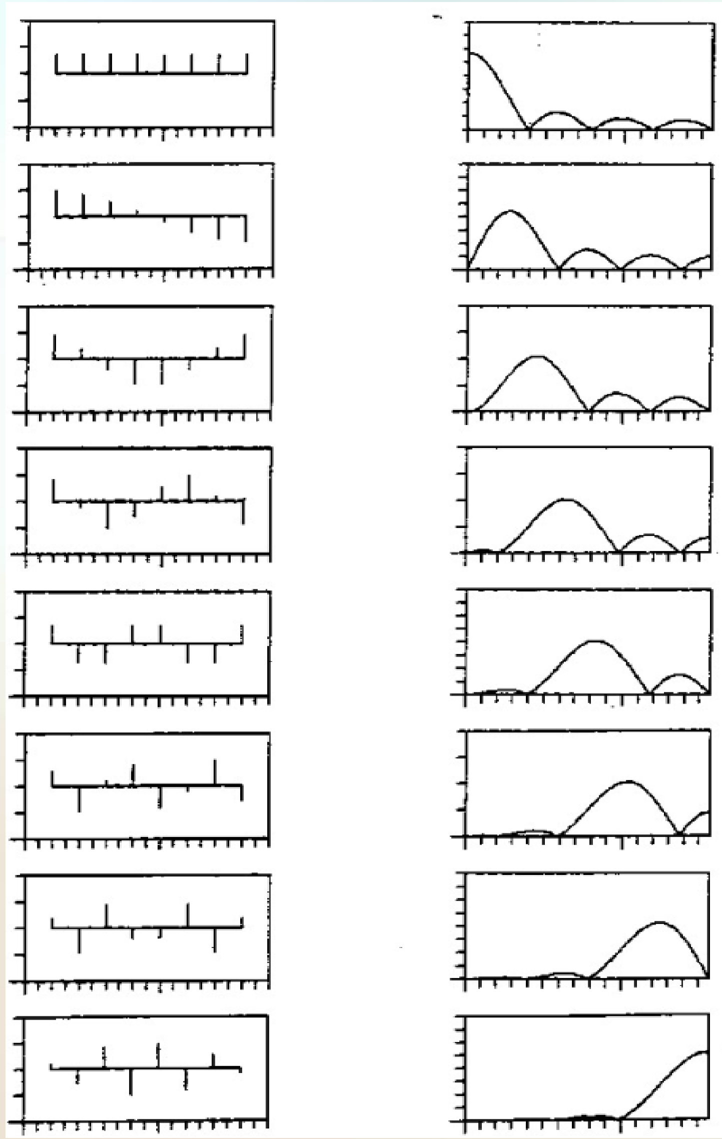
III. Orthogonal Transmultiplexer for Multicarrier Communications: OFDMA, TDMA, CDMA



DFT Amplitude Functions



Walsh Basis (N=8)



Non-linear Phase (Walsh-like) Binary Orthogonal Codes: Design and Performance

- Walsh codes are linear phase, zero mean with unique number of zero crossings in the set. DC code is part of the set
- Features are useful for source coding and not necessary for spread spectrum applications. Hence, such design restrictions are waived in Walsh-like code design.
- For *n*-length binary code, sample space consists of integer numbers up to 2^n-1. First basis function is selected by representing any integer number in the sample space in radix 2 format with [1,-1] elements

Walsh-like Codes Design

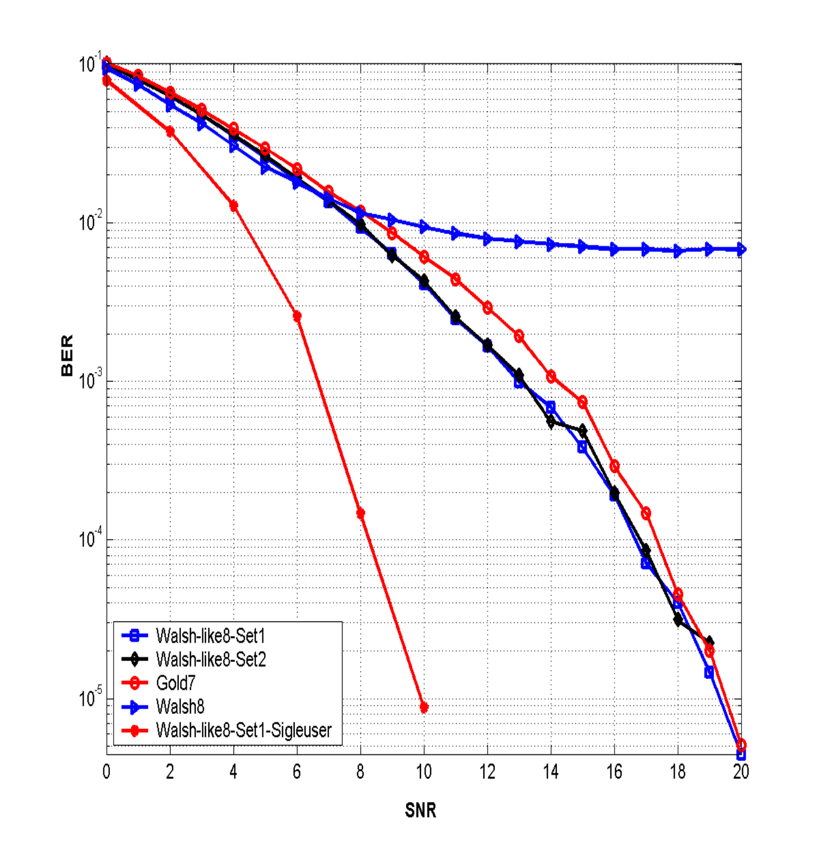
- Select the next basis function by checking the orthogonality with the first basis function and maximum normalized cross-correlation value between the pair is less than 1 for all possible delays
- Repeat this process (n -1) times to get n orthogonal codes
- A number of orthogonal sets are formed with first basis function as common basis function for all the orthogonal sets

Walsh-like Codes Design

- By choosing different integer as the first basis function, unique orthogonal sets can be formed. Number of $n \times n$ orthogonal sets with multiples of 4 as their lengths are obtained in our simulation (8,12,16,20,…)
- Complexity of this algorithm is $n.(2^n-1)$ for n-length code

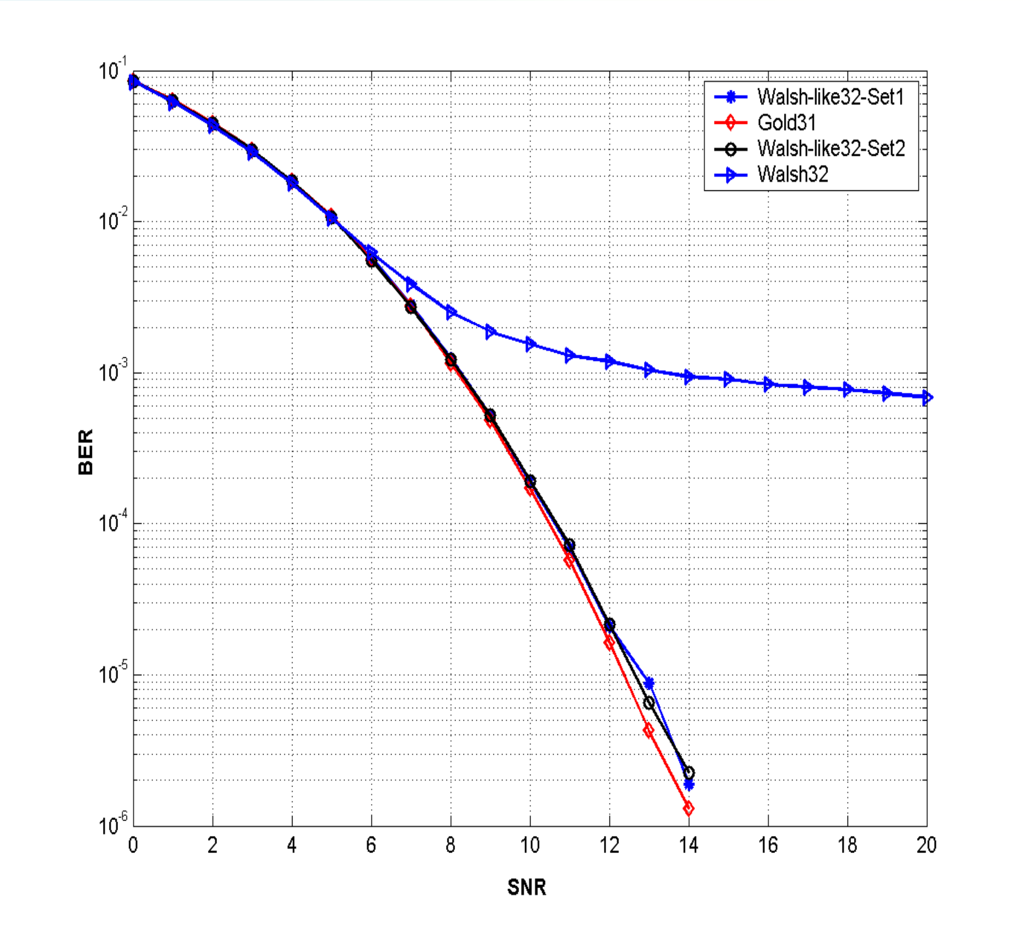
Asynchronous BER Performance Comparison of 8-Length Walsh-like Code Sets in AWG Noise(2 Users)

- BER performance of 8-length Walsh-like codes is marginally better than 7-length Gold codes and far exceeds that of Walsh codes
- Number of orthogonal code sets are available with similar performance



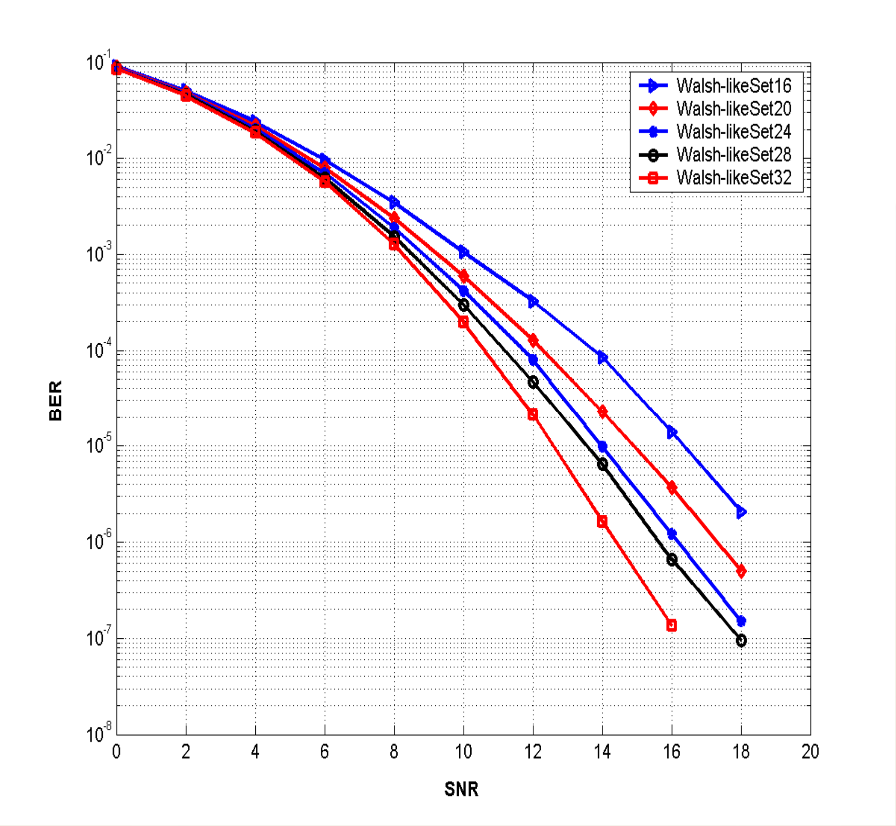
Asynchronous BER Performance Comparison of 32-Length Walsh-like Code Sets in AWG Noise(2 Users)

•BER performance of Walsh-like codes exceeds Walsh codes and closely matches with Gold codes performance at all lengths of the code



Async. BER Performance of 16,20,24,28,32 Length Walsh-like Codes in AWG Noise (2 Users)

- Number of Walsh-like code sets are available for all lengths that are multiples of 4



NJIT

Rayleigh Flat - Slow Fading Channel Description

- Multipath reflections of the symbol occur in the same symbol interval. This implies coherent bandwidth of the channel is greater than the symbol bandwidth (Flat fading)
- Channel conditions are assumed to remain same during symbol interval (Slow fading)

Rayleigh Flat - Slow Fading Channel Description

- Amplitude of the received signal modeled as

$$y(t) = h(t) * s(t) + n(t),$$

$s(t)$ transmitted signal, $n(t)$ AWG noise, $h(t)$ channel impulse response and $y(t)$ received signal

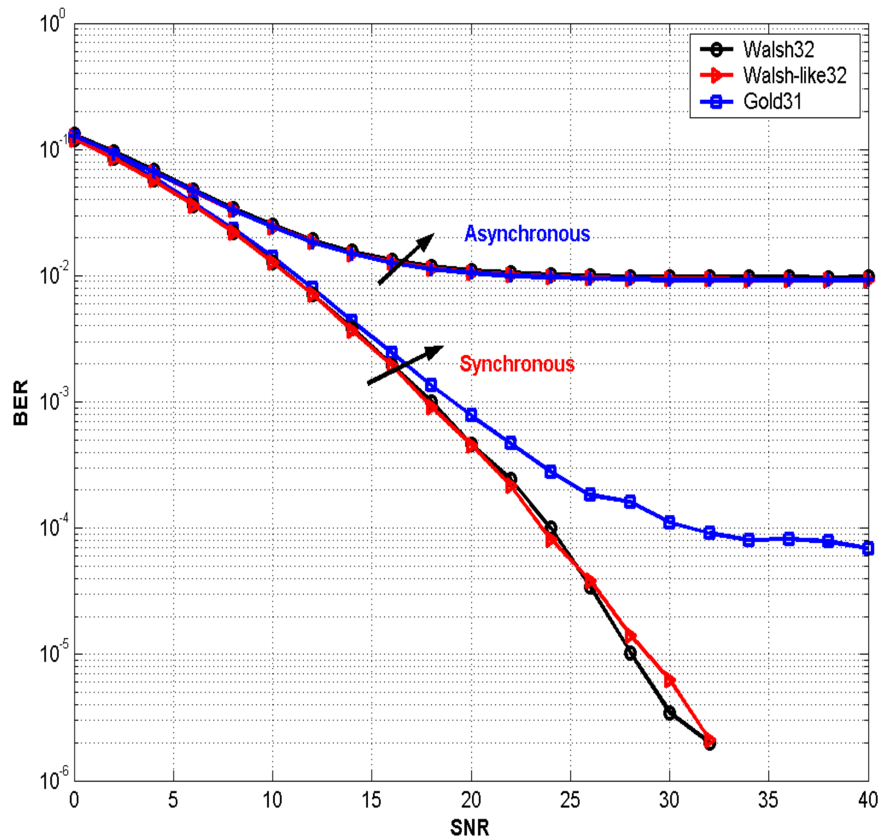
Rayleigh Flat - Slow Fading Channel Description

- For flat fading channel, *h(t)* – single tap filter with zero delay
- *h(t)* WSS complex Gaussian waveform with zero mean and unity variance whose amplitude varies as Rayleigh PDF variable
- Fading channel modeled separately for each user in uplink

NJIT

Sync / Async BER Performance Comparison for Length-32 Walsh-like Codes (2 Users)-Rayleigh Channel

- Performance of orthogonal Walsh and Walsh-like codes is similar in all Rayleigh flat fading conditions
- Performance of non-orthogonal Gold codes is poor in synchronous conditions



NJIT

Multiple Level Code : Design and Performance

- For p level coding, sample space is p^n for an n-length code. Represent numbers in sample space using radix p elements. Map radix elements into corresponding PAM level chip amplitudes
- 4 level representation requires radix 4 elements (0,1,2,3) and PAM chip levels {-3, -1, 1, 3}. Weights of the individual elements for 8-length code are $\{ 4^7, 4^6, \ldots, 4^1, 4^0 \}$

NJIT

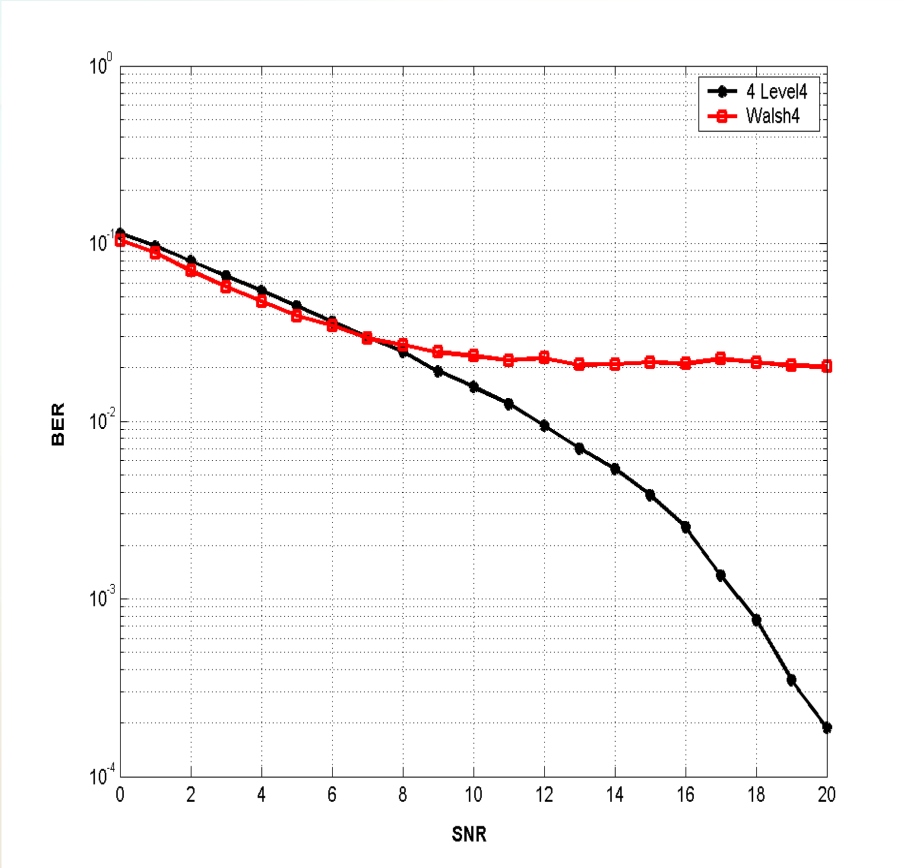
Multiple Level Code : Design and Performance

- As an example, number 125 in radix 4 is represented as {0,0,0,0,1,3,3,1}. After PAM mapping, code becomes {-3, -3, -3, -3, -1, 3, 3, -1}
- Number of unique orthogonal code sets are obtained by brute force search method in the sample space with code features similar to Walsh-like code sets
- In the search algorithm, additional constraint of same norm for all basis functions within the set is also imposed

Asynchronous BER performance Comparison of 4-Level , 4-Length Codes in AWG Noise(2 Users)

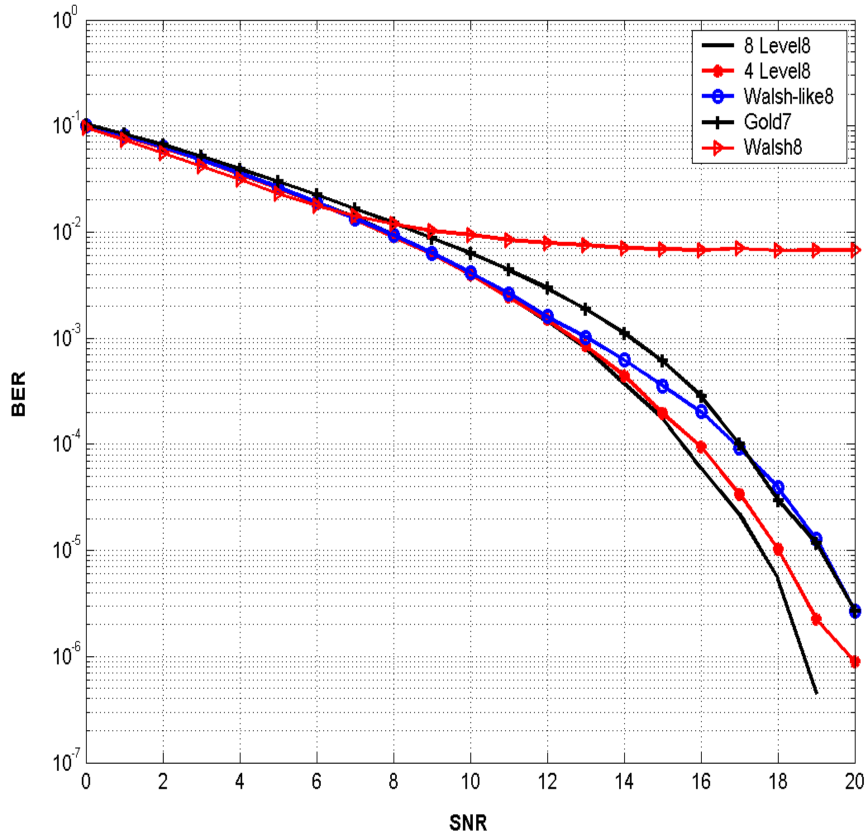
- BER performance of 2- level, 4-length orthogonal codes (Walsh) is poor where as 4-level codes give good performance
- Sample 4-level, 4-length code

$$\begin{bmatrix} -3 & -1 & 1 & 3 \\ -1 & 3 & -3 & 1 \\ -1 & 3 & 3 & -1 \\ 3 & 1 & 1 & 3 \end{bmatrix}$$



Asynchronous BER Performance Comparison of Multiple Level, 8-Length Codes in AWG Noise (2 Users)

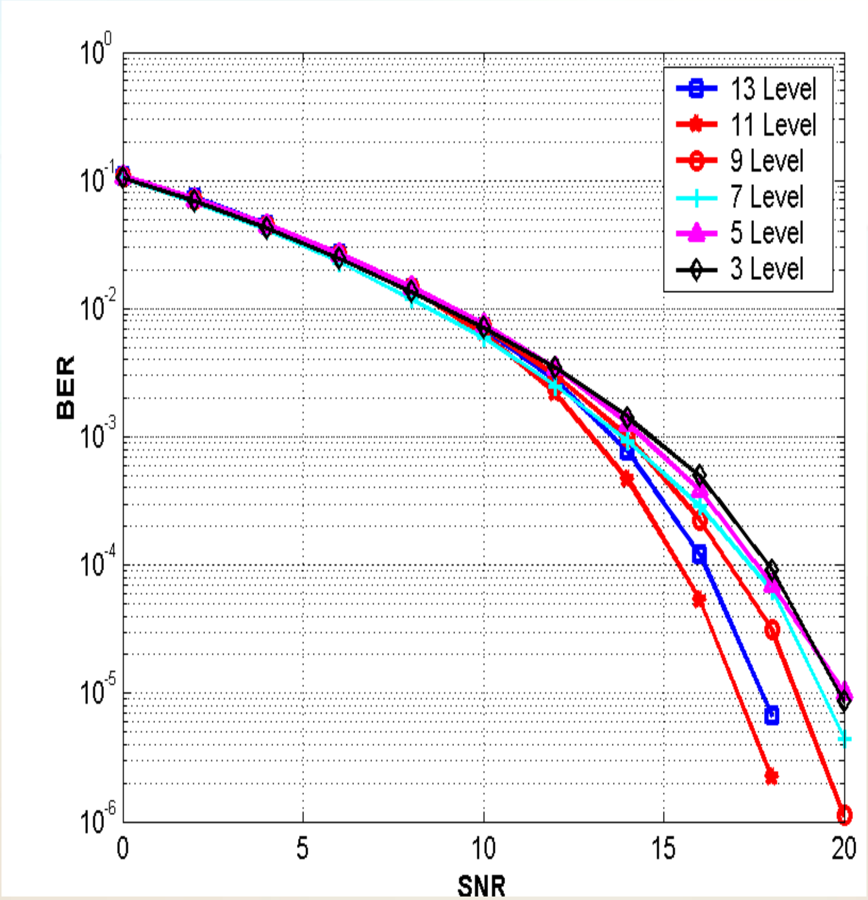
•BER performance improves as the number of coding levels increase



Multiple Level, 6-Length Codes

3-Level Codes Basis Elements $\{-1,0,1\}$ Norm^2-4	5-Level Codes Basis Elements $\{-2,-1,0,1,2\}$ Norm^2-10	7-Level Codes Basis Elements $\{-3,-2,-1,0,1,2,3\}$ Norm^2-20	9-Level Codes Basis Elements $\{-4,-3,-2,-1,0,1,2,3,4\}$ Norm^2-26	11-Level Codes Basis Elements $\{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$ Norm^2-50	13-Level Codes Basis Elements $\{-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6\}$ Norm^2-50
22	182	2858	9832	17968	95570
396	8378	51974	163032	143082	2433484
404	8476	60218	218158	847144	2469152
524	10660	73438	272578	847218	2779564
528	10714	77530	310372	885134	3144890
670	12932	103798	372786	885216	3356242

Asynchronous BER Performance of Multiple Level, 6-Length Codes in AWG Noise (2 Users)



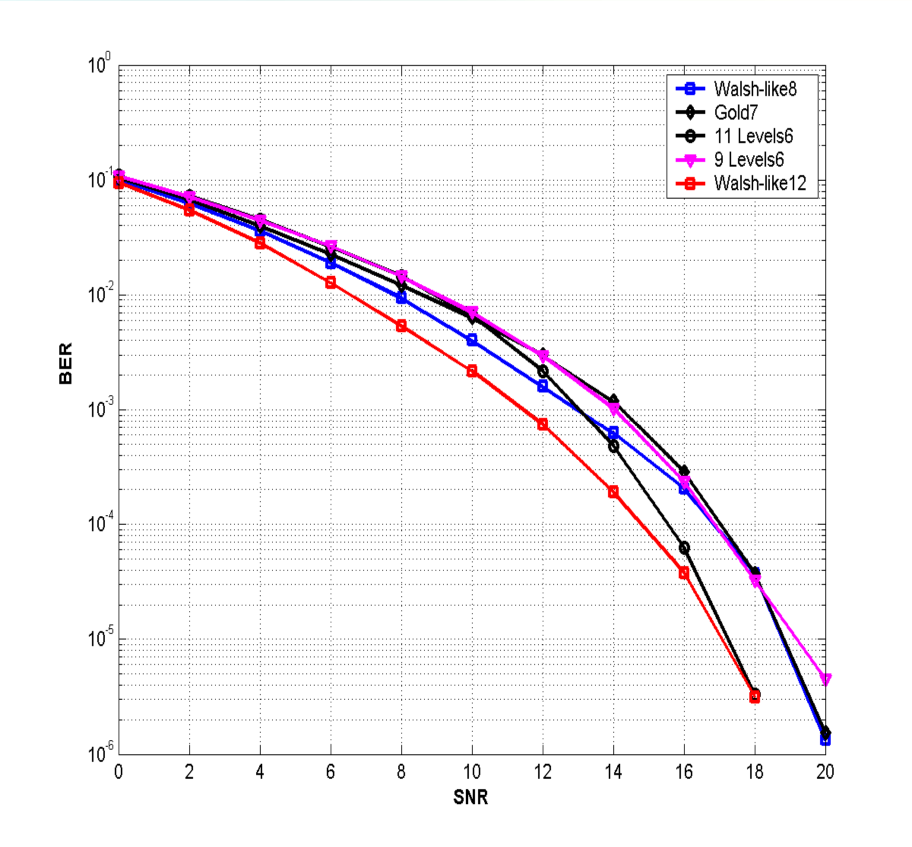
•BER performance improves as the number of levels increase upto a certain level

Normalized Cross Correlation Metrics for Multiple Level 6-Length Codes

Parameter	3 Level	5 Level	7 Level	9 Level	11 Level	13 Level
Max Even Correlation	.75	.7	0.8	.73	.64	.72
Max Odd Correlation	.75	.8	.65	.73	.64	.70
Max Aperiodic Correlation	.75	.7	0.7	.73	.64	.70
Sum of Square of Even Correlations	11	13.1	10	13	14	13
Sum of Square of Odd Correlations	12	13.5	13	13	13	14
Sum of Square of Aperiodic Correlations	12	13.3	11	13	14	13

Comparison of Multiple Level and Binary Level Codes

•Shorter length codes with higher chip levels perform as good as longer length codes with binary spreading levels



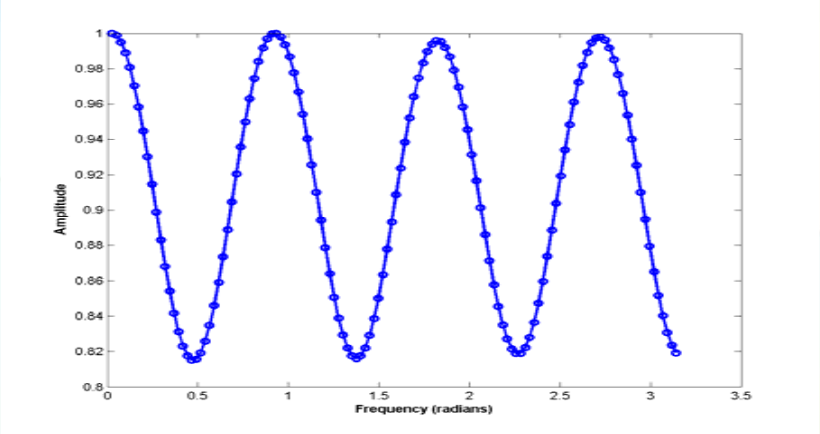
Spread Spectrum KLT Codes

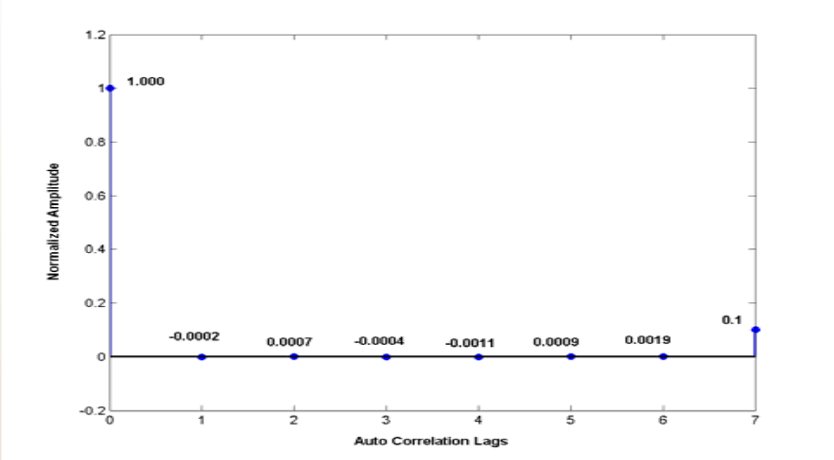
- Integer binary and multiple level spread spectrum codes are obtained by brute force search method
- Karhunen-Loeve Transform (KLT) based analytical method is used to generate multiple value spread spectrum codes for a given covariance or power spectral density (PSD) function
- PSD function can be modeled using AR, ARMA methods, giving many solutions with variable code lengths

NJIT

Example: Power Spectral Density and 8-Length Auto Correlation sequence

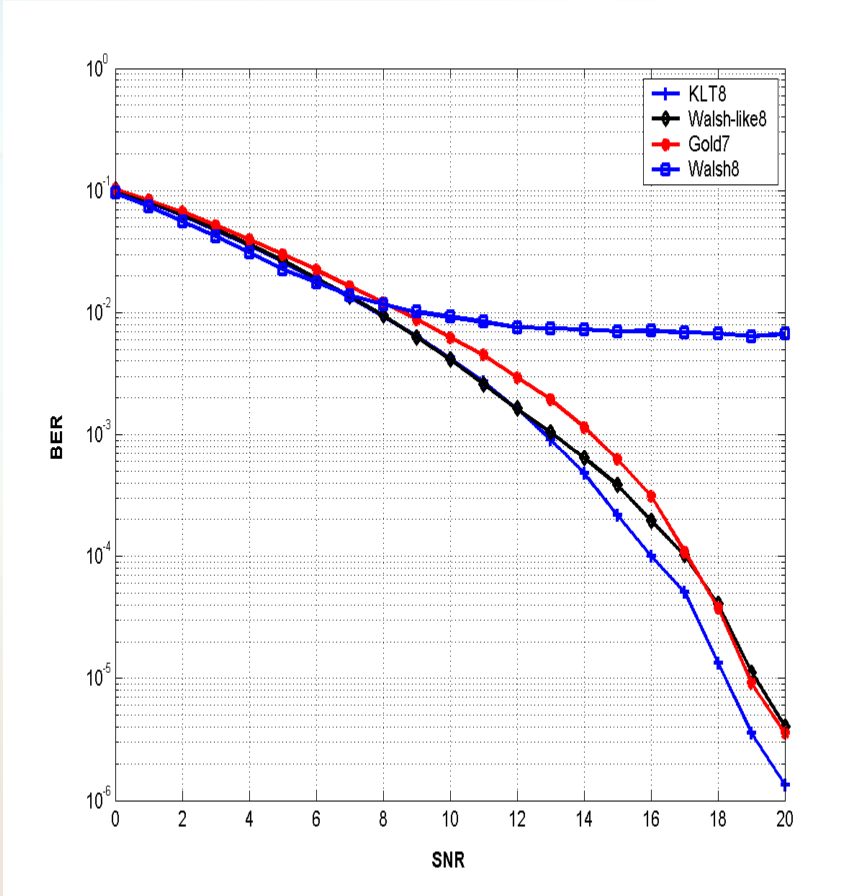
- Power spectral density is first modeled as AR sequence
- Eigen vectors generated from covariance matrix are used as spread spectrum codes





Asynchronous 2 User BER Performance of 8-Length Spread spectrum KLT Codes in AWG Noise

•BER performance of multiple valued 8-length KLT codes is better than binary valued codes.

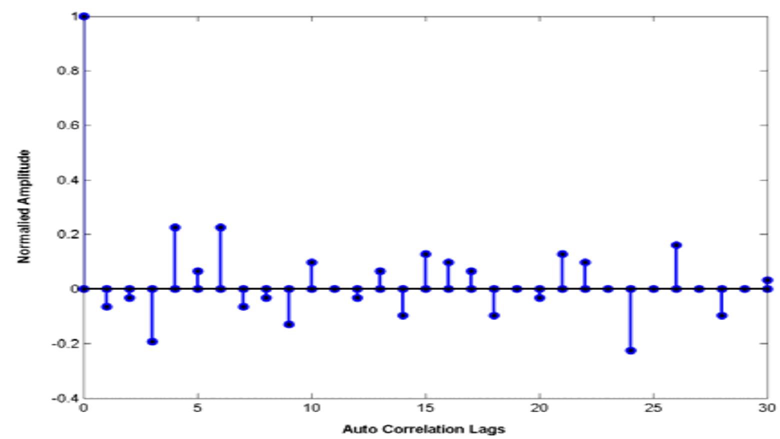


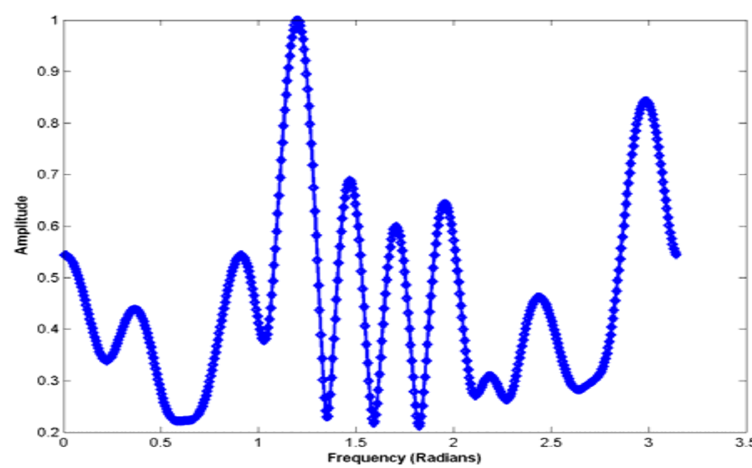
Typical 31-Length Gold Code and its Auto Correlation Sequence

$$\begin{bmatrix} -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & \end{bmatrix}$$

$$\begin{bmatrix} 1.0000 & -0.0645 & -0.0323 & -0.1935 & 0.2258 & 0.0645 & 0.2258 & -0.0645 \\ -0.0323 & -0.1290 & 0.0968 & 0.0000 & -0.0323 & 0.0645 & -0.0968 & 0.1290 \\ 0.0968 & 0.0645 & -0.0968 & 0.0000 & -0.0323 & 0.1290 & 0.0968 & 0 \\ -0.2258 & 0 & 0.1613 & 0 & -0.0968 & 0.\ 0000 & 0.0323 & \end{bmatrix}$$

31-Length Spread Spectrum KLT Auto Correlation Sequence and Power Spectrum

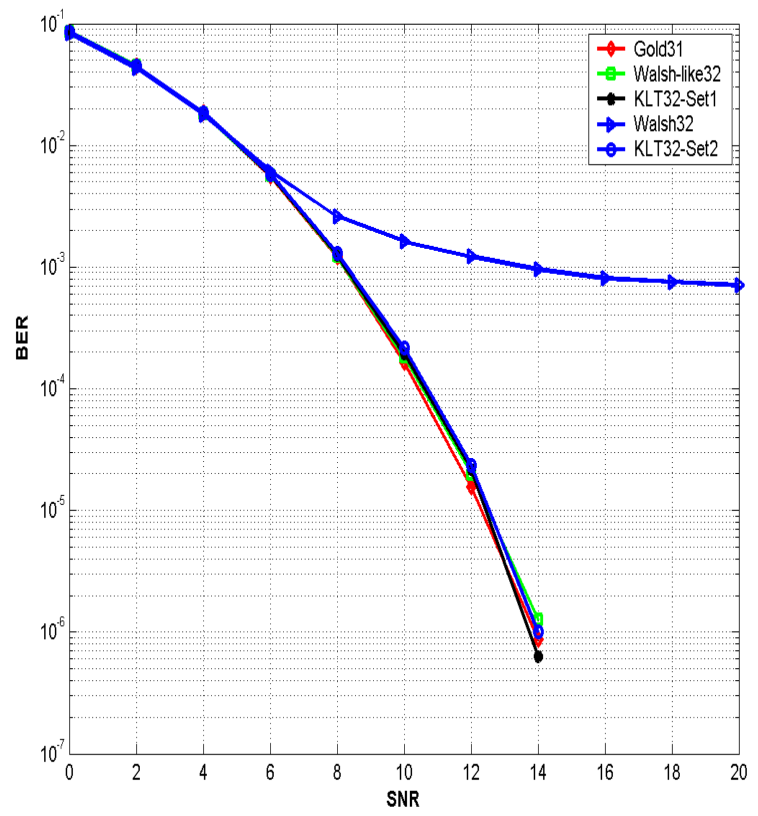




•Normalized auto- and cross-correlation sequences of Gold codes or Walsh-like sequences are used to generate eigen vectors

Asynchronous 2 User BER Performance of 31-Length, Spread Spectrum KLT Codes in AWG Noise

- BER performance of spread spectrum KLT codes matches with Gold codes performance
- Number of unique orthogonal sets can be generated by taking different auto- and cross-correlation sequences of Gold / Walsh-like codes



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IV. Correlation Performance Metrics

Aperiodic Correlation Function (ACF)

$$d_{k,l}(m) = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1-m} e_k(n)e_l^*(n+m), & 0 < m \le N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+m} e_k(n-m)e_l^*(n), & 1-N < m \le 0 \\ 0, & |m| \ge N \end{cases}$$

Max of Auto- and Cross-Correlation Sequences

$$d_{\max} = \max\left\{d_{am}, d_{cm}\right\}$$

$$d_{am} = \max\left\{\left|d_k(m)\right|\right\}$$
$$0 \le k < M$$
$$1 \le m < M$$

$$d_{cm} = \max\left\{\left|d_{k,l}(m)\right|\right\}$$
$$0 \le k, l < M \quad k \ne l$$
$$0 \le m < M$$

MS of Auto- and Cross-Correlation Sequences

$$R_{AC} = \frac{1}{M} \sum_{k=1}^{M} \sum_{\substack{m=1-N \\ m \neq 0}}^{N-1} \left| d_{k,k}(m) \right|^2$$

$$R_{CC} = \frac{1}{M(M-1)} \sum_{k=1}^{M} \sum_{\substack{l=1 \\ l \neq k}}^{M} \sum_{m=1-N}^{N-1} \left| d_{k,l}(m) \right|^2$$

Merit Factor (F_k)

$$F_k = \frac{d_k(0)}{2\sum_{m=1}^{N-1}\left|d_k(m)\right|^2}$$

MS of Auto- and Cross-Correlation Sequences

$$R_{AC} = \frac{1}{M} \sum_{k=1}^{M} \sum_{\substack{m=1-N \\ m \neq 0}}^{N-1} \left| d_{k,k}(m) \right|^2$$

$$R_{CC} = \frac{1}{M(M-1)} \sum_{k=1}^{M} \sum_{\substack{l=1 \\ l \neq k}}^{M} \sum_{m=1-N}^{N-1} \left| d_{k,l}(m) \right|^2$$

Peak-to-Average Power Ratio (PAPR)

$$PAPR = \frac{\max \left| x(n) \right|^2}{E\left[\left| x(n) \right|^2 \right]}$$

V. GDFT with Nonlinear Phase for Auto- and Cross-Correlation Improvements

- Motivation
- GDFT
- Design Metrics and Efficient Implementation
- Performance Improvements in BER, PAPR
- Dynamic Code Basis Hopping

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Motivation

- Constant modulus transforms are valuable for several applications including communications
- DFT is very popular since everyone uses it
- Can I have my DFT-like transforms?
- Emerging radio applications particularly SW based ones

  for sensing and P2P communications might benefit from flexible code/carrier libraries
- Introducing dynamic code/basis assignments offers additional improvements for system security (scrambler)

Generalized DFT: Mathematical Preliminaries

An *Nth* root of unity is a complex number satisfying the equation

$$z^N = 1 \qquad N = 1, 2, 3 , \ ......$$

If *Z* hold this equation but

$$z_p^m \neq 1 \qquad m = 1,2,3,..., N-1$$

then Z_p is defined as a *primitive Nth root of unity where m and N are coprime integers*

Generalized DFT: Mathematical Preliminaries

$$z_1 = e^{j(2\pi/N)}$$

is the primitive Nth root of unity with the smallest positive argument.

The are N distinct Nth roots of unity and expressed as

$$z_k = (z_p)^k \quad k = 1, 2, 3, \ \dots\ , N \quad \forall p$$

Generalized DFT: Mathematical Preliminaries

As an example, for N=4 there are two primitive N^{th} roots of unity expressed as

$$z_1 = e^{j\frac{2\pi}{4}} \qquad z_2 = e^{j\frac{3\pi}{2}}$$

Generalized DFT: Mathematical Preliminaries

All primitive Nth roots of unity satisfy the unique summation property of a geometric series expressed as follows

$$\sum_{n=0}^{N-1}(z_p)^n = \frac{(z_p)^N - 1}{z_p - 1} = \begin{Bmatrix} 1, N = 1 \\ 0, N > 1 \end{Bmatrix} \qquad \forall p$$

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Generalized DFT: Mathematical Preliminaries

We now define a periodic, with the period of N, constant modulus, complex sequence as the *rth power* of the first *primitive Nth roots of unity* raised to the *nth power* as

$$e_r(n) \triangleq (z_1^r)^n = e^{j(2\pi r/N)n}$$

$$n = 0,1,2,\ldots,N-1 \text{ and } r = 0,1,2,\ldots,N-1$$

Generalized DFT: Mathematical Preliminaries

This complex sequence over a finite discrete-time interval in a geometric series is expressed

$$z_1 = e^{j\omega_0} \qquad \omega_0 = 2\pi / N$$

$$\frac{1}{N}\sum_{n=0}^{N-1} e_r(n) = \frac{1}{N}\sum_{n=0}^{N-1} (z_1^r)^n = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi r / N)n}$$

$$= \begin{cases} 1, & r = mN \\ 0, & r \neq mN \\ & m = \text{integer} \end{cases}$$

Generalized DFT: Mathematical Preliminaries

One defines the discrete Fourier transform (DFT) set with the factorization into two orthogonal exponential functions where

$$< e_k(n), e_l^*(n) > = \frac{1}{N}\sum_{n=0}^{N-1} e_k(n) e_l^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi k/N)n} e^{-j(2\pi l/N)n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-l)n} = \begin{Bmatrix} 1, & k-l=r=mN \\ 0, & k-l=r\neq mN \\ & m=\text{integer} \end{Bmatrix}$$

Generalized DFT

Let's generalize by introducing a product function in the phase defined as

$$\varphi(n) = \varphi_k(n) - \varphi_l(n)$$

and expressing a constant amplitude orthogonal set as follows,

Generalized DFT

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi r/N)n} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi\varphi(n)/N]n}$$

$$= \left\{ \begin{array}{ll} 1, & \varphi(n) = \varphi_k(n) - \varphi_l(n) = r = mN \\ 0, & \varphi(n) = \varphi_k(n) - \varphi_l(n) = r \neq mN \\ & m, n = \text{integer} \end{array} \right\}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi(\varphi_k(n)-\varphi_l(n))/N)]n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{j[2\pi\varphi_k(n)/N]n} e^{-j[2\pi\varphi_l(n)/N]n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e_k(n)e_l^*(n) =< e_k(n), e_l^*(n) >$$

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Generalized DFT

Hence, the basis functions of Generalized DFT (GDFT) are defined as

$$e_k(n) \triangleq e^{j(2\pi/N)\varphi_k(n)n}$$

$$k, n = 0, 1, \ldots, N-1$$

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Generalized DFT

As an example, one might define

$$\varphi_k(n) = \frac{N_k(n)}{D_k(n)} = \frac{\sum_{j=1}^{N} a_{kj} n^{b_{kj}}}{\sum_{j=1}^{M} c_{kj} n^{d_{kj}}} \qquad N \le M; \quad k = 0, 1, \ldots, N-1$$

$$D_k(n) = 1$$

$$\varphi_k(n) = N_k(n) = \sum_{j=1}^{N} a_{kj} n^{b_{kj}} = a_{k1} n^{b_{k1}} + a_{k2} n^{b_{k2}} + a_{k3} n^{b_{k3}} + \ldots + a_{kN} n^{b_{kN}}$$

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Remarks on Generalized DFT

1) DFT is a special solution of GDFT with

$$\varphi_k(n) = a_{k1} = k \quad \text{and} \quad a_{k2} = a_{k3} = \dots = a_{kN} = 0$$

$$b_{k1} = b_{k2} = \dots = b_{kN} = 0$$

Having constant valued $\{\varphi_k(n)\}$ functions

makes DFT a linear-phase transform

$$\{e_k(n)\} \triangleq e^{j(2\pi / N)kn} \qquad k, n = 0, 1, \dots, N-1$$

Remarks on Generalized DFT

2) Popular Walsh and Nonlinear Phase Walsh-like orthogonal binary transforms are special solutions of GDFT. As an example,

$$A_{WALSH} = A_{GDFT} = A_{DFT} G_{WALSH}$$

$$G_{WALSH} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0.71e^{j\pi 0.25} & 0.71e^{-j\pi 0.25} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.71e^{-j\pi 0.25} & 0.71e^{j\pi 0.25} \end{bmatrix}$$

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Remarks on Generalized DFT

$$\{b_{k1} = 0, b_{k2} = 1, b_{k3} = 2, b_{k4} = 3 \quad \text{for} \;\; k = 0,1,2,3\}$$

$$\left\{ \begin{array}{l} \varphi_0^{WALSH}(n) = 0 \\ \varphi_1^{WALSH}(n) = 6.22n - 5.33n^2 + 1.11n^3 \\ \varphi_2^{WALSH}(n) = -1.278n + 1.667n^2 - 0.389n^3 \\ \varphi_3^{WALSH}(n) = 4.5n - 3n^2 + 0.5n^3 \end{array} \right\} \quad n = 0,1,2,3$$

Remarks on Generalized DFT

3) There are infinitely many possible GDFT sets available in the phase space with constant power where one can design the optimal basis for the desired figure of merit.

The availability of rich library of orthogonal constant amplitude transforms with good performance allows us to design adaptive systems where basis assignments as well as code allocations are made dynamically and intelligently to exploit the current channel conditions in order to deliver better communications performance and improved physical layer security.

Remarks on Generalized DFT

4) Oppermann, Frank-Zadoff and Chu Sequences are the special cases of his code family.

$$A_{GDFT} = A_{OPP} = A_{DFT} G_{OPP}$$

$$a_1 = \frac{k^m + kN}{2}$$

$$b_1 = 0; \quad a_2 = \frac{1}{2}; \quad b_2 = n - 1$$

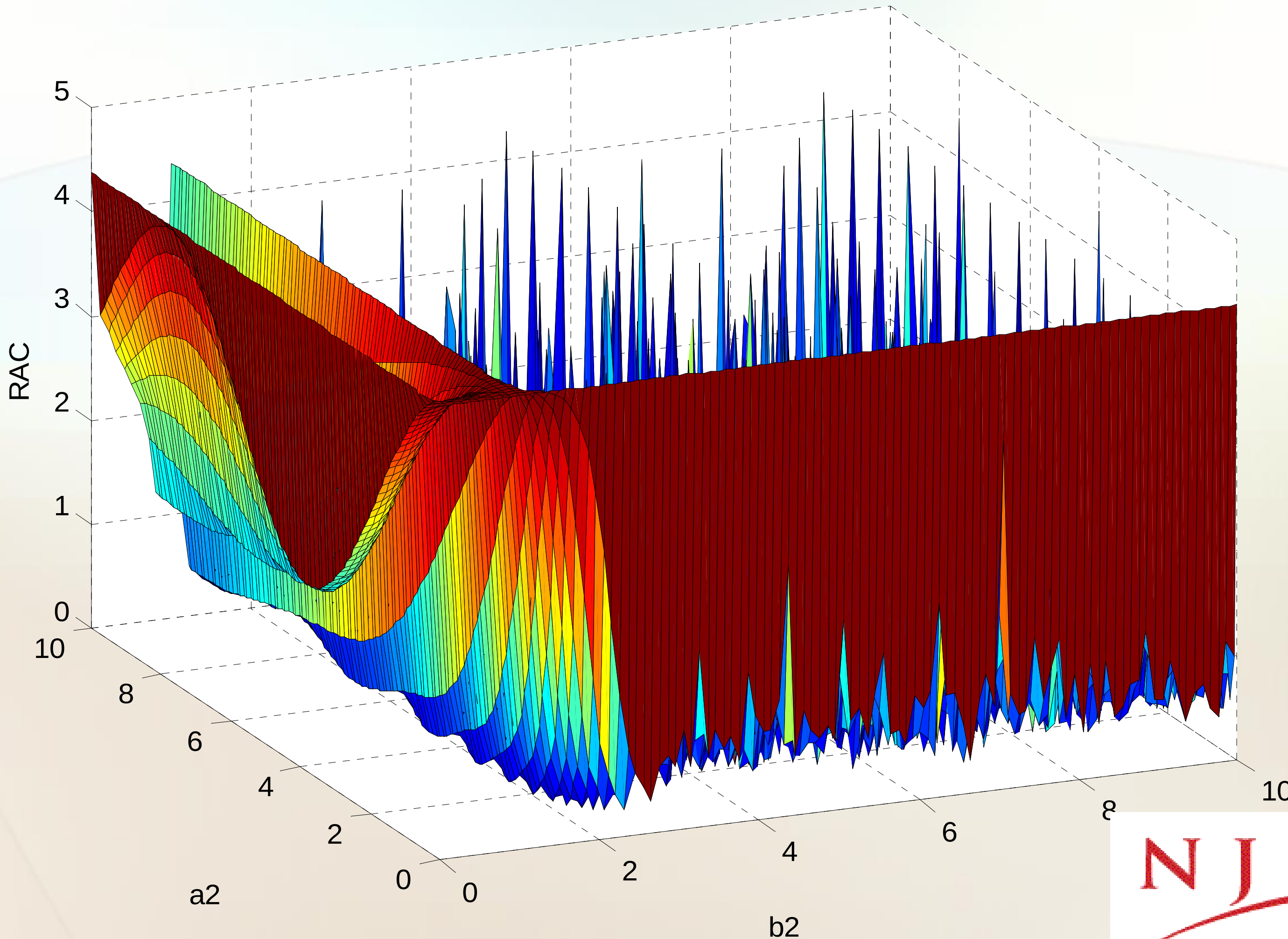
$$A_{OPP}(k,i) = (-1)^{ki} \exp\left(\frac{j\pi(k^m i^p + i^n)}{N} \right)$$

$$k, i = 1, 2, \ldots, N$$

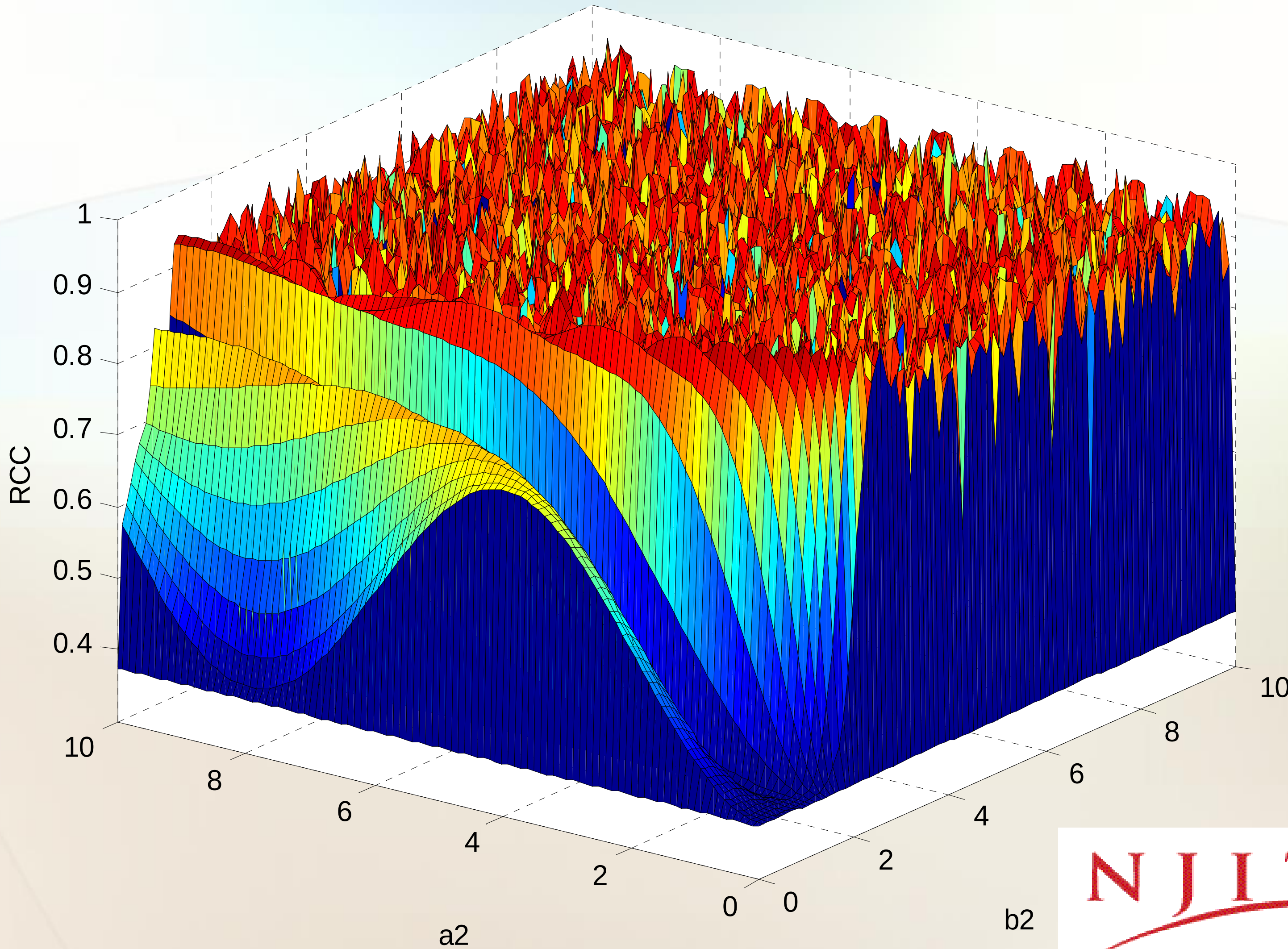
Remarks on Generalized DFT

$$G_{OPP} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & e^{-j\pi 0.87} \\ 0 & 0 & 0 & 0 & e^{-j\pi} & 0 & 0 \\ 0 & 0 & e^{-j\pi 0.8} & 0 & 0 & 0 & 0 \\ e^{-j\pi 0.71} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{j\pi 0.63} & 0 \\ 0 & 0 & 0 & e^{-j\pi 0.54} & 0 & 0 & 0 \\ 0 & e^{-j\pi 0.59} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Variations of Auto-correlation Metric for Parametric GDFT Solutions (N=8)



Variations of Cross-correlation Metric for Parametric GDFT Solutions (N=8)



Matrix Representation

$$A_{GDFT} = G_1 A_{DFT} G_2$$

$$A_{GDFT} A_{GDFT}^{*T} = I$$

$$G_1 G_1^{*T} = I \text{ and } G_2 G_2^{*T} = I$$

$$G_1(k,n) = \begin{cases} e^{j\theta_{kk}}, & k = n \\ 0, & k \neq n \\ & k,n = 0,1,\ldots N \end{cases}$$

$$G_2(k,n) = \begin{cases} e^{j\gamma_{nn}}, & n = k \\ 0, & n \neq k \\ & k,n = 0,1,\ldots N \end{cases}$$

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GDFT Kernel (Diagonal G1 & G2)

$$A_{GDFT} = \{e_k(n)\} \triangleq \left\{ e^{j[(2\pi/N)kn + \theta_{kk} + \gamma_{nn}]} \right\}$$

$$k,\ n\ =\ 0,\ 1,\ .\ .\ .\ ,\ N\ -\ 1$$

DESIGN METRICS

Aperiodic Correlation Function (ACF)

$$d_{k,l}(m) = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1-m} e_k(n) e_l^*(n+m), & 0 < m \le N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+m} e_k(n-m) e_l^*(n), & 1-N < m \le 0 \\ 0, & |m| \ge N \end{cases}$$

Max of Auto- and Cross-Correlation Sequences

$$d_{\max} = \max\left\{d_{am}, d_{cm}\right\}$$

$$d_{am} = \max\left\{\left|d_k(m)\right|\right\}$$
$$0 \le k < M$$
$$1 \le m < M$$

$$d_{cm} = \max\left\{\left|d_{k,l}(m)\right|\right\}$$
$$0 \le k, l < M \quad k \ne l$$
$$0 \le m < M$$

MS of Auto- and Cross-Correlation Sequences

$$R_{AC} = \frac{1}{M} \sum_{k=1}^{M} \sum_{\substack{m=1-N \\ m \neq 0}}^{N-1} \left| d_{k,k}(m) \right|^2$$

$$R_{CC} = \frac{1}{M(M-1)} \sum_{k=1}^{M} \sum_{\substack{l=1 \\ l \neq k}}^{M} \sum_{m=1-N}^{N-1} \left| d_{k,l}(m) \right|^2$$

Merit Factor (F_k)

$$F_k = \frac{d_k(0)}{2\sum_{m=1}^{N-1}\left|d_k(m)\right|^2}$$

MS of Auto- and Cross-Correlation Sequences

$$R_{AC} = \frac{1}{M} \sum_{k=1}^{M} \sum_{\substack{m=1-N \\ m \neq 0}}^{N-1} \left| d_{k,k}(m) \right|^2$$

$$R_{CC} = \frac{1}{M(M-1)} \sum_{k=1}^{M} \sum_{\substack{l=1 \\ l \neq k}}^{M} \sum_{m=1-N}^{N-1} \left| d_{k,l}(m) \right|^2$$

Peak-to-Average Power Ratio (PAPR)

$$PAPR = \frac{\max |x(n)|^2}{E\left[|x(n)|^2 \right]}$$

Optimal Design of Phase Shaping Function

$$\hat{\varphi}_k(n) = \varphi_k(n)n = kn + \psi(n) \quad \text{for } k = 0,1,\ldots,N-1 \text{ and } n = 1,\ldots,N-1$$

$$\psi(n) = \hat{\varphi}_k(n) - kn = [\varphi_k(n) - k]n \quad \text{for } k = 0,1,\ldots,N-1 \text{ and } n = 1,\ldots,N-1$$

$$\psi(0) \in \mathbb{R} \quad \hat{\varphi}_k(0) = \psi(0)$$

Cross Correlation:

$$R_{\hat{\varphi}_k \hat{\varphi}_l}(m) = \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})\hat{\varphi}_k(n)} \, e^{-j(\frac{2\pi}{N})\widehat{\varphi}_l(n+m)}$$

$$= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})[-lm+(k-l)n+\psi(n)-\psi(n+m)]}$$

Ideal Case for CC (Requires Sinc Functions):

$$R_{\widehat{\varphi}_k \widehat{\varphi}_l}(m) = 0; \quad \forall m$$

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Optimal Design of Phase Shaping Function

$$\hat{\varphi}_k(n) = \varphi_k(n)n = kn + \psi(n)$$

Auto Correlation:

$$R_{\hat{\varphi}_k \hat{\varphi}_k}(m) = \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})\hat{\varphi}_k(n)} e^{-j(\frac{2\pi}{N})\hat{\varphi}_k(n+m)}$$

$$= \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})[-km+\psi(n)-\psi(n+m)]}$$

Ideal Case:

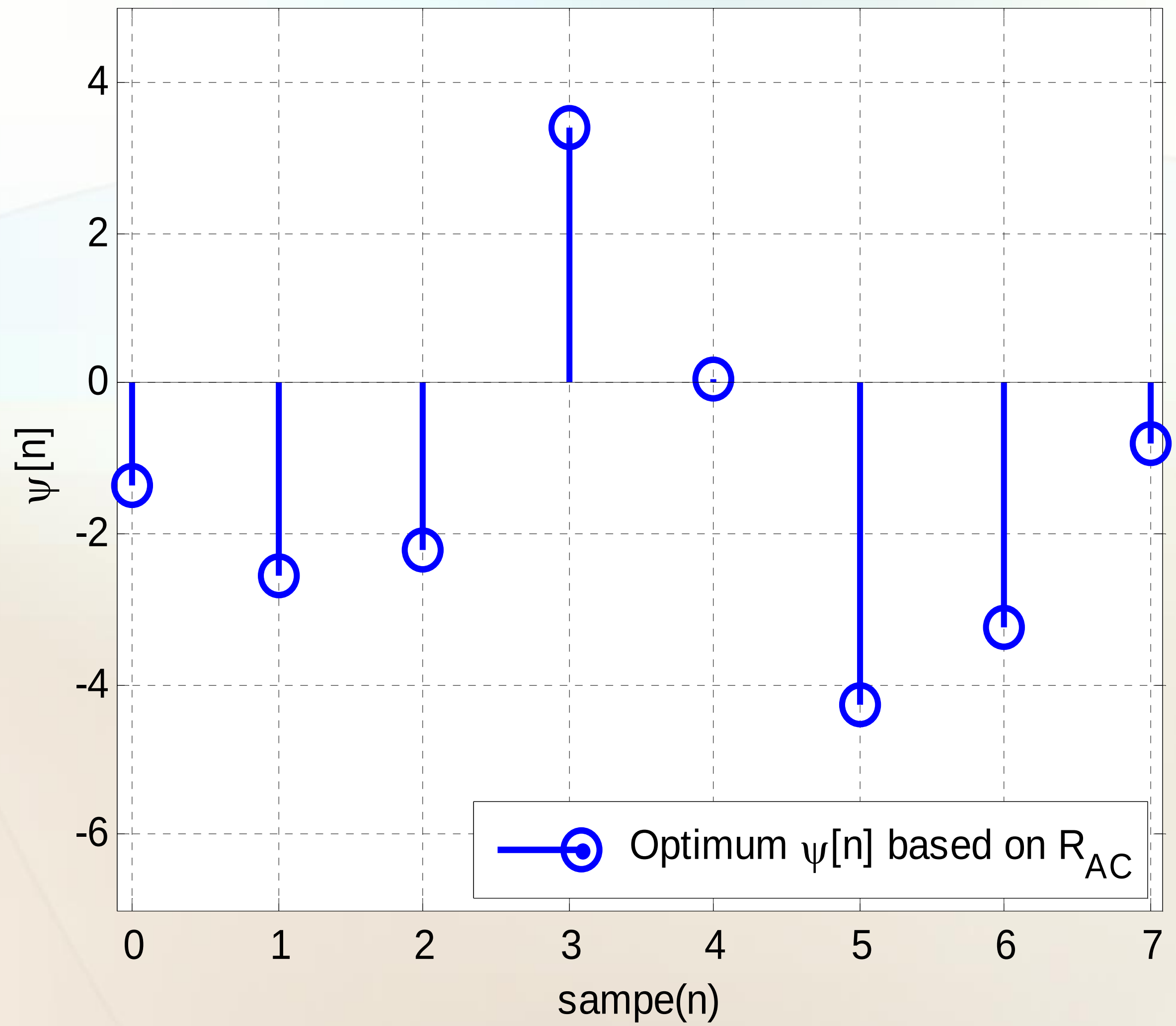
$$R_{\hat{\varphi}_k \hat{\varphi}_k}(m) = \delta(m)$$

Optimal Design of Phase Shaping Function

The first two functions of optimal GDFT sets with N=8 along with their DFT counterparts

Numerical Search Tool and Optimal Phase Shaping Function	OPTIMIZATION METRIC (N=8)	
	R_{AC}	R_{CC}
GDFT (Mathematica, FindMin)	0.0877	0.4219
$\psi(n)$	{ -1.37 -2.53 -2.21 3.39 0.0 -4.21 -3.19 -0.83 }	{ 1.637 -0.79 -0.54 2.01 1.59 -0.83 1.73 2.44}
GDFT (MATLAB,fminsearch)	0.086	0.4205
$\psi(n)$	{ -1.38 -2.56 -2.24 3.42 0.07 -4.27 -3.27 -0.80 }	{ 1.673 -0.87 -0.51 2.02 1.51 -0.86 1.70 2.46 }
DFT	4.375	0.8536

Closed Form Phase Shaping Function
$$\psi(n)$$



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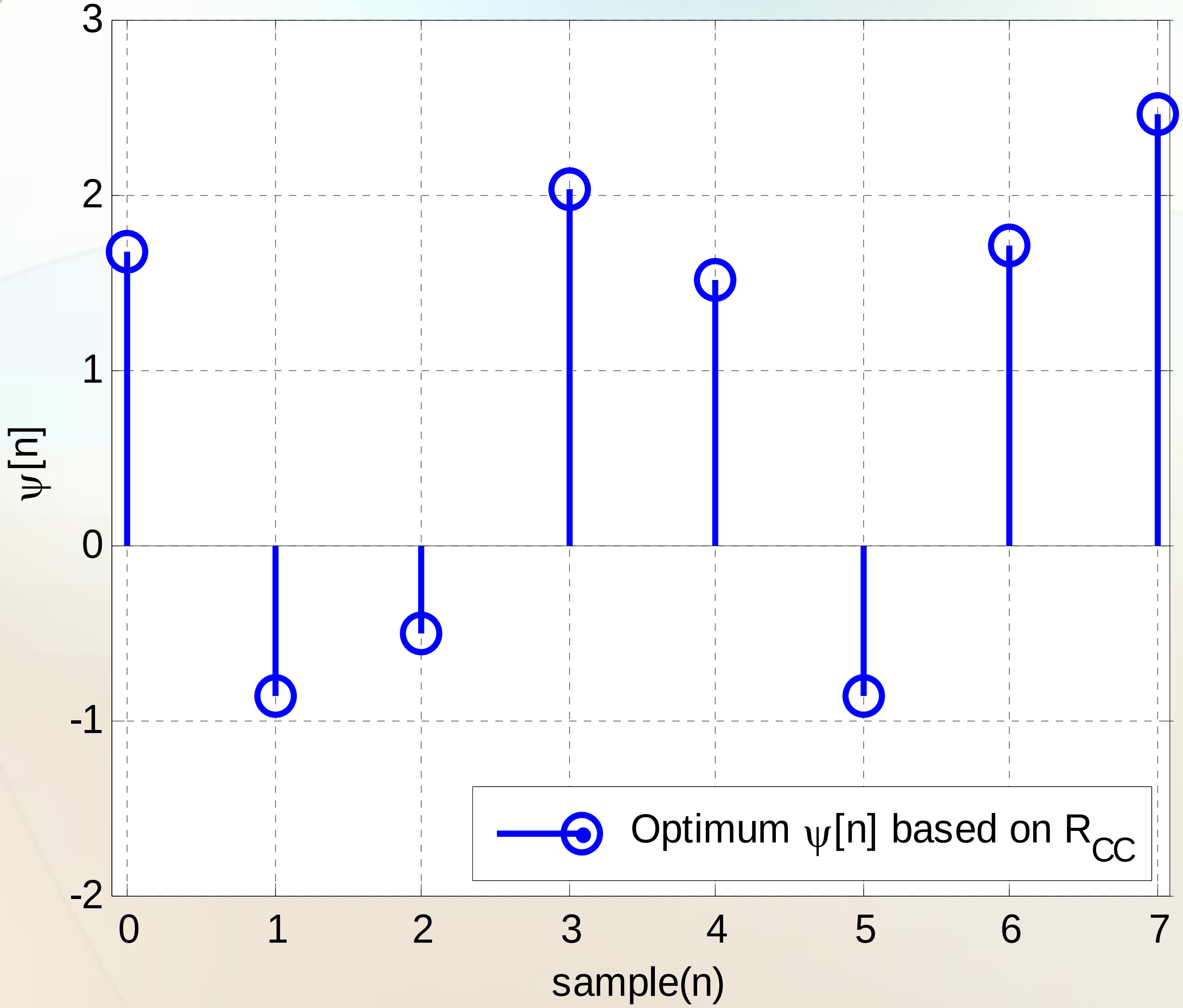
Closed Form Phase Shaping Function $\psi(n)$



Closed Form Phase Shaping Function for GDFT Design

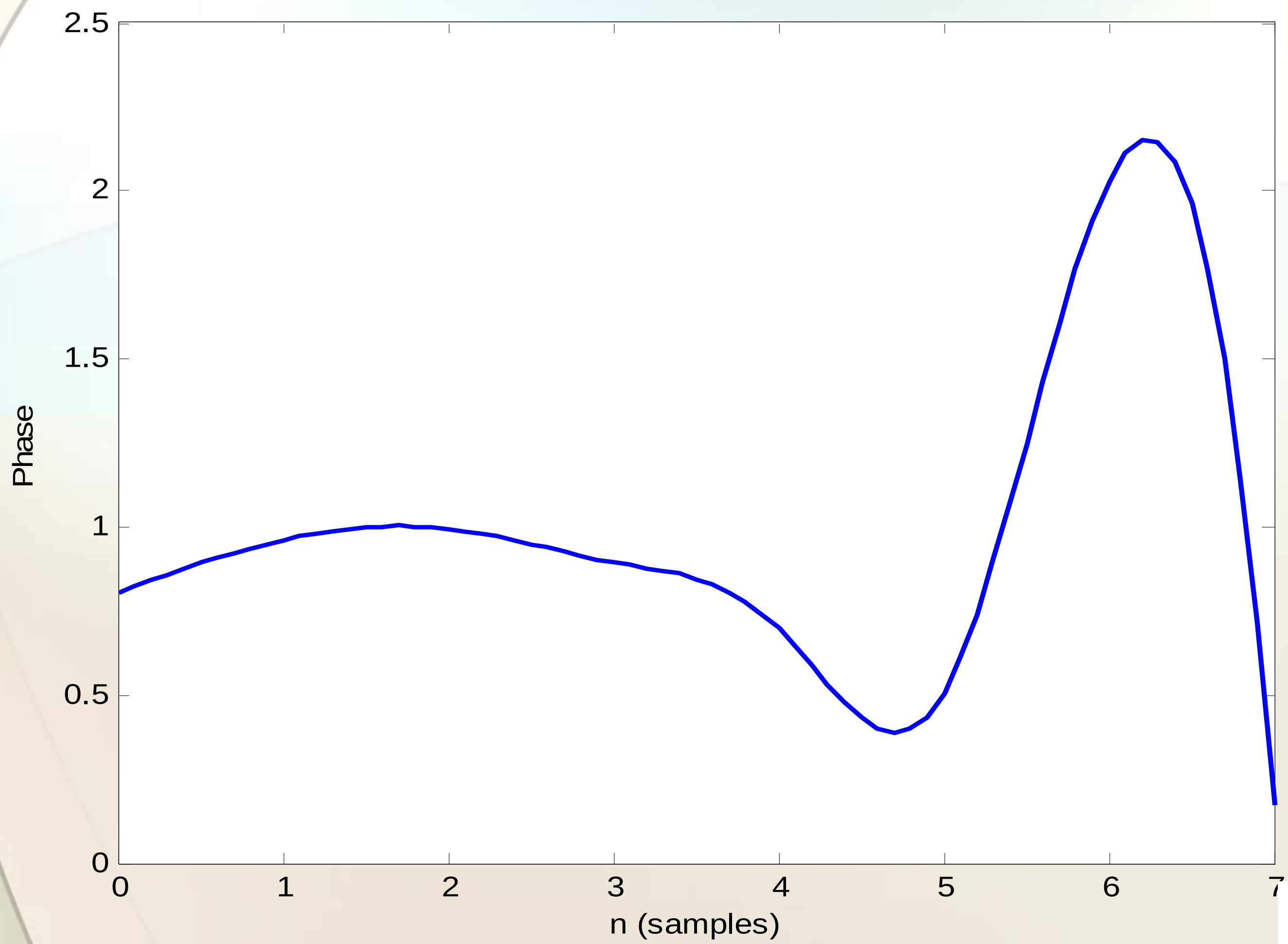
$$\varphi_k(n) = kn + \psi(n)$$

$$\{a_1 = 1, b_1 = 1.75, c_1 = 3.75, a_2 = 1.75, b_2 = 6, c_2 = 0.5\}$$

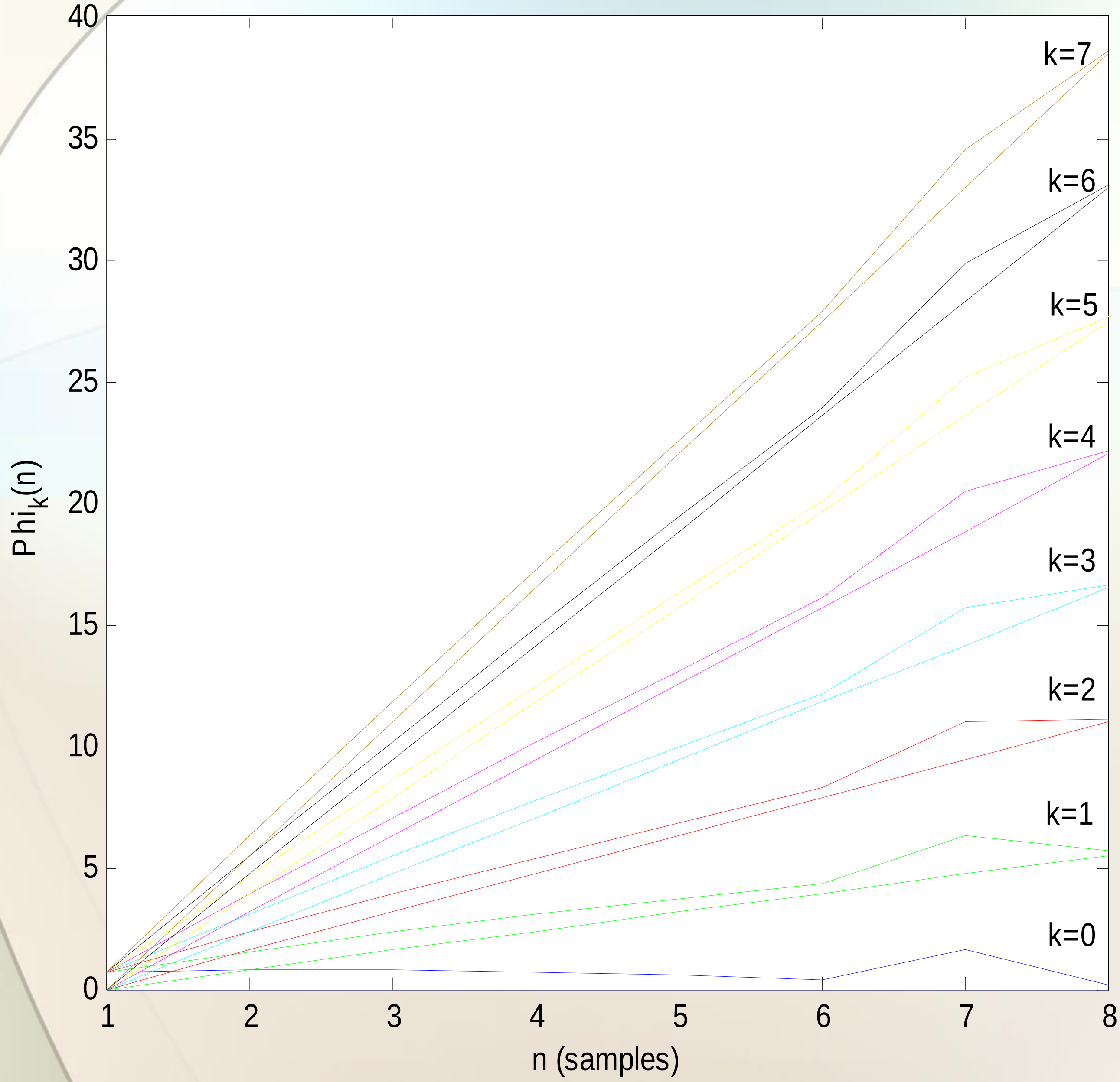
$$\psi(n) = \exp\left(-\left(\frac{n-1.75}{3.75}\right)^2 \right) + 1.75\exp\left(-\left(\frac{n-6}{0.50}\right)^2 \right)$$

Closed Form Phase Shaping Function

$$\psi(n)$$

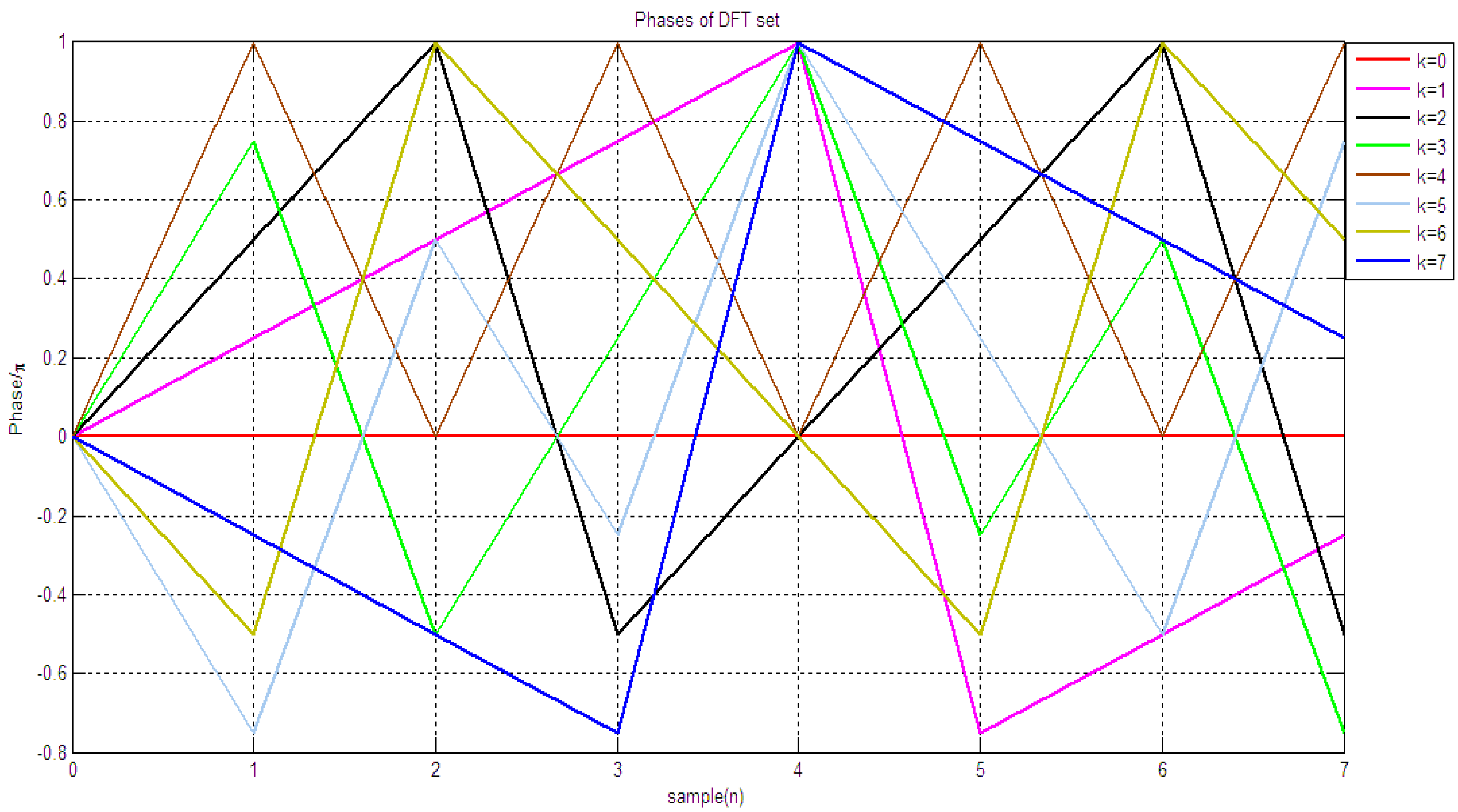


Phase Functions: DFT (Linear) vs GDFT (Nonlinear)

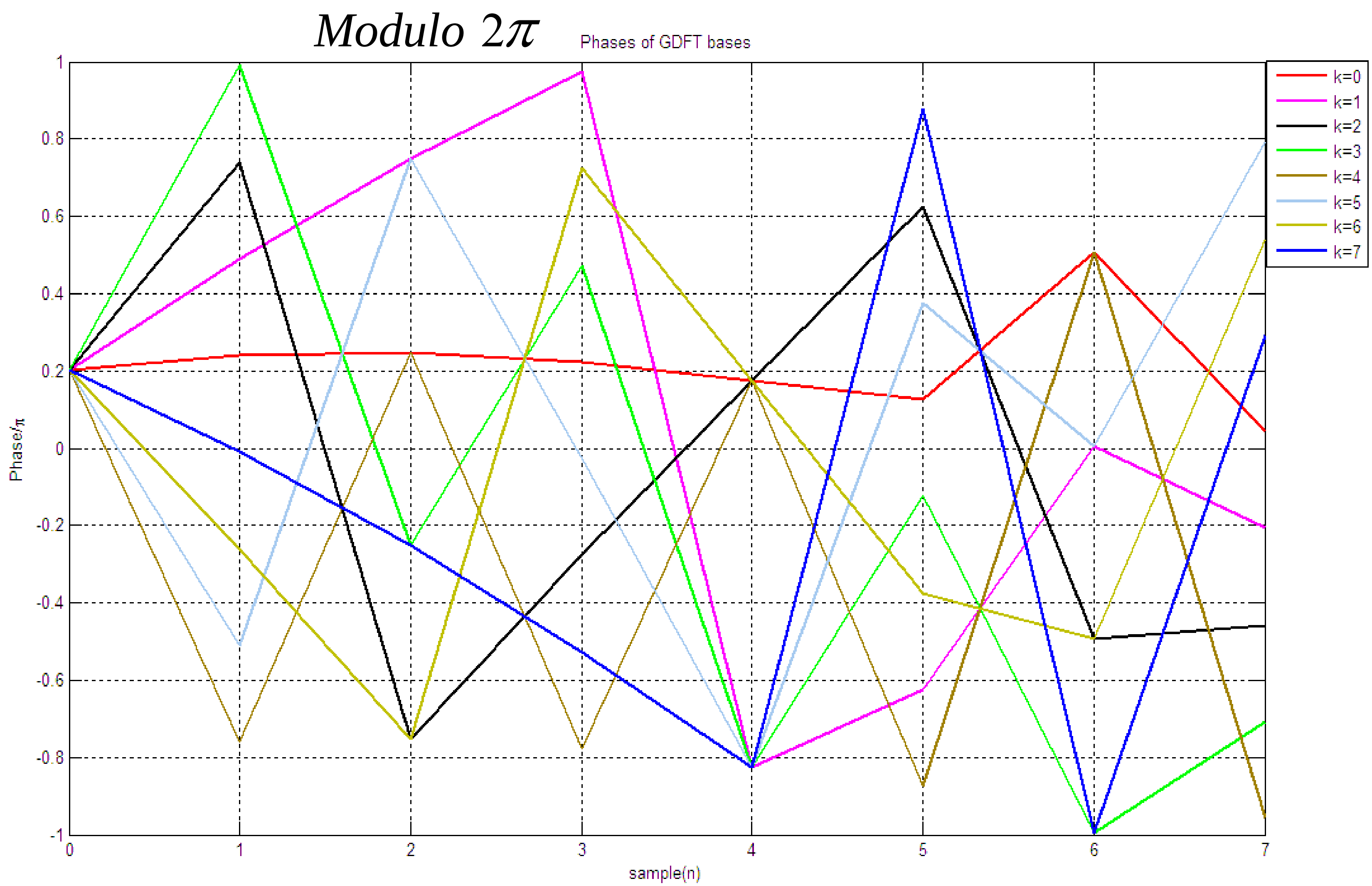


DFT Phase Functions (Linear)

Modulo 2π



GDFT Phase Functions (Nonlinear)



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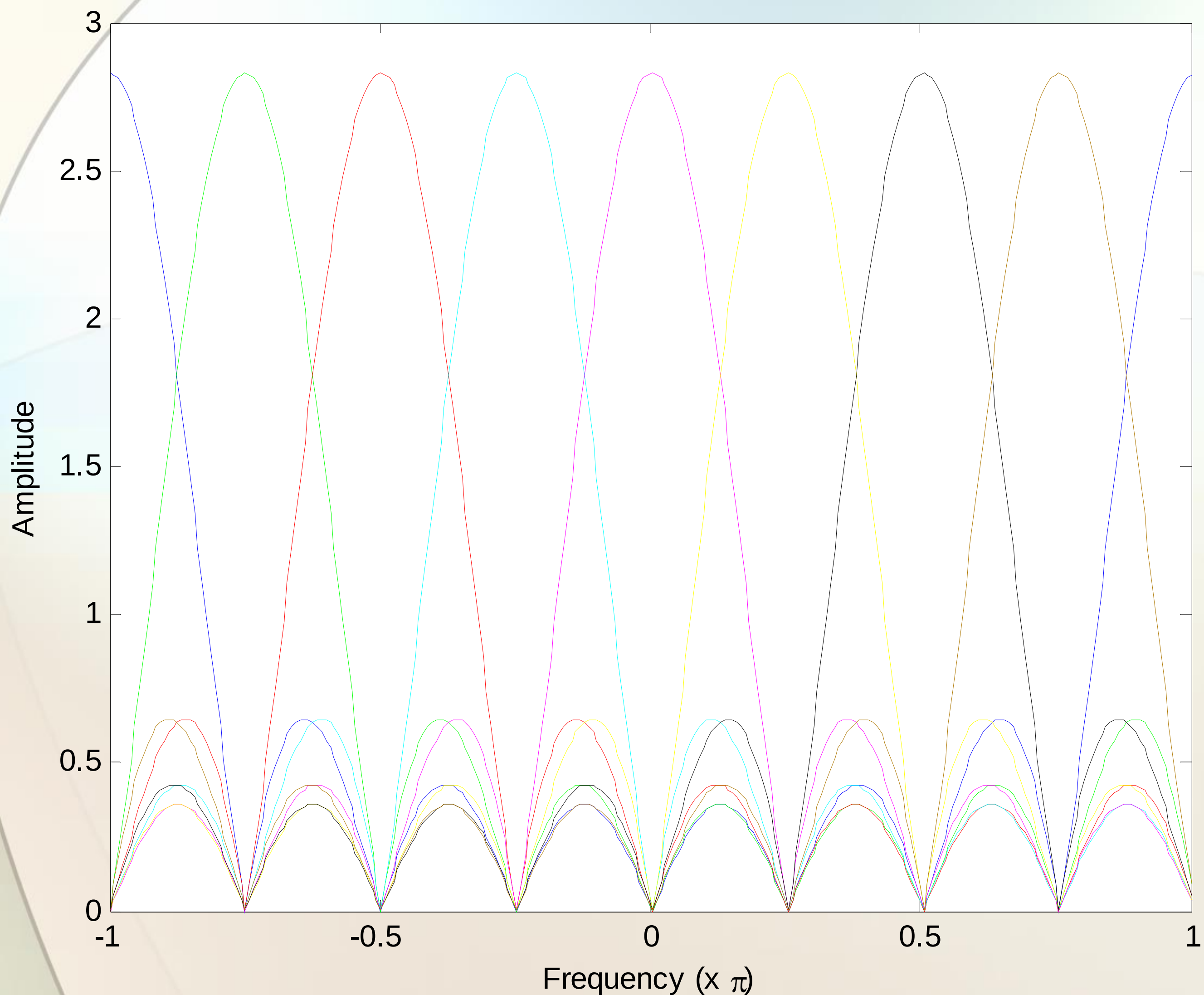
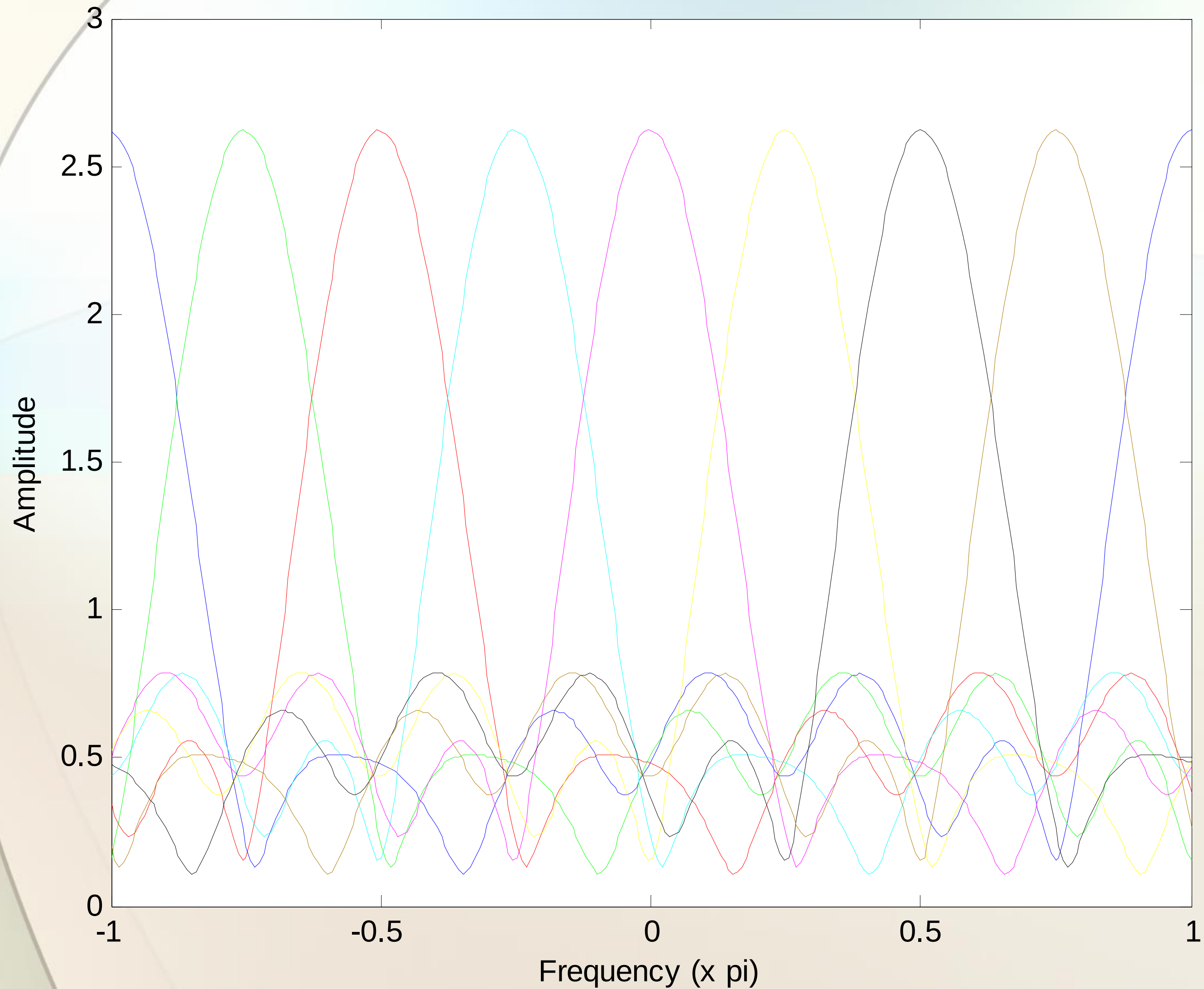
DFT Amplitude Functions



GDFT Amplitude Functions



Correlation metrics: DFT vs GDFT

N=8	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}
DFT	0.875	0.327	0.875	4.375	0.375
GDFT	0.703	0.288	0.703	3.261	0.534

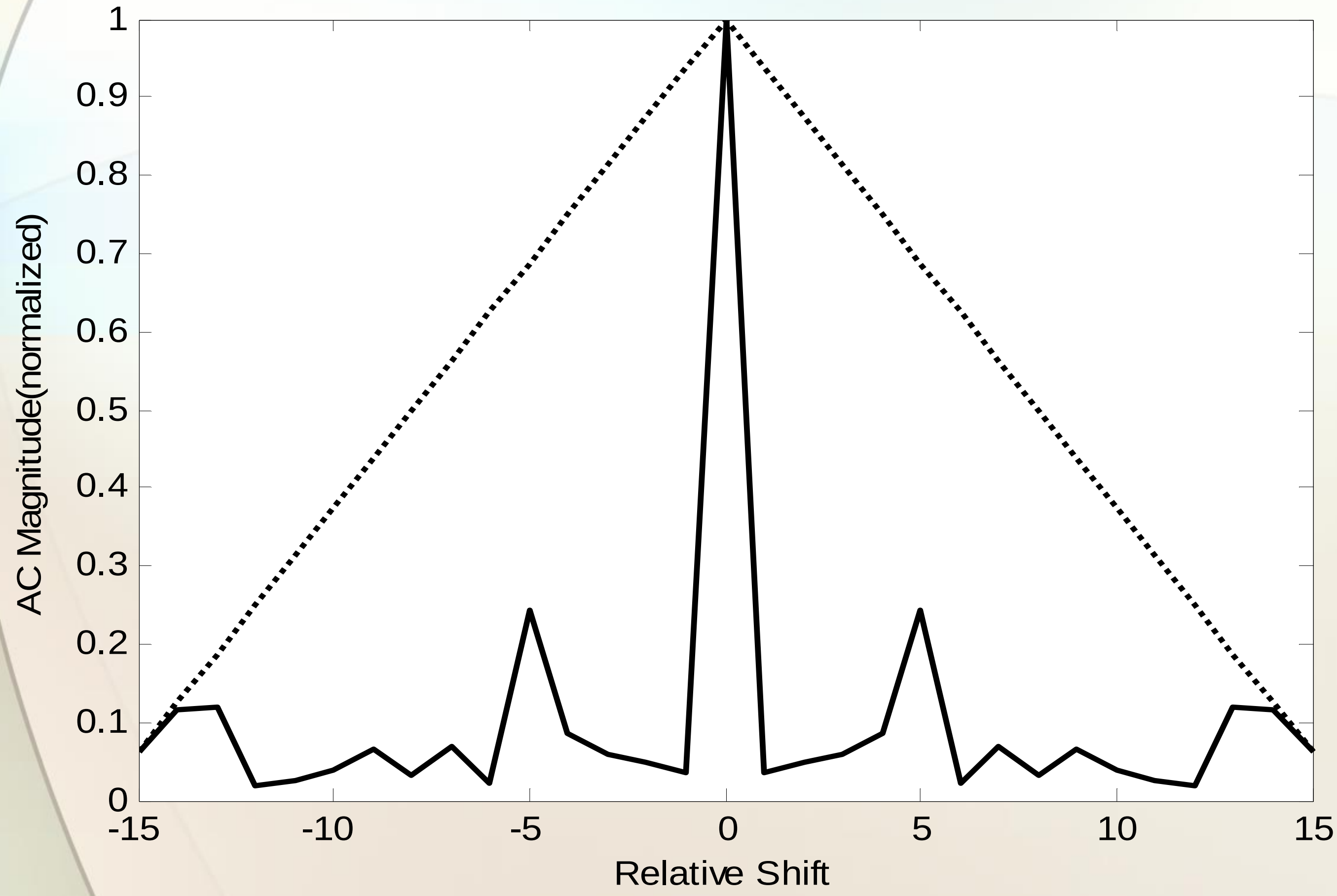
Performance of Various Codes

$$d_{\max} = \max\left\{d_{am}, d_{cm}\right\}$$

Code	d_{am}	d_{cm}	d_{max}	R_{AC}	R_{CC}	F
Walsh [8x8]	0.875	0.875	0.875	2.375	0.661	0.421
Walsh-like [8x8]	0.625	0.625	0.625	0.875	0.875	1.143
DFT [8x8]	0.875	0.327	0.875	4.375	0.375	0.220
7/8 Gold	0.714	0.714	0.714	0.857	0.878	1.167
Opperrman (opt d_{max}) (m=1, p=1, n=2.98, N=7)	0.425	0.419	0.465	1.278	0.787	0.783
GDFT (opt d_{max})	0.376	0.387	0.387	1.095	0.843	0.912

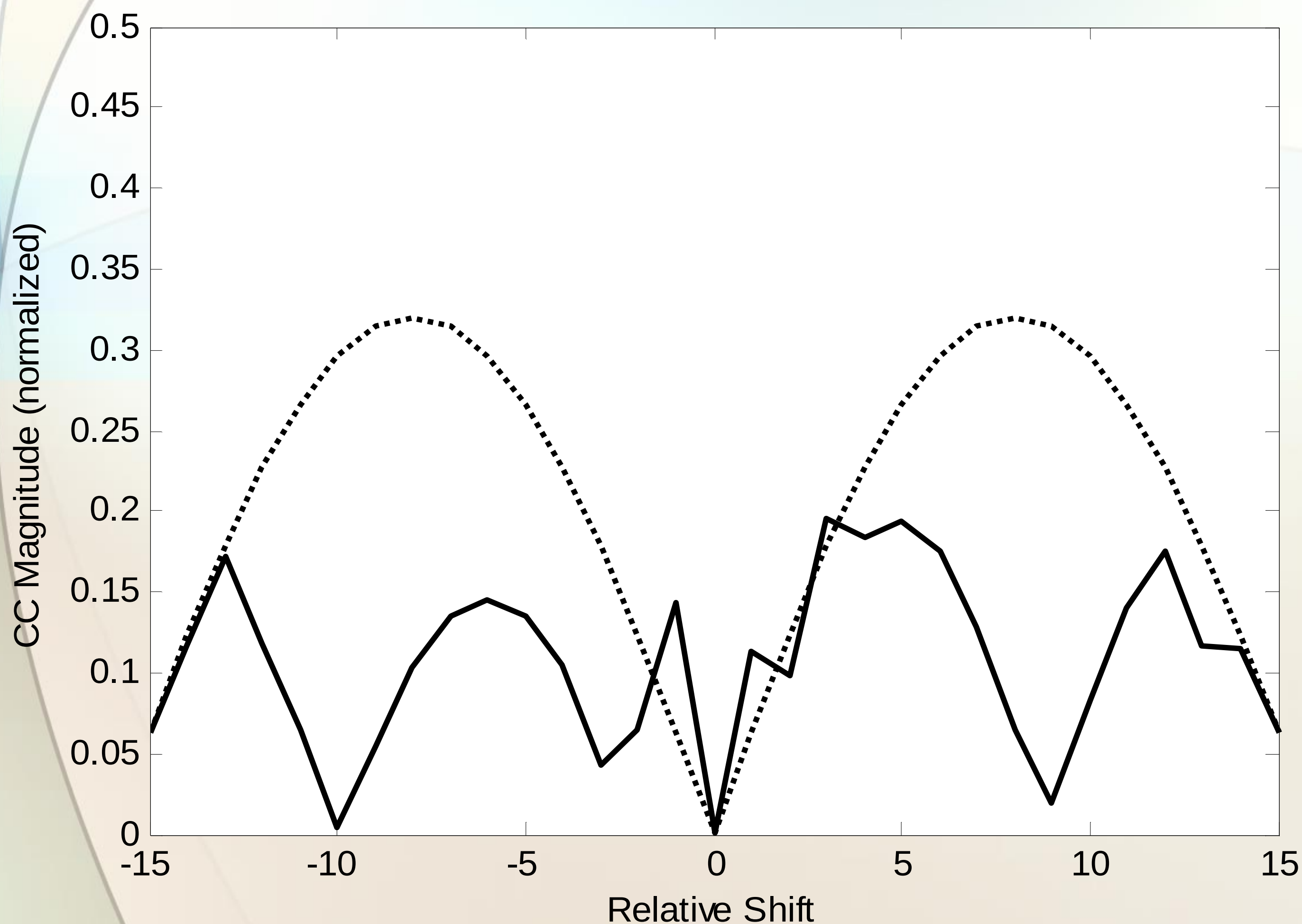
Magnitude of Auto-correlation Functions for GDFT (solid line) and DFT (dashed line) (N=16)

$$R_{AC} = \frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{m=1-N \\ m\neq 0}}^{N-1}\left|d_{k,k}(m)\right|^2$$

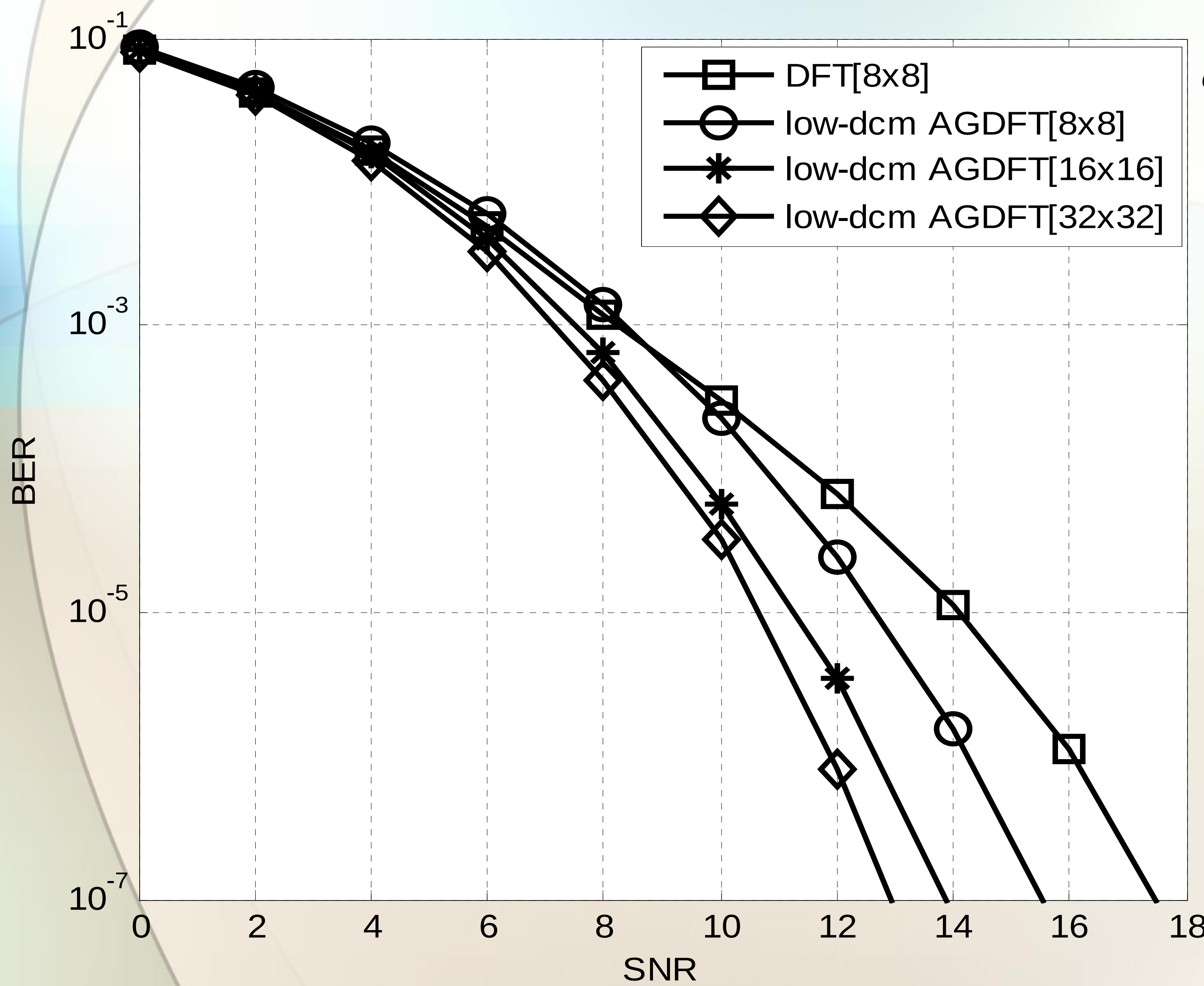


Magnitude of Cross-correlation Functions for GDFT (solid line) and DFT (dashed line) (N=16)

$$R_{CC} = \frac{1}{M(M-1)} \sum_{k=1}^{M} \sum_{\substack{l=1 \\ l \neq k}}^{M} \sum_{m=1-N}^{N-1} \left| d_{k,l}(m) \right|^2$$



DS-CDMA BER Performance
(2 Users, AWGN)



$$d_{cm} = \max\left\{\left|d_{k,l}(m)\right|\right\}$$
$$0 \le k,l < M \;\; k \ne l$$
$$0 \le m < M$$

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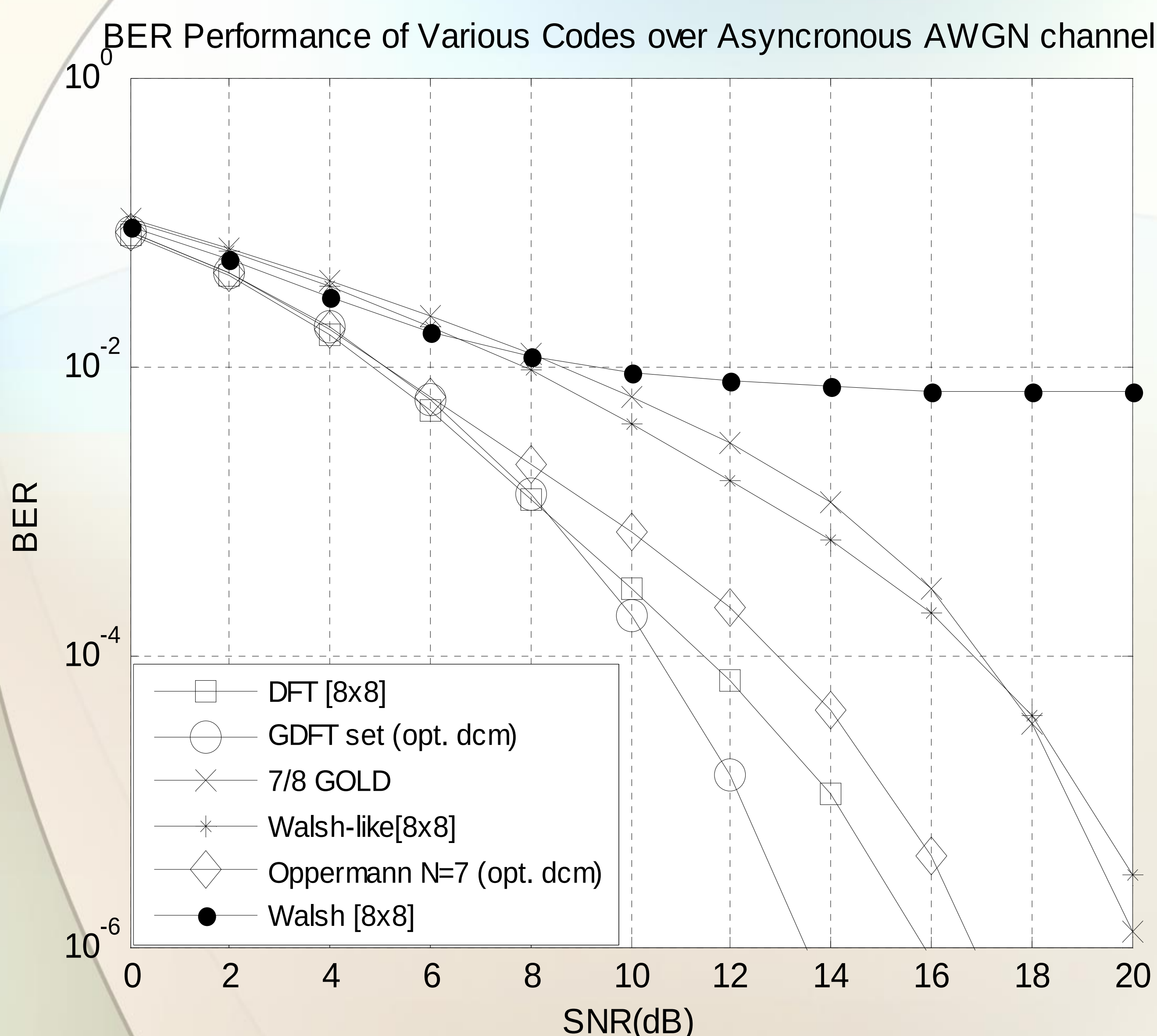
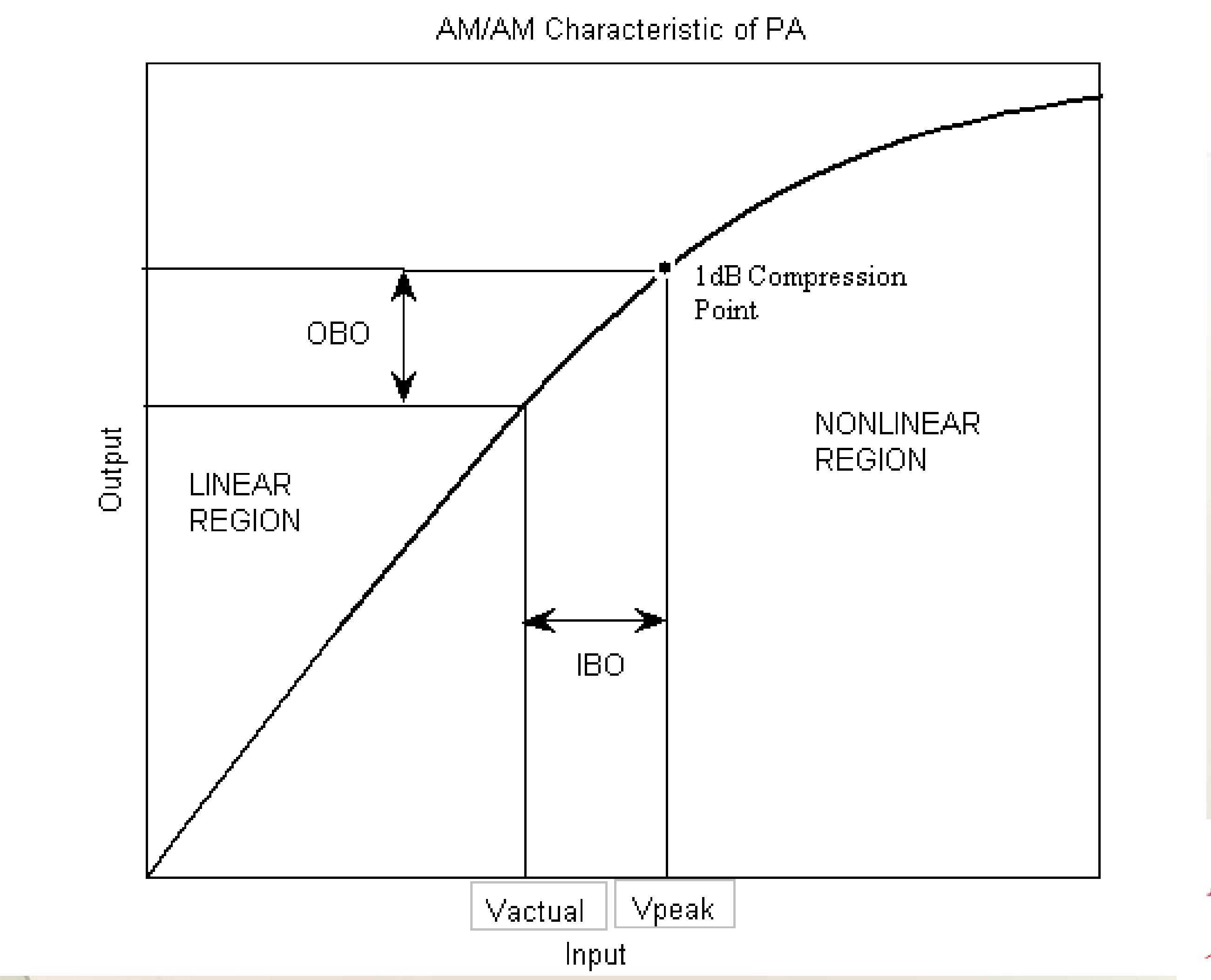
DS-CDMA BER Performance
(2 Users, AWGN)

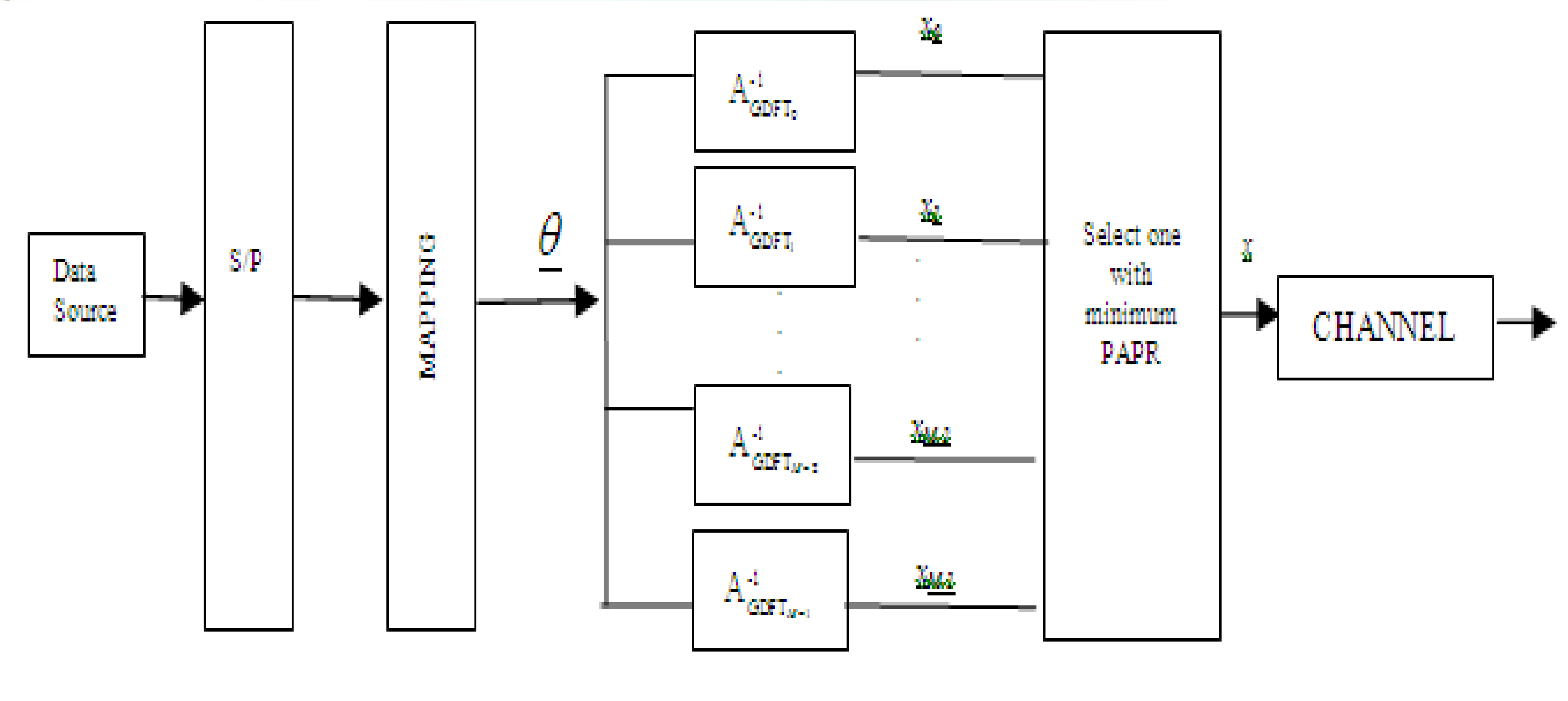


$$d_{cm} = \max\left\{\left|d_{k,l}(m)\right|\right\}$$
$$0 \le k,l < M \;\; k \ne l$$
$$0 \le m < M$$

AM/AM Characteristics of RF PA



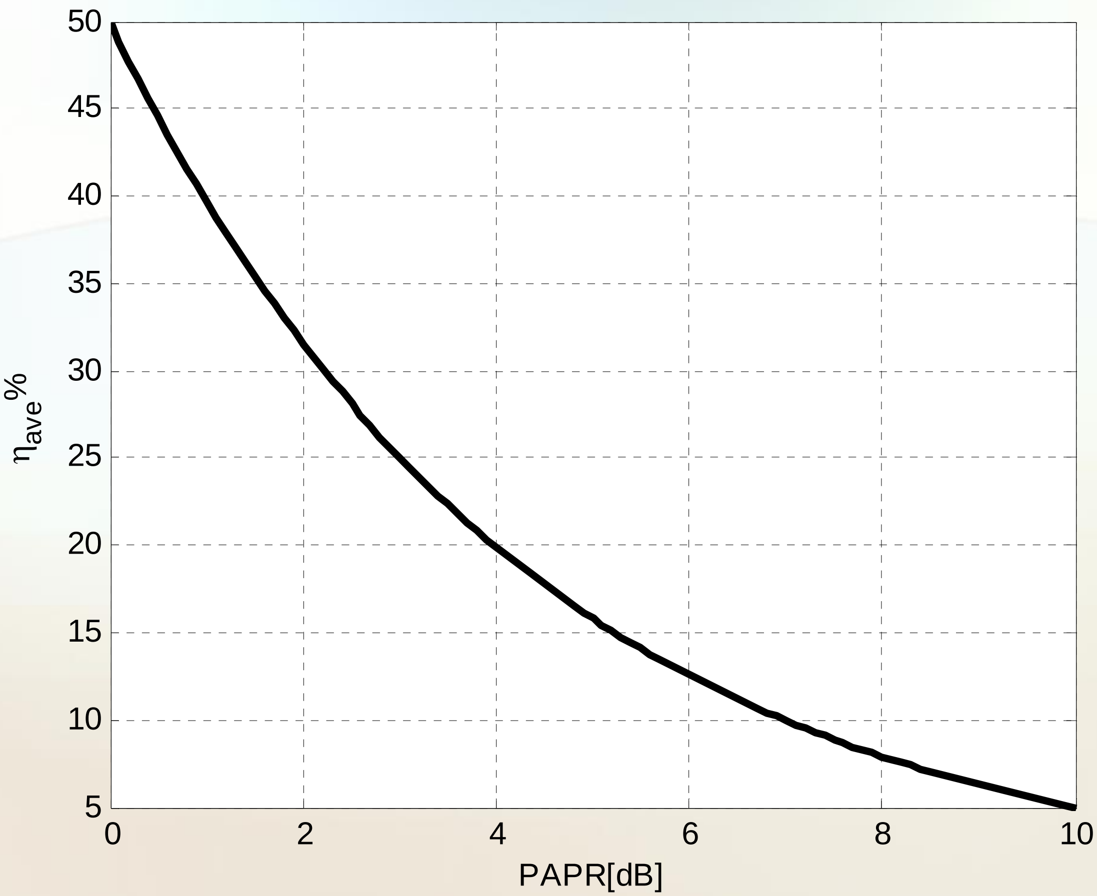
GDFT-SLM BASED PAPR REDUCTION



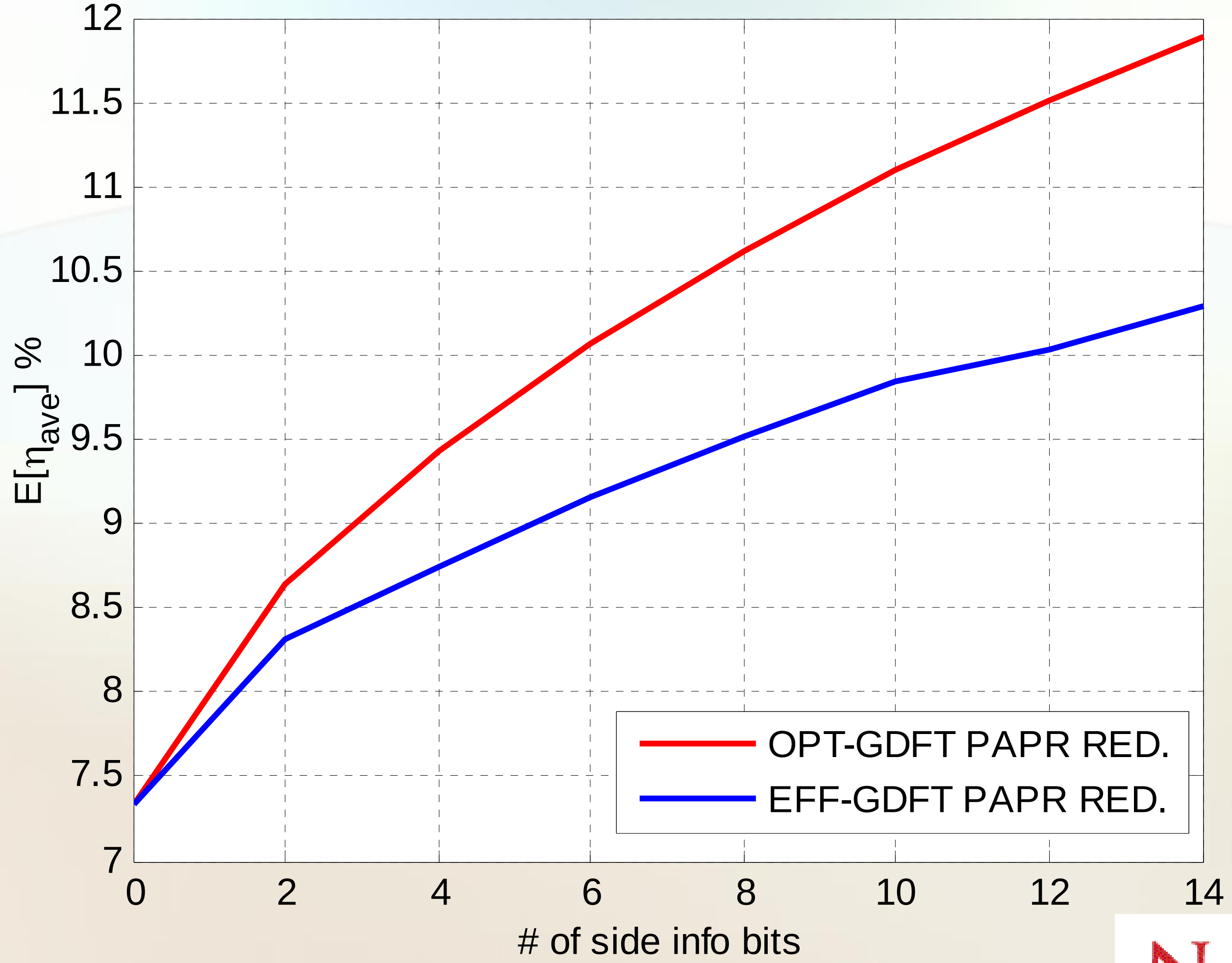
PAPR Reduction Comparisons for N=256, $P(PAPR>\gamma)=10^{-1}$

m	$PAPR_{DFT}$ [dB]	$PAPR_{GDFT}^{EFF}$ [dB]	$PAPR_{GDFT}^{OPT}$ [dB]	$PAPR_{SLM}$ [dB]	$\Delta PAPR$ [dB]		
					EFF-GDFT	OPT-GDFT	SLM
2	10.4	8.6	8.1	8.3	1.8	2.3	2.1
4	10.4	8.3	7.6	7.7	2.1	2.8	2.7
6	10.4	8.1	7.2	7.3	2.3	3.2	3.1
8	10.4	7.9	7.0	7.0	2.5	3.4	3.4
10	10.4	7.7	6.7	6.8	2.6	3.7	3.6
12	10.4	7.6	6.5	6.6	2.8	3.9	3.8
14	10.4	7.5	6.4	6.4	2.9	4	4

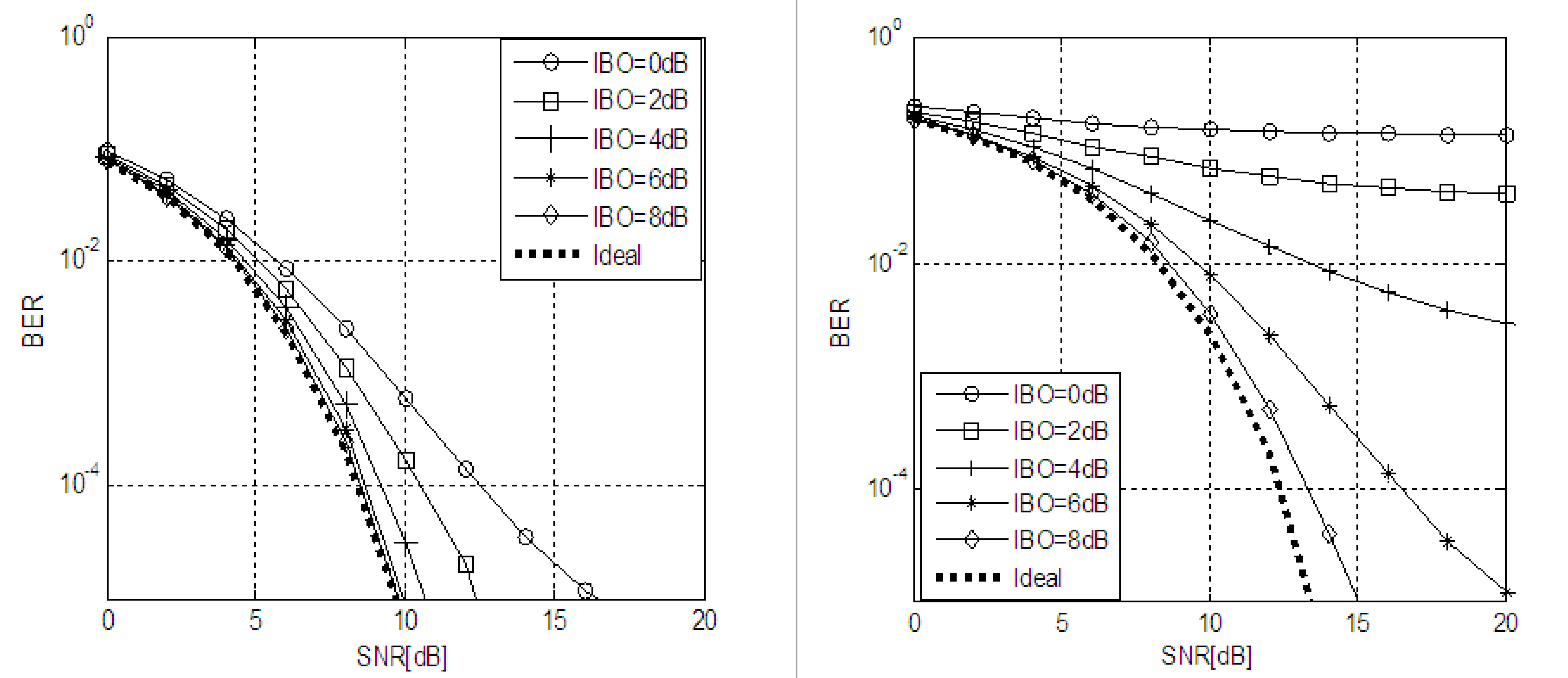
Average efficiency vs. PAPR [dB] for a Class-A Type Power Amplifier



Average PA Efficiency with PAPR Reduction N=256

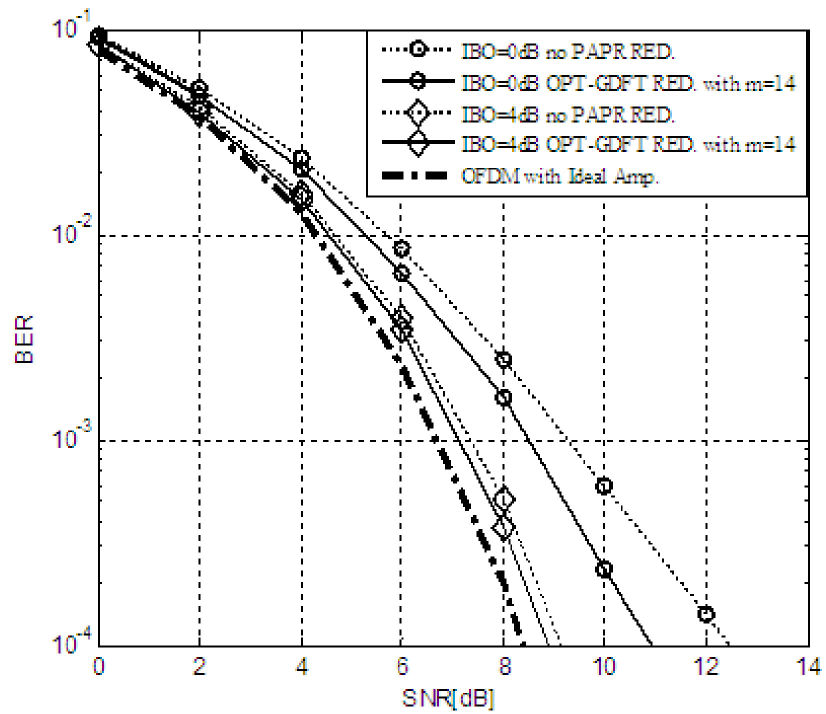


BER with PA Nonlinearities for N=256
(a) QPSK, (b) 16-QAM

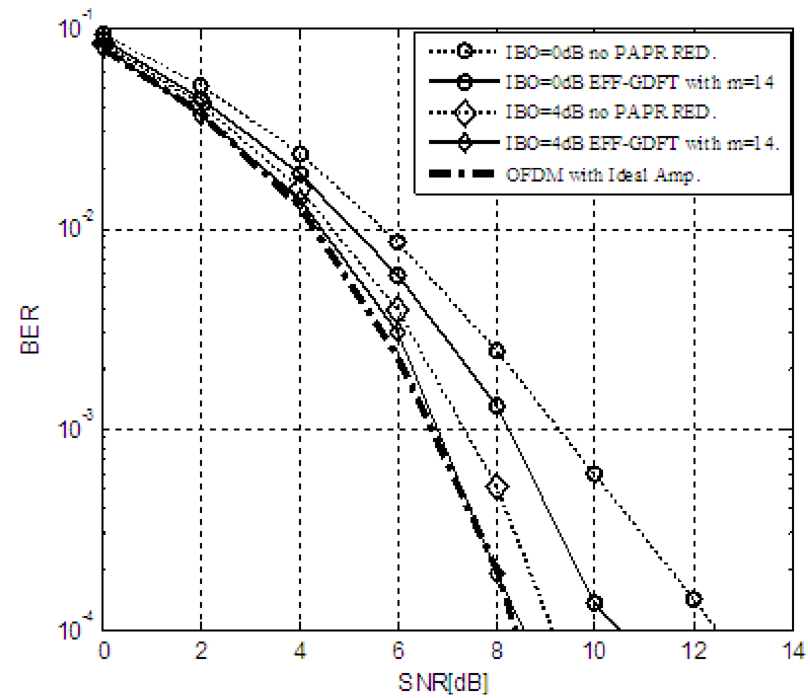


(a) (b)

BER with PA Nonlinearities for N=256 (QPSK)
(a) OPT-GDFT (b) EFF-GDFT

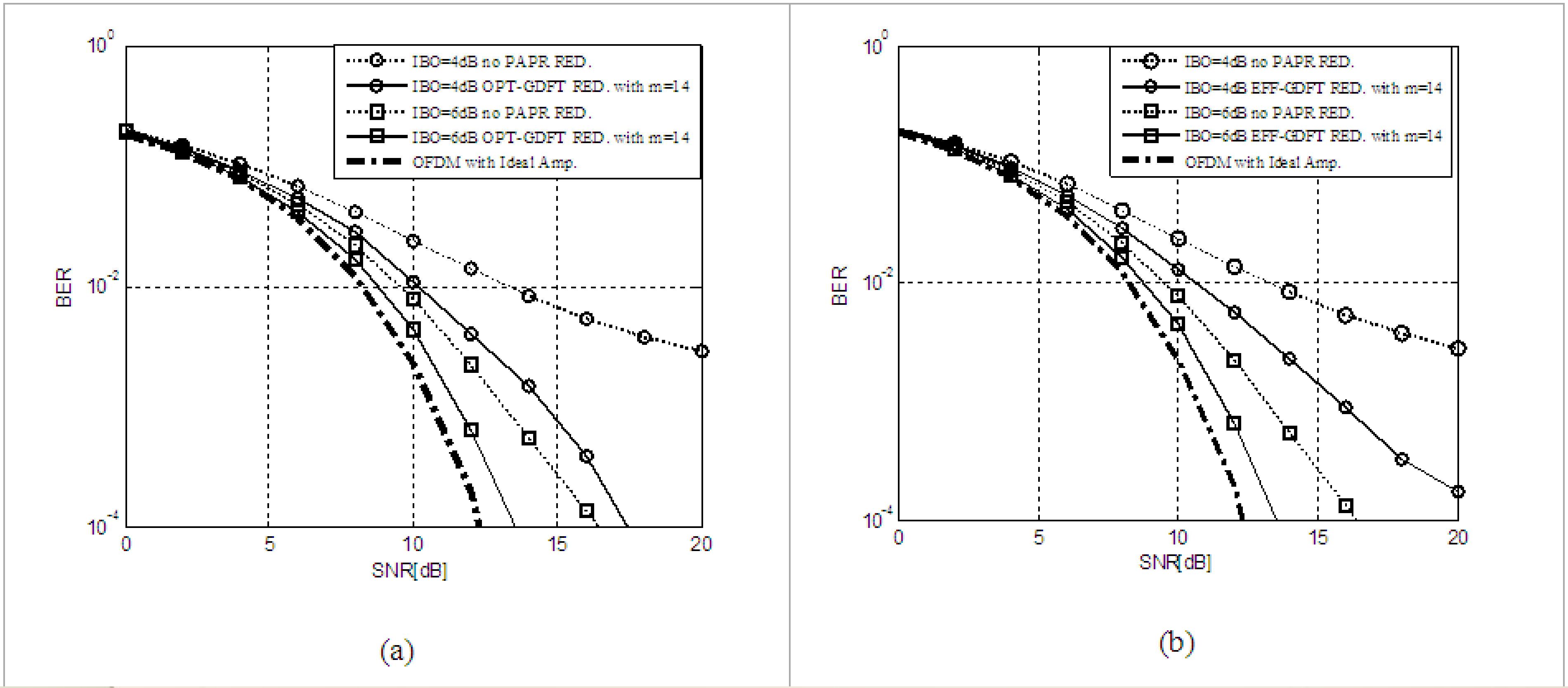


(a)



(b)

BER with PA Nonlinearities for N=256 (16-QAM)
(a) OPT-GDFT (b) EFF-GDFT



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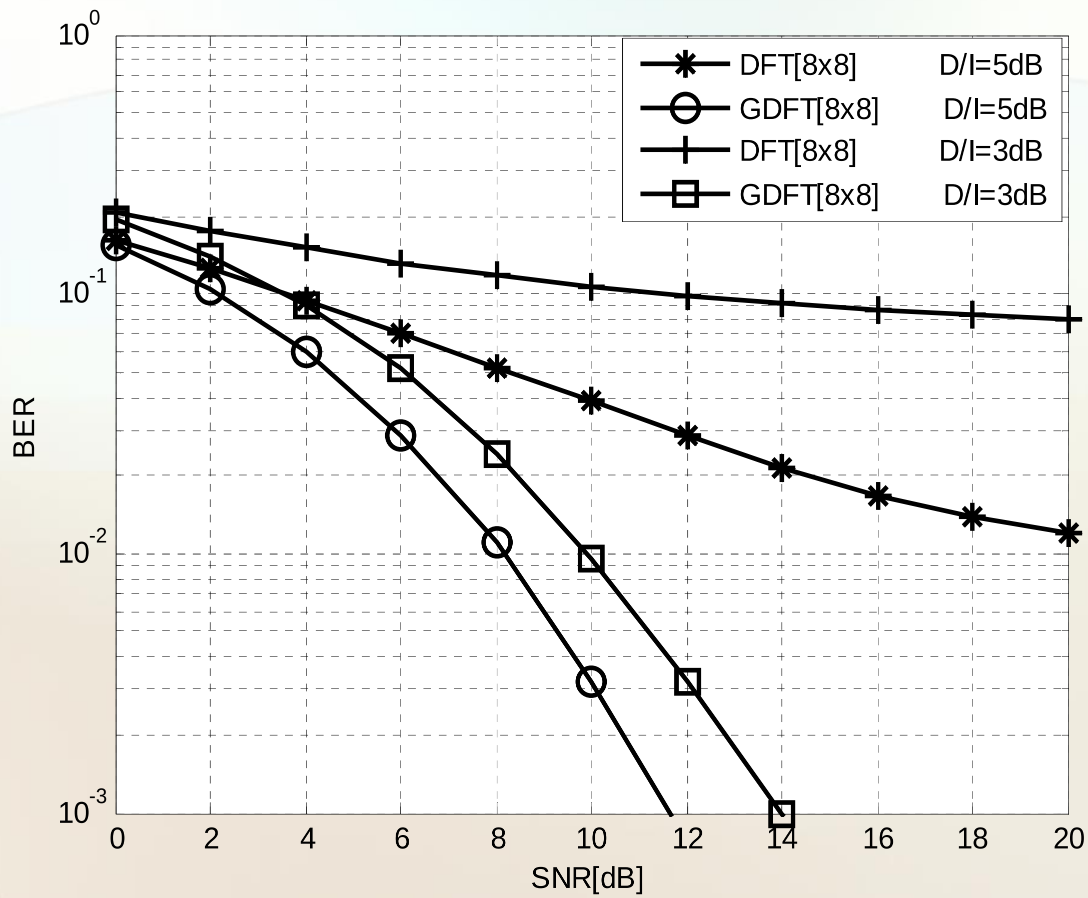
Potential GDFT Applications

- PAPR Reduction
- PAPR-ISI-ICI-Spectrum and Power Efficiency Trade-offs
- OFDM Variations SC-OFDM/MC-OFDM/DS-CDMA-OFDM and LTE Types
- Scrambling & Cryptography
- Basis Hopping

VI. BER Performance of DFT and GDFT CDMA for Rayleigh channel and 2 users



VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA

In DS/CDMA, each user assigned a spreading code. The transmitted signal uses the entire frequency band. Spreading is performed in time-domain.

$$y(t) = \sum_{k=1}^{K} A_k b_k(t) s_k(t) + n(t)$$

$$y(t) = \sum_{k=1}^{K} A_k b_k(t-\tau_k) s_k(t-\tau_k) + n(t) \qquad t \in [0, T]$$

$$Z_i = b_i(0) + \sum_{k \neq 1}^{K} b_k(-1)\rho_{ki}(\tau_k) + b_k(0)\hat{\rho}_{ki}(\tau_k)$$

$$\rho_{ki}(\tau) = \int_0^{\tau} s_i(t) s_k(t-\tau) dt$$

$$\hat{\rho}_{ki}(\tau) = \int_{\tau}^{T} s_k(t-\tau) s_i(t) dt$$

VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA

$$d_{k,i}(l) = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1-l} a_k(n)a_i(n+l), & 0 < l \le N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+l} a_k(n-l)a_i(n), & 1-N < l \le 0 \\ 0, & |l| \ge N \end{cases}$$

$$\rho_{k,i}(\tau) = d_{k,i}(l-N) + [d_{k,i}(l+1-N) - d_{k,i}(l-N)](\tau - l)$$

$$\hat{\rho}_{k,i}(\tau) = d_{k,i}(l) + [d_{k,i}(l+1) - d_{k,i}(l)](\tau - l) \qquad 0 \le l < \tau < 1 + l \le N$$

$$\theta_{k,i}(l) = \sum_{n=0}^{N-1} a_k(n)a_i(n+l)$$

$$\theta_{k,i}(l) = d_{k,i}(l) + d_{k,i}(l-N)$$

$$\hat{\theta}_{k,i}(l) = d_{k,i}(l) - d_{k,i}(l-N) \quad \text{for } 0 \le l \le N$$

VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA

Multiple user interference in DS/CDMA system depends on the even and odd correlations between the user spreading codes.

If $b_k(-1) \neq b_k(0)$

$$Z_i = b_i(0) + \sum_{k \neq i}^{K} b_k(0)\hat{\theta}_{k,i}(l_k) + [\hat{\theta}_{k,i}(l_k+1) - \hat{\theta}_{k,i}(l_k)](\tau_k - l_k)$$

If $b_k(-1) = b_k(0)$

$$Z_i = b_i(0) + \sum_{k \neq i}^{K} b_k(0)\theta_{k,i}(l_k) + [\theta_{k,i}(l_k+1) - \theta_{k,i}(l_k)](\tau_k - l_k)$$

VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA

- With the advances in Digital Signal Processing and success of Multicarrier Modulation in early broadcast applications motivated researchers to investigate the suitability of multicarrier modulation in mobile wireless communications.
- Multicarrier CDMA and MC-DS/CDMA are introduced emerging spreading spectrum techniques with multicarrier modulation to serve multiple users even on frequency selective channels.
- In MC-CDMA, the spreading is performed in frequency domain whereas in MC-DS/CDMA, the spreading is in time-domain. In both methods, all users shared the same available bandwidth simultaneously

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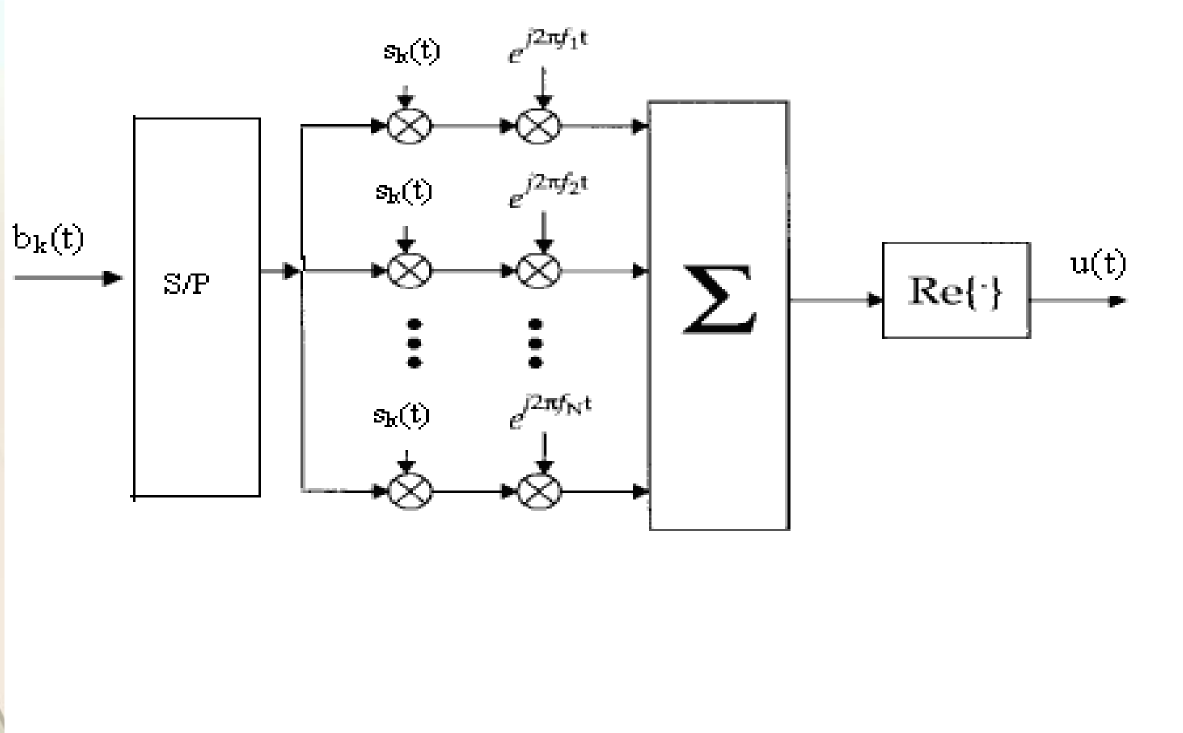
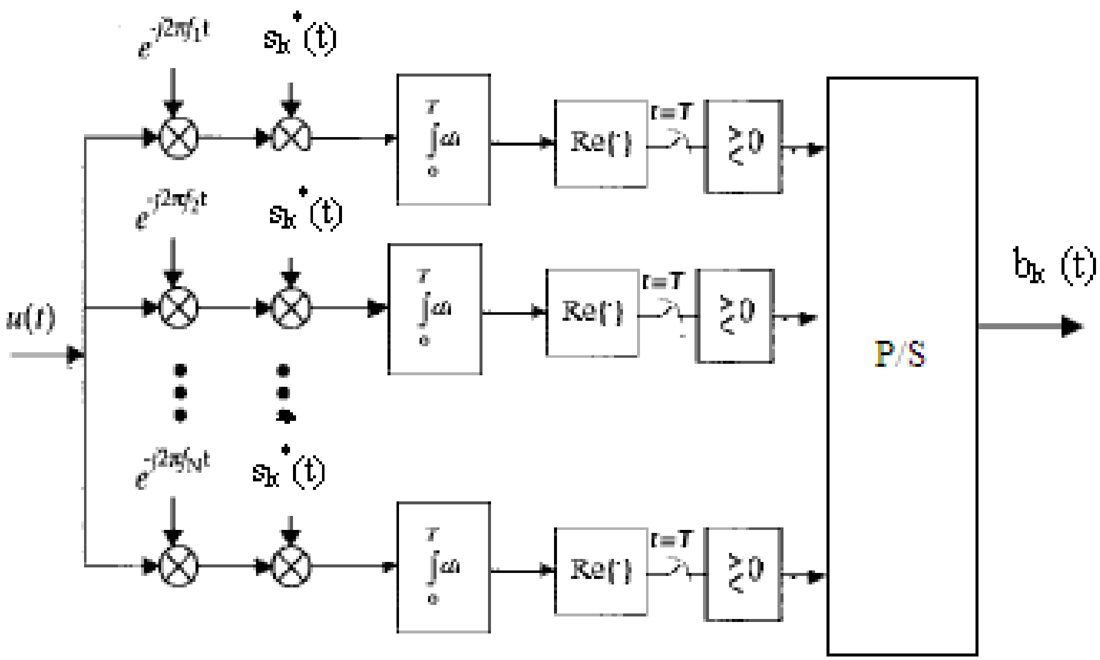
VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA

MC-DS-CDMA offers better performance on Rayleigh channel due to the use of orthogonal sub-carriers each having equally spaced bandwidth in the frequency. MC-DS-CDMA may be considered as N-channel DS-CDMA system.



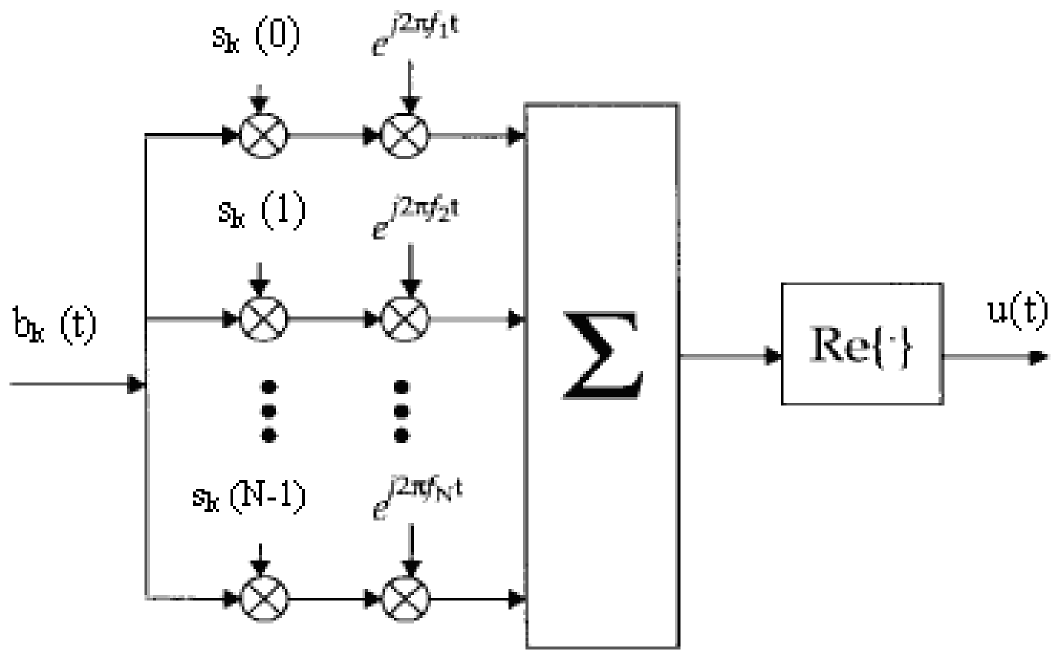
MC-DS-CDMA Transmitter

VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA



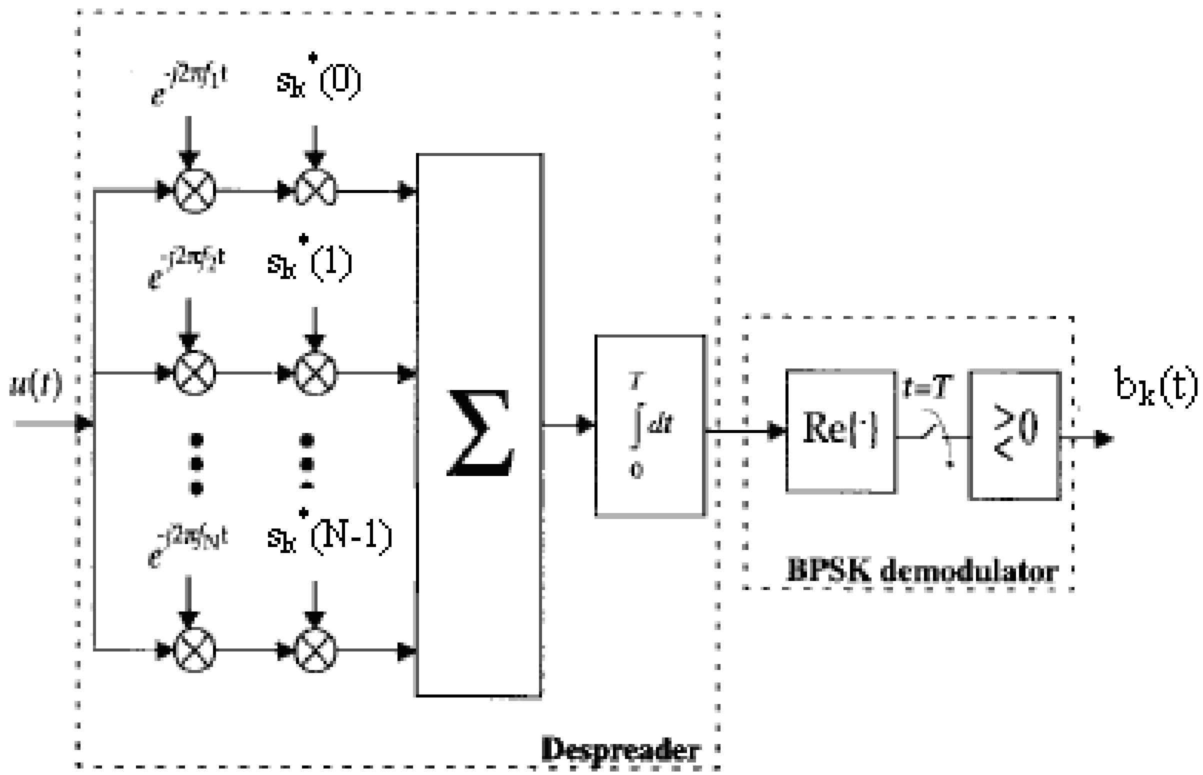
MC-DS-CDMA Receiver

VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA



MC-CDMA
Transmitter

VII. Variations of CDMA Communications: From DS-CDMA to MC-DS-CDMA



MC-CDMA Receiver

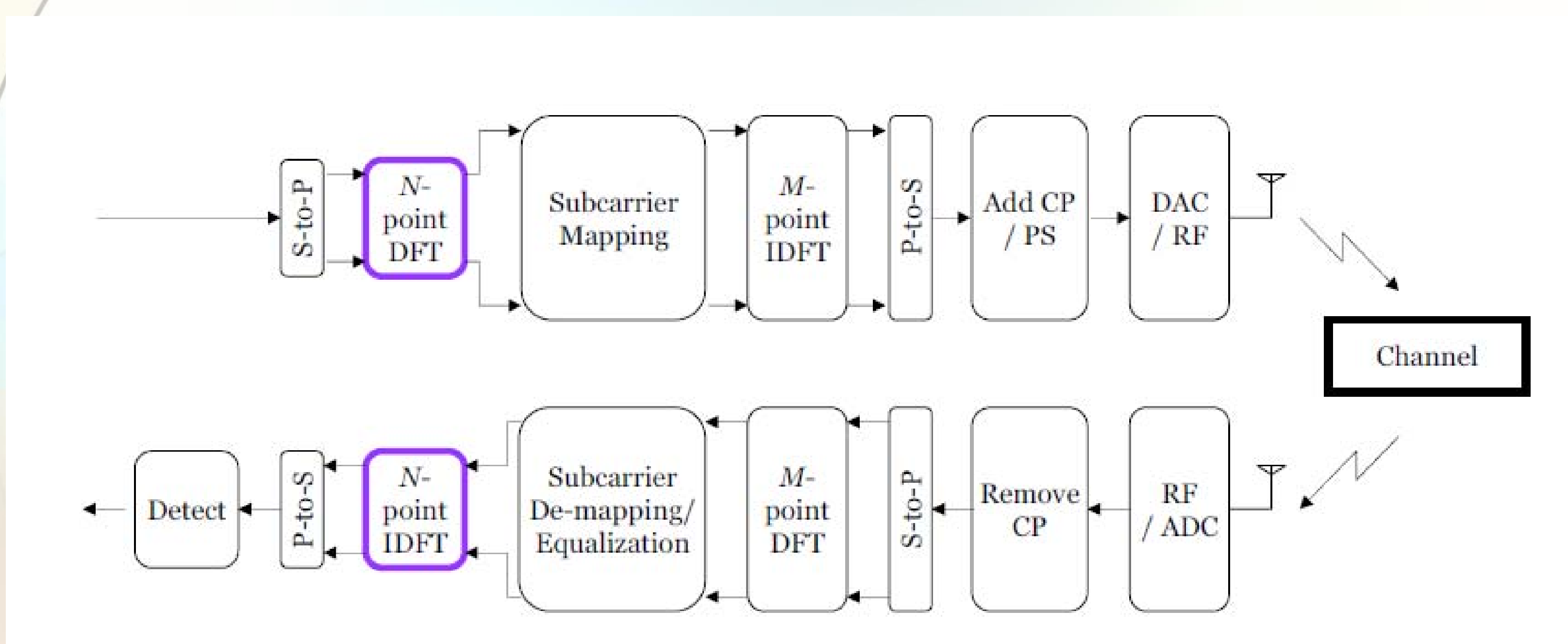
VIII. Emerging 3GPP LTE Mobile Phone Standard

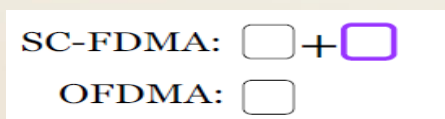
- 4G Technologies aim to increase cell capacity, cell radius, scalability of bandwidth and data rates deploying a completely new technology or emerging existing 3G networks.
- Two parallel standardization efforts are IEEE 802.16 (Wimax) and 3GPP LTE.

VIII. Emerging 3GPP LTE Mobile Phone Standard

- Both standards use OFDMA for downlink. Wimax uses OFDM for uplink whereas 3GPP LTE suggest using SC-FDMA for uplink. SC-FDMA is preferred due to its advantage of low PAPR over OFDM systems. Both employ MIMO with multiple antennas for UL and DL.

VIII. Emerging 3GPP LTE Mobile Phone Standard





*

* H. G. Myung and D. J. Goodman, *Single Carrier FDMA: A New Air Interface for Long-Term Evolution*. John Wiley & Sons, Nov. 2008.

VIII. Emerging 3GPP LTE Mobile Phone Standard

- Roaming framework for Mobile Wimax is completely new whereas 3GPP LTE is based on existing GSM/UMTS communications systems.
- Legacies for Wimax is IEEE 802.16a through IEEE 802.16d, the legacies for 3GPP LTE GSM, GPRS, UMTS, EGPRS, HSPA.

VIII. Emerging 3GPP LTE Mobile Phone Standard

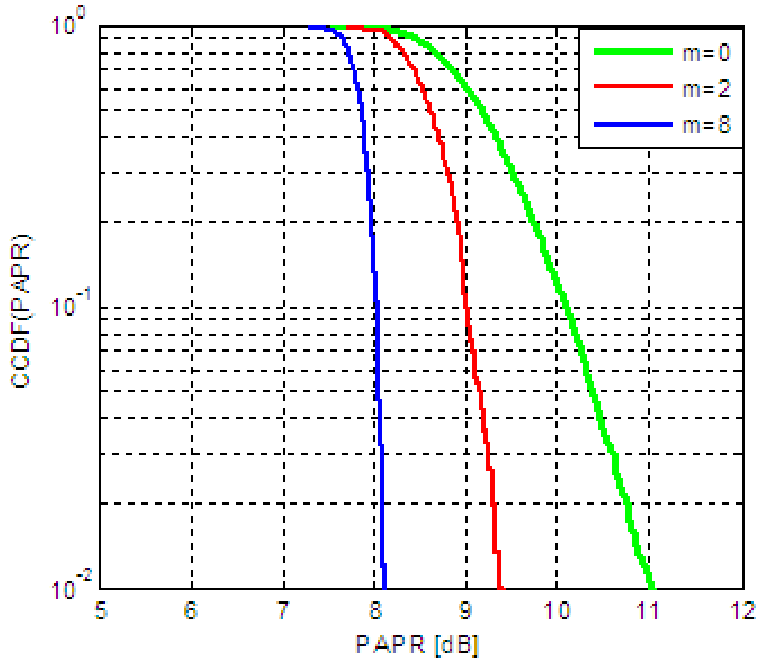
- Nokia Siemens Network, Motorola, Ericson, Freescale Semiconductor and NTT DoCoMo demonstrated successful implementations of LTE networks.
- AT&T, T-Mobile, Verizon Wireless, Vodafone, France Telecom are among those companies which announced their intension to upgrade their current networks to LTE.

VIII. Emerging 3GPP LTE Mobile Phone Standard

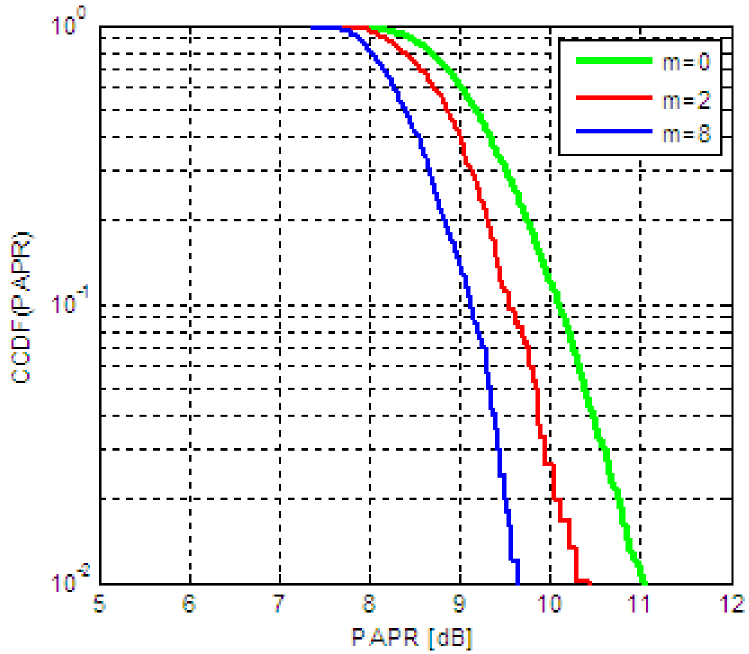
SPECS OF WIMAX SYSTEM SIMULATED

FFT size	1024
Number of data subcarriers	720
Number of pilot subcarriers	120
Number of null-Guard subcarriers	184
Channel bandwidth	10MHz

VIII. Emerging 3GPP LTE Mobile Phone Standard



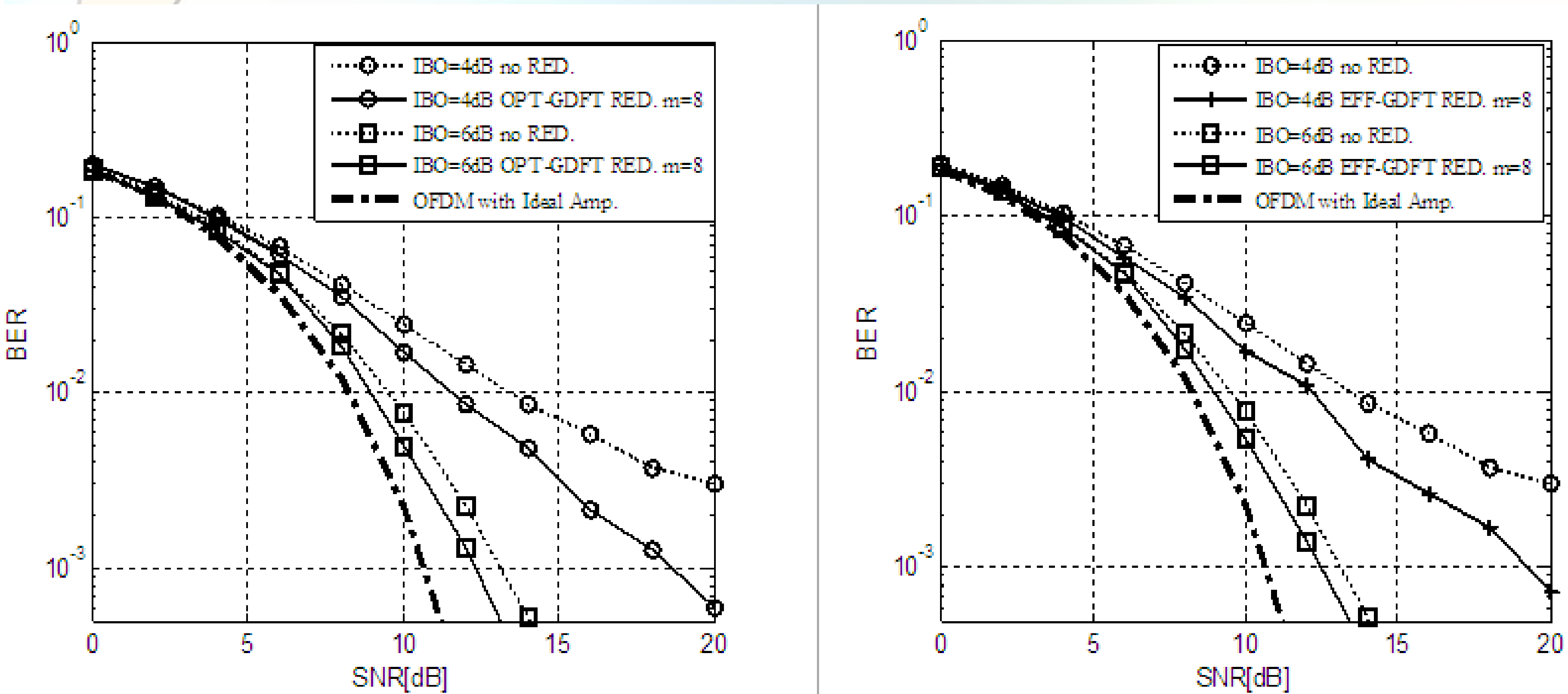
(a)



(b)

CCDF of PAPR for Mobile Wimax System with GDFT

VIII. Emerging 3GPP LTE Mobile Phone Standard



(a) *(b)*

BER Performance of Mobile Wimax System with GDFT

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IX. Discussions and Future Research Directions

- GDFT is a continuation of early work on Nonlinear Phase Walsh-like transforms (binary phase grid)
- We introduced Generalized DFT (GDFT) framework with Nonlinear Phase and Efficient Design Methods (any phase grid) for design of constant modulus sets
- Marked departure from block-circulant correlation and eigen-structures of DFT

IX. Discussions and Future Research Directions

- Methodically interconnects constellation, DFT and OFDM frame of interest
- Graceful departure from OFDM to CDMA or any TF-MA
- Basis hopping for better fit to channel (loading) and/or code level security (built-in scrambler) is inherent

IX. Discussions and Future Research Directions

- Design flexibilities for possible improvements (e.g. BER, PAPR) over DFT with efficient add on to FFT
- Next Generation Multicarrier Communications Systems (SW based) might benefit from GDFT family
- Currently looking into radar applications including MIMO Radar employing GDFT concepts

IX. Discussions and Future Research Directions

- Extensions to filters banks being studied for complex MUX
- Hilbert pair interpretation of a GDFT subset being formalized

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