

RISK MANAGEMENT FOR TRADING IN MULTIPLE FREQUENCIES

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ABSTRACT

We present fundamental concepts of risk and propose two methods for risk management of a portfolio in this paper. Moreover, we introduce their novel extensions to trading in multiple frequencies. We use stocks listed in NASDAQ 100 index as the investment universe for our back-testing to highlight the merit of the proposed portfolio risk management methods.

Index Terms— Risk Measurement, Risk Management, Multiple Frequency Trading

1. INTRODUCTION

Risk management is an essential layer of an asset trading system. Variance of the portfolio return is a widely used metric for investment risk. Investors want to keep their portfolios balanced according to the cross-correlation of assets to lower the risk. For example, the scenario of being long in two assets (buying the assets due to the expectation that the prices will increase) that are positively correlated and move simultaneously in the same direction is a highly risky bet and should be avoided. In practice, size of the portfolio is much larger than two assets. In order to construct the empirical correlation matrix for N assets, a number of $N(N-1)/2$ unknowns need to be estimated using a limited amount of data. However, defining the time-frame for the estimation is a challenging problem since a short time frame results in unstable estimation whereas a long time frame misses time-local events. Moreover, it was shown that the empirical correlation matrix contains a significant amount of measurement noise and that Principal Component Analysis (PCA), also called eigen filtering in the literature, offers a practical noise filtering method prior to risk calculations [1].

In Markowitz's modern portfolio theory, risk is calculated and used in the portfolio management layer. The portfolio is formed by minimizing the risk under the constraint that the portfolio return is a desired constant [2]. We propose two

risk management methods used to track risk dynamics of a portfolio. In reality, different assets in a portfolio might be rebalanced at different time points. Moreover, Epp's effect is a well known phenomenon in finance where an increase in sampling rate leads to a drop in cross-correlation of asset returns [3]. Therefore, we introduce novel extensions of risk management methods to the scenario of trading in multiple frequencies. The merit of the proposed methods is shown by Profit and Loss (PNL) back-testing performance for stocks listed in NASDAQ 100 index.

2. RISK MEASUREMENT

2.1. Definition of Risk

Consider an investment portfolio comprised of two assets. The return of this two-asset portfolio is expressed as

$$R_p(n) = q_1(n)R_1(n) + q_2(n)R_2(n) \quad (1)$$

where n is the discrete time variable, q_i is the amount invested in the i^{th} asset, and R_i is the return of the i^{th} asset defined as

$$R_i(n) = \frac{P_i(n)}{P_i(n-1)} - 1 \quad (2)$$

where $P_i(n)$ is the price of the asset at discrete time n . We omit the time index n in further discussions, knowing that each variable in an equation is a function of time. The investment amount, q_i in Eq. 1, can be dimensionless or its unit may be a currency. This choice reflects itself into the unit of measured risk. The risk of a two-asset portfolio is defined as the standard deviation of the portfolio return where its variance is expressed as

$$\sigma_p^2 = E[R_p^2] = q_1^2\sigma_1^2 + 2q_1q_2\sigma_1\sigma_2\rho_{12} + q_2^2\sigma_2^2 \quad (3)$$

σ_p is the risk of the portfolio, σ_i is the volatility, i.e. standard deviation of the returns of the i^{th} asset, and ρ_{ij} is the correlation coefficient between the returns of i^{th} and j^{th} assets. It is straightforward to generalize this concept to a portfolio

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consisting of N assets. The return of the N -asset portfolio is expressed as

$$R_p = \sum_{i=1}^N q_i R_i \quad (4)$$

Similarly, variance of an N -asset portfolio is calculated from

$$\sigma_p^2 = E [R_p^2] = \mathbf{q}^T \Sigma^T \mathbf{C} \Sigma \mathbf{q} = \sum_{i=1}^N \sum_{j=1}^N q_i q_j \rho_{ij} \sigma_i \sigma_j \quad (5)$$

where superscript T is the matrix transpose operator, $\mathbf{q} = [q_1 \ q_2 \ \dots \ q_N]^T$ is the investment vector, Σ is an N -by- N diagonal matrix with the elements corresponding to the volatility of individual assets, and \mathbf{C} is the correlation matrix with the elements $[C]_{ij} = \rho_{ij}$. Note that all elements on the main diagonal of \mathbf{C} are equal to one. Furthermore, \mathbf{C} is a symmetric and non-negative definite matrix.

2.2. Eigen Filtering of Noise in Measured Correlation Matrix

Let us decompose the measured correlation matrix into its eigenvectors with corresponding eigenvalues. We can express the correlation matrix \mathbf{C} as

$$\mathbf{C} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T \quad (6)$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ is a diagonal matrix with the eigenvalues as its elements, λ_k as the k^{th} eigenvalue and $\lambda_k \geq \lambda_{k+1}$. $\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_N]$ is an N -by- N matrix composed of N eigenvectors as its columns, and \mathbf{v}_k is the N -by-1 eigenvector corresponding to the k^{th} eigenvalue, λ_k . Note that due to the non-negative definite property of the matrix \mathbf{C} , all eigenvalues are non-negative, i.e. $\lambda_k \geq 0 \ \forall k$. Eq. 6 is substituted into Eq. 5, and the variance of the portfolio is formulated as

$$\sigma_p^2 = \mathbf{q}^T \Sigma^T \mathbf{V} \mathbf{\Lambda} \mathbf{V}^T \Sigma \mathbf{q} \quad (7)$$

We follow the approach of using eigen filtered correlation matrix as

$$\tilde{\mathbf{C}} = \sum_{k=1}^L \lambda_k \mathbf{v}_k \mathbf{v}_k^T + \mathbf{E} \quad (8)$$

where L is the number of selected factors (eigenvalues) with $L \ll N$, and \mathbf{E} is a diagonal noise matrix added into the equation to preserve the total variance. The elements of \mathbf{E} are defined as

$$[E]_{ij} = \epsilon_{ij} = \begin{cases} 1 - \sum_{k=1}^L \lambda_k v_i^{(k)} v_j^{(k)} & i = j \\ 0 & i \neq j \end{cases} \quad (9)$$

where $v_i^{(k)}$ is the i^{th} element of the k^{th} eigenvector. Note that the error term of Eq. 9 is required to keep the trace of

the correlation matrix, $\tilde{\mathbf{C}}$, to be equal to N . From Eqs. 8 and 9 we express the elements of noise filtered correlation matrix as

$$[\tilde{\mathbf{C}}]_{ij} = \tilde{\rho}_{ij} = \sum_{k=1}^L \lambda_k v_i^{(k)} v_j^{(k)} + \epsilon_{ij} \quad (10)$$

Let us substitute the filtered version of ρ_{ij} , Eq. 10, into Eq. 5, and we have

$$\tilde{\sigma}_p^2 = \sum_{k=1}^L \lambda_k \left(\sum_{i=1}^N q_i v_i^{(k)} \sigma_i \right)^2 + \sum_{i=1}^N \epsilon_{ii} q_i^2 \sigma_i^2 \quad (11)$$

It is intuitive to expect more robust risk estimation due to the eigen filtering of noise in measured statistics. Note that specifying L , the number of significant factors, is also an important task [1].

3. RISK MANAGEMENT

Consider an independent investment strategy constantly rebalancing a given portfolio according to a performance metric. Our goal is to manage the portfolio risk by filtering the decisions of the underlying investment strategy based on a predetermined risk limit. We propose two risk management methods called ‘‘stay in the ellipsoid (SIE)’’ and ‘‘stay on the ellipsoid (SOE)’’, respectively. Although the first method is simpler to implement, the performance of the second one is superior.

3.1. Stay in the Ellipsoid Algorithm

The geometric interpretation of Eq. 5 is of an ellipsoid in N dimensional space. This ellipsoid is centered at the origin, and its shape is defined by the correlation matrix \mathbf{C} . Depending on the investment vector \mathbf{q} , portfolio can be in, out of, or on the ellipsoid. SIE tries to keep the portfolio risk anywhere inside the predefined risk ellipsoid by checking the risk of the target portfolio, and rejecting any new investment positions that violate this requirement. We assume that once a *signal to enter* a new position is rejected by the risk manager, the underlying strategy does not create another signal until the *signal to exit* is generated. SIE risk management method is expressed as

$$\mathbf{q}_{t+\Delta t} \leftarrow \begin{cases} \mathbf{q}_{t+\Delta t} & \sigma_{t+\Delta t} < \sigma_{MAX} \\ \mathbf{q}'_{t+\Delta t} & \sigma_{t+\Delta t} \geq \sigma_{MAX} \end{cases} \quad (12)$$

where Δt is the time interval between the two re-balancing of the portfolio, σ_{MAX} is a predetermined maximum allowable risk threshold, and $\mathbf{q}'_{t+\Delta t}$ is the modified investment vector achieved according to the new investment rules as stated

$$[q'_{t+\Delta t}]_i = \begin{cases} 0 & q_{t,i} = 0 \text{ and } |q_{t+\Delta t,i}| > 0 \\ q_{t+\Delta t,i} & \text{otherwise} \end{cases} \quad (13)$$

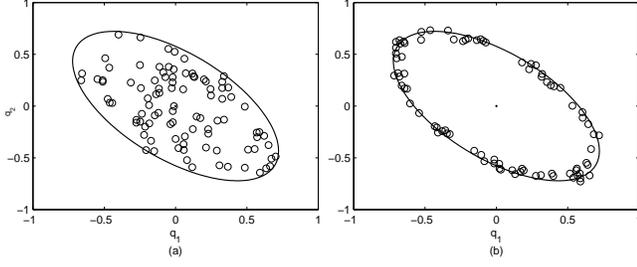


Fig. 1. (a) Stay in the ellipsoid (SIE), (b) Stay on the ellipsoid (SOE) risk management methods.

where $q_{t,i}$ is the investment amount in the i^{th} asset at time t . It is observed from Eqs. 12 and 13 that the proposed method rejects any new investment position in the target portfolio when it has a target risk higher than the risk limits.

3.2. Stay on the Ellipsoid Algorithm

We propose the second risk management method which tries to keep the portfolio risk not only in the ellipsoid but also as close to it as possible. The difference between the two proposed methods is observed from Fig. 1 for the case of a two-asset portfolio. The second method maximizes the utilization of risk limits and is formulated as follows

$$\mathbf{q}_{t+\Delta t} \leftarrow \begin{cases} \mathbf{q}_{t+\Delta t} & \sigma_{t+\Delta t} < \sigma_{MAX} \\ \mathbf{q}'_{t+\Delta t} & \sigma_{t+\Delta t} \geq \sigma_{MAX} \end{cases} \quad (14)$$

where $\mathbf{q}'_{t+\Delta t}$ might be modified using various search algorithms minimizing the risk distance

$$\mathbf{q}'_{t+\Delta t} = \arg \min_{\mathbf{q}} |\sigma_{MAX} - \sigma(\mathbf{q})| \quad (15)$$

$\sigma(\mathbf{q})$ is the calculated risk for investment vector \mathbf{q} given that its elements are limited to

$$q_i \in \begin{cases} \{0, q_{t+\Delta t,i}\} & q_{t,i} = 0 \text{ and } |q_{t+\Delta t,i}| > 0 \\ \{q_{t+\Delta t,i}\} & \text{otherwise} \end{cases} \quad (16)$$

where the notation $\{\cdot\}$ defines a set.

Eqs. 15 and 16 suggest to look for a specific combination of signals to open new investment positions taking the portfolio risk level as close to risk limits as possible. The intuition here is to maintain a relatively dynamic and diverse portfolio while keeping the risk within a desired limit. For the two-asset portfolio case, solution for the optimization problem is trivial. However, optimization in an N -asset portfolio might become computationally intensive, especially when N is large.

4. TRADING IN MULTIPLE FREQUENCIES

Trading at multiple frequencies, i.e. re-balancing the investment amounts in different assets at different times, may be de-

sirable for some investors. For example, an asset might show no trend when sampled at every m time units, but a larger trend might become visible when it is sampled at every n time units. Hence, an investor who employs a trend-following strategy, might explore different frequencies for different assets. Therefore, risk management framework is extended into trading at various frequencies.

Assuming that the asset prices change according to the geometric Brownian motion where volatilities measured at different sampling frequencies (trading frequencies) have the following relationship [4]

$$\sigma_{k_i} = \sqrt{k/l} \sigma_l = m \sigma_l \quad (17)$$

where σ_k and σ_l are the volatilities measured at sub-sampling rates kT_s and lT_s with $l > k$, respectively, and T_s is the base sampling rate. Hence, it is possible to measure portfolio risk at a certain frequency, and manage risk of assets by re-balancing at different sampling rates, by modifying the original risk formula given in Eq. 5 as follows

$$\sigma_p^2 = \mathbf{q}^T \Sigma^T \mathbf{M}^T \mathbf{C} \mathbf{M} \Sigma \mathbf{q} \quad (18)$$

where $\mathbf{M} = \text{diag}(m_1, m_2, \dots, m_N)$ and m_i is the scaling factor of Eq. 17 for asset i , provided that Σ and \mathbf{C} matrices are estimated at lT_s sampling rate. This framework not only allows re-balancing of a portfolio by opening positions in multiple frequencies but also provides investors the flexibility of using empirical correlation and volatility matrices estimated at different sampling rates. Due to the Epp's effect [3], \mathbf{C} must be estimated at a reasonably low frequency although the portfolio might be re-balanced at higher trading frequencies.

5. BACK-TESTING AND PNL PERFORMANCE

The investment universe for back-testing performance reported in this paper consists of stocks listed in NASDAQ 100 index. The time span considered is from April 1, 2010 to May 31, 2010 with the sampling rate of 5 minutes. Correlation matrix is estimated at each sample by looking at returns of past multiple days. A simple trading strategy generating 50% long orders and 50% short orders in the course of the day is employed. At each entering point 4% of the capital is invested in a particular stock.

PNL curve for the test strategy without any risk management method is displayed in Fig. 2.a (solid line). Similarly, PNL curves for the risk managed cases are displayed in Fig. 2.a with dashed and dash-dot lines for SIE and SOE methods, respectively. In both methods, risk threshold is set to 3 bps / sample (~25 bps / day). The day after the flash crash of May 6, 2010 [5] is of special interest to us since risk managed strategies avoid the 1.8% draw-down that the strategy without any risk management suffered. The measured risk values are displayed in Fig. 2.b for all scenarios considered. It is observed

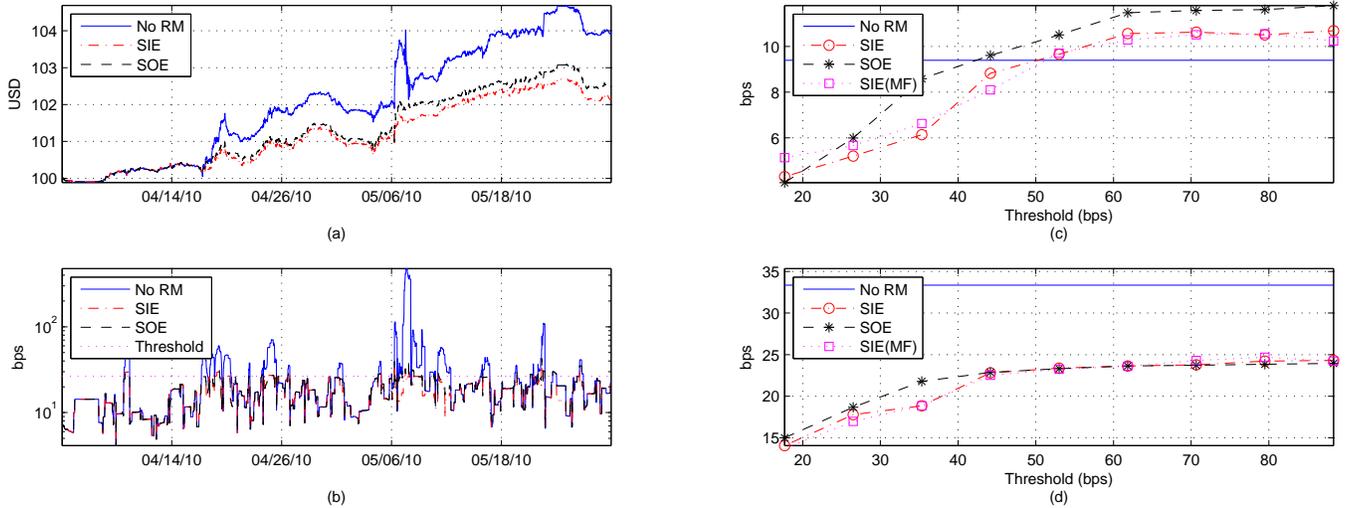


Fig. 2. (a) PNLs for no risk management case along with the proposed Stay in the Ellipsoid (SIE), and Stay on the Ellipsoid (SOE) based risk management methods. Their expected daily returns are of 9.4 bps, 5.2 bps, and 6 bps, and daily volatilities are of 33.4 bps, 17.8 bps, and 18.7 bps, respectively, (b) Corresponding measured daily risks, (c) Expected daily return versus daily risk threshold for SIE, SOE, and multiple frequency SIE methods along with the expected daily return of no risk management case, and (d) Daily volatilities.

from the figures that SOE method outperforms SIE method in terms of the daily expected return while keeping the daily volatility at a desired level. Both methods perform well in terms of keeping the portfolio risk bounded with trade-off of reduced return. However, a less risk-averse investor may easily set the daily risk threshold to a higher level to increase the daily expected return.

This experiment is repeated by changing the risk threshold from 2 to 10 bps / sample (from ~ 17 bps / day to ~ 88 bps / day). Expected daily return and daily volatility of the PNLs for SIE and SOE methods are shown in Fig. 2.c and Fig. 2.d, respectively. The daily expected return and daily volatility of the strategy without any risk management are also presented for comparison purposes. It is observed from these performance figures that the SOE algorithm yields significantly higher returns with a negligible increase in volatility than the others for a given risk level. The PNL performance for the multiple frequency trading strategy formulated in Eqs. 17 and 18 with the sub-sampling rates of $k = 1$, $l = 3$, and $T_s = 5 \text{ min}$ is displayed in Fig. 2.c and Fig. 2.d. Note that, in this scenario, all the assets in the portfolio are traded at the same frequency although the framework introduced in this paper allows investors to trade different assets at different frequencies. The trivial multiple frequency trading results are presented to highlight the flexibility of the proposed framework rather than emphasizing a superior performance. It is evident from the figures that the portfolio risk management and re-balancing may be performed at multiple trading frequencies with more flexibility by utilizing the framework proposed in this paper.

6. CONCLUSIONS

We proposed two portfolio risk management methods in this paper. Moreover, we presented their extensions to the case of trading in multiple frequencies. The merit of the proposed methods is highlighted by their PNL back-testing performance for the investment universe of stocks listed in NASDAQ 100 index.

7. REFERENCES

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