An Efficient Method to Derive Explicit KLT Kernel for First-Order Autoregressive Discrete Process

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***Abstract*—Signal dependent Karhunen-Loève transform (KLT), also called factor analysis or principal component analysis (PCA), has been of great interest in applied mathematics and various engineering disciplines due to optimal performance. However, implementation of KLT has always been the main concern. Therefore, fixed transforms like discrete Fourier (DFT) and discrete cosine (DCT) with efficient algorithms have been successfully used as good approximations to KLT for popular applications spanning from source coding to digital communications. In this paper, we propose a simple method to derive explicit KLT kernel, or to perform PCA, in closed-form for first-order autoregressive, AR(1), discrete process. It is a widely used approximation to many real world signals. The merit of the proposed technique is shown. The novel method introduced in this paper is expected to make real-time and data-intensive applications of KLT, and PCA, more feasible.**

***Index Terms*—Covariance analysis, principal component analysis (PCA), eigenanalysis, factor analysis, first-order autoregressive process, signal dependent transform, explicit Karhunen-Loève Transform (KLT) kernel.**

I. Introduction

Orthogonal block transform has been one of the pillars of engineering mathematics that found its high impact use in popular technologies including digital communications, image and video compression, search engines, finance, and social network analytics. The Karhunen-Loève transform (KLT), also called principal component analysis (PCA), is the optimal block transform where its basis functions are generated based on a given signal covariance matrix. Hence, it is a signal dependent transform. KLT has three steps to implement with a computational cost attached to each one. First, the statistical measurement of the random vector process is performed in order to define the covariance matrix. Second, eigenvectors (eigenmatrix) and eigenvalues for the given covariance matrix are calculated. And finally, incoming random signal vector is mapped to the eigenspace (subspace) by using the pre-calculated eigenmatrix for the given covariance matrix. In contrast, the celebrated transforms like discrete cosine transform (DCT) have their fixed kernels to define orthogonal matrix of size $N \times N$, and only require the third step regardless of signal statistics [1, 2]. Although the former offers the best orthogonal block transform, defining KLT basis for a given signal demands prohibitive computational resources in many cases [3]. Therefore, fixed transforms with efficient implementations and acceptable performance have been the most practical option for most engineering applications [2].

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Fast implementation of KLT is of great interest to several disciplines, and there were prior attempts to derive closed-form kernel expressions for certain classes of stochastic processes reported in the literature [4, 5]. In particular, such a kernel in its implicit form for processes with exponential correlation was reported in multiple references [3–8]. That form requires one to solve a transcendental tangent equation by using either numerical techniques with convergence concerns, or to implement complex methods for explicit expression of KLT kernel. In this paper, we revisit an efficient root finding method for a transcendental equation, and propose a simple method to derive explicit KLT kernel for first-order autoregressive, AR(1), discrete stochastic process. The derivation of explicit KLT kernel for AR(1) process, introduced first in its implicit form in [5], is also summarized in the paper. Furthermore, we present a detailed implementation procedure for the proposed technique in order to highlight its merit.

The structure of the paper is as follows. First, we treat the the fundamentals of orthogonal block transforms in the next section. Eigenanalysis of AR(1) signal model along with the implicit closed-form KLT kernel expression utilizing root locations of a transcendental tangent equation is highlighted in Sec. III. The mathematical derivation steps to arrive at the transcendental tangent equation of concern by analyzing the characteristic values and functions of a continuous process with exponential autocorrelation is presented in Sec. IV. In Sec. V, we summarize an explicit root finding method for transcendental equations introduced by Luck and Stevens in order to address the problem at hand [9]. Then, we introduce an efficient method to derive explicit expression for KLT kernel of a discrete AR(1) process in Sec. VI. In Sec. VII, we compare energy compaction performance of separable 2D KLT, derived by using the proposed kernel derivation method, and DCT for a test image. We also emphasize implementation advantages of the proposed method in this section. The contribution of the paper and concluding remarks are given in the last section.

II. Orthogonal Transforms

A family of linearly independent N orthonormal discrete-time sequences, $\{\phi_k(n)\}$, on the interval $0 \le n \le N-1$ satisfies the inner product relationship [2]

$$\sum_{n=0}^{N-1} \phi_k(n)\phi_l^*(n) = \delta_{k-l} = \begin{cases} 1, k=l \\ 0, otherwise \end{cases} . \quad (1)$$

Equivalently, the orthogonality is also expressed on the unit circle of the complex plane, $z = e^{j\omega}; -\pi \le \omega \le \pi$, as follows

$$\sum_{n=0}^{N-1} \phi_k(n)\phi_l^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_k(e^{j\omega})\Phi_l^*(e^{j\omega})d\omega = \delta_{k-l}, \quad (2)$$

where $\Phi_k(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of $\phi_k(n)$. In matrix form, $\{\phi_k(n)\}$ are the rows of the transform matrix, and also called basis functions

$$\Phi = [\phi_k(n)] : k, n = 0, 1, ..., N-1, \quad (3)$$

with the matrix orthogonality property stated as

$$\Phi\Phi^{-1} = \Phi\Phi^{*\mathrm{T}} = \mathbf{I}, \quad (4)$$

where $*\mathrm{T}$ indicates conjugated and transposed version of a matrix and $\mathbf{I}$ is $N \times N$ identity matrix. A signal vector

$$\mathbf{x} = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-1) \end{bmatrix}^{\mathrm{T}}, \quad (5)$$

is mapped into the orthonormal space (subspace) through forward transform operator

$$\boldsymbol{\theta} = \Phi\mathbf{x}, \quad (6)$$

where $\boldsymbol{\theta}$ is transform coefficients vector as given

$$\boldsymbol{\theta} = \begin{bmatrix} \theta(0) & \theta(1) & \cdots & \theta(N-1) \end{bmatrix}^{\mathrm{T}}. \quad (7)$$

Similarly, the inverse transform yields the signal vector

$$\mathbf{x} = \Phi^{-1}\boldsymbol{\theta}. \quad (8)$$

We assume that the vector $\mathbf{x}$ is populated by a wide-sense stationary (WSS) stochastic process. Then, we have

$$\begin{aligned} &E\{x(n)\} = \mu(n) = \mu \\ &E\{x(n)x^*(n+m)\} = R_{xx}(m) \end{aligned}, \quad (9)$$

where $E\{\cdot\}$ is the expectation operator and $R_{xx}(m)$ is the autocorrelation sequence of the WSS process $x(n)$. The correlation and covariance matrices of such a random vector process $\mathbf{x}$ are defined, respectively,

$$\begin{aligned} \mathbf{R}_x &= E\left\{\mathbf{x}\mathbf{x}^{*\mathrm{T}}\right\}, \\ \mathbf{R}_x &= \begin{bmatrix} R_{xx}(0) & R_{xx}(1) & \cdots & R_{xx}(N-1) \\ R_{xx}(1) & R_{xx}(0) & \cdots & R_{xx}(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(N-1) & R_{xx}(N-2) & \cdots & R_{xx}(0) \end{bmatrix}, \\ \mathbf{C}_x &= E\left\{(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^{*\mathrm{T}}\right\} = \mathbf{R}_x - \boldsymbol{\mu}\boldsymbol{\mu}^{*\mathrm{T}}, \end{aligned} \quad (10)$$

where

$$\boldsymbol{\mu}\boldsymbol{\mu}^{*\mathrm{T}} = \mu^2 \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \quad (11)$$

Note that $\mathbf{R}_x = \mathbf{C}_x$ for a zero-mean WSS process where $\mu = 0$. Hence, one can derive the covariance matrix of transform coefficients as follows

$$\begin{aligned} \mathbf{R}_\theta &= E\left\{\boldsymbol{\theta}\boldsymbol{\theta}^{*\mathrm{T}}\right\} \\ &= E\left\{\Phi\mathbf{x}\mathbf{x}^{*\mathrm{T}}\Phi^{*\mathrm{T}}\right\} \\ &= \Phi E\left\{\mathbf{x}\mathbf{x}^{*\mathrm{T}}\right\}\Phi^{*\mathrm{T}} \\ &= \Phi\mathbf{R}_x\Phi^{*\mathrm{T}}. \end{aligned} \quad (12)$$

Energy preserving property of an orthonormal transform allows the equality between signal variance and the average of transform coefficient variances to maintain the following equality

$$\begin{aligned} &E\left\{\boldsymbol{\theta}^{*\mathrm{T}}\boldsymbol{\theta}\right\} = E\left\{\mathbf{x}^{*\mathrm{T}}\mathbf{x}\right\}, \\ &E\left\{\boldsymbol{\theta}^{*\mathrm{T}}\boldsymbol{\theta}\right\} = \sum_{k=0}^{N-1} E\left\{\theta_k^2\right\} = \sum_{k=0}^{N-1}\sigma_k^2, \\ &E\left\{\mathbf{x}^{*\mathrm{T}}\mathbf{x}\right\} = \sum_{n=0}^{N-1}\sigma_x^2(n) = N\sigma_x^2, \\ &\sigma_x^2 = \frac{1}{N}\sum_{k=0}^{N-1}\sigma_k^2. \end{aligned} \quad (13)$$

We focus on covariance matrix of first-order autoregressive stochastic process in the next section of the paper. It will lead us to the proposed derivation method for explicit KLT kernel as follows.

III. Eigenanalysis of AR(1) Process

Random processes and information sources are mathematically described by a variety of signal models including autoregressive (AR), moving average (MA), and autoregressive moving average (ARMA) types. AR source models, also called all-pole models, have been successfully used in applications including speech processing for decades [10]. First-order AR model, AR(1), is a first approximation to many natural signals. It has been successfully employed in applications including the modeling of digital images [2, 11] and financial signals [12–15], as well as trend forecasting in economics [16]. AR(1) signal is generated through the first-order regression formula written as [2]

$$x(n) = \rho x(n-1) + \xi(n), \quad (14)$$

where $\xi(n)$ is a white noise sequence with zero-mean, i.e. $E\{\xi(n)\xi(n+k)\} = \sigma_\xi^2\delta_k$, $E\{\xi(n)\} = 0$. The first-order correlation coefficient, ρ, is real in the range of $-1 < \rho < 1$, and the variance of $x(n)$ is given as follows

$$\sigma_x^2 = \frac{1}{(1-\rho^2)}\sigma_\xi^2. \quad (15)$$

Autocorrelation sequence of $x(n)$ is expressed as

$$R_{xx}(k) = E\{x(n)x(n+k)\} = \sigma_x^2\rho^{|k|}; k = 0, \pm1, \pm2, \ldots. \quad (16)$$

The resulting Toeplitz correlation matrix of size $N \times N$ is shown as

$$\mathbf{R}_x = \sigma_x^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{N-1} \\ \rho & 1 & \rho & \cdots & \rho^{N-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \cdots & 1 \end{bmatrix}. \quad (17)$$

From linear algebra, it is known that an eigenvalue λ and an eigenvector $\boldsymbol{\phi}$ with size $N \times 1$ of a matrix $\mathbf{R}_x$ with size $N \times N$ must satisfy the eigenvalue equation [2–4]

$$\mathbf{R}_x \boldsymbol{\phi} = \lambda \boldsymbol{\phi}. \quad (18)$$

It is rewritten as

$$\mathbf{R}_x \boldsymbol{\phi} - \lambda \mathbf{I} \boldsymbol{\phi} = (\mathbf{R}_x - \lambda \mathbf{I}) \boldsymbol{\phi} = \mathbf{0}, \quad (19)$$

such that $(\mathbf{R}_x - \lambda \mathbf{I})$ is singular where $\mathbf{0}$ is an $N \times 1$ vector with its elements all equal to zero. Namely,

$$\det (\mathbf{R}_x - \lambda \mathbf{I}) = 0. \quad (20)$$

Since $\mathbf{R}_x$ is a real and symmetric matrix, its eigenvectors with different eigenvalues are linearly independent. Hence, this determinant is a polynomial in λ of degree N, (20) has N roots and (19) has N solutions for $\boldsymbol{\phi}$ that result in the eigenpair set $\{\lambda_k, \boldsymbol{\phi}_k\}$ where $0 \leq k \leq N-1$. Therefore, we can write the eigendecomposition for $\mathbf{R}_x$ with distinct eigenvectors as follows

$$\begin{aligned} \mathbf{R}_x \mathbf{A}_{KLT}^{*\mathrm{T}} &= \mathbf{A}_{KLT}^{*\mathrm{T}} \boldsymbol{\Lambda}, \\ \mathbf{R}_x &= \mathbf{A}_{KLT}^{*\mathrm{T}} \boldsymbol{\Lambda} \mathbf{A}_{KLT} = \sum_{k=0}^{N-1} \lambda_k \boldsymbol{\phi}_k \boldsymbol{\phi}_k^{*\mathrm{T}}, \end{aligned} \quad (21)$$

where $\boldsymbol{\Lambda} = diag(\lambda_k); k = 0, 1, \ldots, N-1$, and k*th* column of $\mathbf{A}_{KLT}^{*\mathrm{T}}$ matrix is the k*th* eigenvector $\boldsymbol{\phi}_k$ of $\mathbf{R}_x$ with the corresponding eigenvalue λ_k.

Note that $\{\lambda_k = \sigma_k^2\} \forall k$, for the given $\mathbf{R}_x$ where σ_k^2 is the variance of the k*th* transform coefficient, θ_k. The eigenvalues of $\mathbf{R}_x$ for an AR(1) process defined in (17) are derived to be in the closed-form [5]

$$\sigma_k^2 = \lambda_k = \frac{1-\rho^2}{1 - 2\rho\cos(\omega_k) + \rho^2}; 0 \leq k \leq N-1, \quad (22)$$

where $\{\omega_k\}$ are the positive roots of the following transcendental equation

$$\tan(N\omega) = -\frac{(1-\rho^2)\sin(\omega)}{\cos(\omega) - 2\rho + \rho^2 \cos(\omega)}, \quad (23)$$

that is equivalent to (See Appendix)

$$\begin{aligned} &\left[\tan\left(\omega \frac{N}{2}\right) + \gamma \tan\left(\frac{\omega}{2}\right)\right] \left[\tan\left(\omega \frac{N}{2}\right) - \frac{1}{\gamma} \cot\left(\frac{\omega}{2}\right)\right] = 0 \\ &\gamma = (1+\rho)/(1-\rho), \end{aligned} \quad (24)$$

and the resulting KLT matrix of size $N \times N$ is expressed in the kernel as [5]

$$\begin{aligned} \mathbf{A}_{KLT} &= [A(k,n)] = c_k \sin\left[\omega_k \left(n - \frac{N-1}{2}\right) + \frac{(k+1)\pi}{2}\right] \\ c_k &= \left(\frac{2}{N+\lambda_k}\right)^{1/2}, \ 0 \leq k, n \leq N-1. \end{aligned} \quad (25)$$

Note that the roots of the transcendental tangent equation in (24), $\{\omega_k\}$, are required in the KLT kernel expressed in (25). There are a few well-known numerical methods like *secant method* [17] to approximate roots of the tangent equation given in (24) in order to solve it implicitly rather than explicitly. Therefore, we focus on a root finding method described in Sec. V to find explicit solutions to transcendental equations including the tangent equation of (24). That method leads us to an explicit definition of KLT kernel given in (25) for an AR(1) process. However, we will first discuss the derivation of characteristic values and functions of a continuous random process with exponential autocorrelation in the next section since it provides some very useful insights for AR(1) family. The detailed derivations of results summarized in (22) and (23) are given in Appendix.

IV. Characteristic Values and Functions of a Continuous Random Process with Exponential Autocorrelation

In this section we revisit the classic problem of deriving explicit solutions for characteristic values and functions of a continuous random process with exponential autocorrelation function since it offers useful insights for the problem at hand. Note that this problem is discussed in detail on page 99 of [8]. Similar discussions can also be found in [6, 7, 18] and references therein.

We assume a continuous random process $x(t)$ with zero-mean, i.e. $E\{x(t)\} = 0$, and exponential autocorrelation function

$$R_{xx}(\tau) = E\{x(t)x(t+\tau)\} = e^{-\alpha|\tau|}, \quad (26)$$

where $-\infty < \tau < \infty$. We will find the orthogonal expansion of $x(t)$ for the interval $-T/2 < t < T/2$. Therefore, we need to derive the characteristic values and functions that satisfy the following integral equation

$$\int_{-T/2}^{T/2} e^{-\alpha|t-s|} \phi(s) ds = \lambda \phi(t), \quad (27)$$

where λ is the characteristic value and $\phi(t)$ is the corresponding characteristic function. Note that in [8] the integral is defined on the interval $-T < t < T$. In our discussion the interval is from $-T/2$ to $T/2$ in order to be consistent with the discrete case. The derivation steps for the discrete case are given in the Appendix. Integral equation in (27) can be solved by finding a linear differential equation that $\phi(t)$ must satisfy, and then substituting the general solution of the differential equation back in (27) to determine the value of λ. In order to drop the magnitude operator, (27) can be rewritten as [8]

$$\lambda \phi(t) = \int_{-T/2}^{t} e^{-\alpha(t-s)} \phi(s) ds + \int_{t}^{T/2} e^{-\alpha(s-t)} \phi(s) ds. \quad (28)$$

Then, both sides of the equality are differentiated to obtain

$$\lambda\phi'(t) = -\alpha \int_{-T/2}^{t} e^{-\alpha(t-s)}\phi(s)ds + \alpha \int_{t}^{T/2} e^{-\alpha(s-t)}\phi(s)ds, \quad (29)$$

where $f'(t)$ is the first-order derivative of $f(t)$. We differentiate one more time and use Leibniz integral rule to obtain

$$\lambda\phi''(t) = \alpha^2 \int_{-T/2}^{T/2} e^{-\alpha|t-s|}\phi(s)ds - 2\alpha\phi(t). \quad (30)$$

It follows from (27) and (30) that

$$\phi''(t) + \frac{\alpha\,(2-\alpha\lambda)}{\lambda}\phi(t) = 0. \quad (31)$$

Note that the characteristic function, $\phi(t)$ must satisfy the linear homogeneous differential equation of (31) in order to satisfy the integral equation given in (27). It is shown on pages 99-101 of [8], and pages 187-190 of [18] that (31) has solution only in the range of $0 < \lambda < \frac{2}{\alpha}$ as given

$$\phi(t) = c_1 e^{jbt} + c_2 e^{-jbt}, \quad (32)$$

where $b^2 = -\alpha\,(2-\alpha\lambda)/\lambda$ and $0 < b^2 < \infty$. It was also shown in [8] and [18] that substituting (32) into (27) reveals that solution is possible only when $c_1 = \pm c_2$. For $c_1 = c_2$, b satisfies the following equation

$$b\tan\left(b\frac{T}{2}\right) = \alpha. \quad (33)$$

It follows from (32) that for every positive b_k that satisfies the transcendental equation in (33), we have the characteristic function given as [8]

$$\phi_k(t) = c_k\cos\left(b_k t\right), \quad (34)$$

where integer $k \geq 0$. Similarly, for $c_1 = -c_2$, b satisfies the following equation

$$b\cot\left(b\frac{T}{2}\right) = -\alpha. \quad (35)$$

Again, for every positive b_k that satisfies the transcendental equation in (35), we have the characteristic function given as [8]

$$\phi_k(t) = c_k\sin\left(b_k t\right). \quad (36)$$

For both cases the corresponding eigenvalues are expressed as [8]

$$\lambda_k = \frac{2\alpha}{\alpha^2 + b_k^2}. \quad (37)$$

Note that the roots of transcendental equations given in (33) and (35) provide the even and odd indexed characteristic values and functions, respectively. These two equations can be combined as a product

$$\left[b\tan\left(b\frac{T}{2}\right) - \alpha\right]\left[b\cot\left(b\frac{T}{2}\right) + \alpha\right] = 0. \quad (38)$$

Similarly, for every positive b_k that satisfies the transcendental equation in (38), we have the characteristic function given as

$$\phi_k(t) = c_k\sin\left(b_k t + \frac{(k+1)\,\pi}{2}\right). \quad (39)$$

Note that (37), (38), and (39) are the continuous analogs of (22), (24), and (25), respectively. Moreover, it is worth noting that the constant, c_k, in (39) can be found by normalizing the characteristic functions such that

$$\|\phi_k(t)\|^2 = \int_{-T/2}^{T/2} \phi_k(t)\phi_k^*(t) = 1, \quad (40)$$

leading to

$$c_k = \left(\frac{2}{T + \lambda_k}\right)^{1/2}. \quad (41)$$

The definition of the constant given in (41) is analogous to the one in (25).

V. An Efficient Method for Explicit Solution of a Transcendental Equation

Finding solutions of a transcendental equation has always been of great interest in various fields [19–21]. There are a number of numerical methods reported in the literature offering approximate solutions to such equations. Although these techniques are sufficient for many cases, it is quite desirable to have exact explicit solutions for this class of equations leading to analytical treatment of problems at hand as reported in [22, 23]. Most of them are based on the approach of formulating a Riemann problem and finding the solution for the resulting transcendental equation by utilizing a canonical solution of the problem. It was shown that the implementation of this method may easily become quite difficult [23–26]. Therefore, the topic has been active and several researchers proposed new techniques to improve the efficiency of finding solutions for a transcendental equation [19–23]. Note that such a tool is needed to solve (23) in order to write an explicit expression for KLT kernel of (25). Therefore, we employ a relatively new method to handle such a problem. It is computationally efficient and easy to implement as described below.

Herein, we focus on a simple method of formulating exact explicit solution for the roots of transcendental equations using Cauchy's integral theorem from complex analysis [27] that was introduced by Luck and Stevens in [9]. The method determines the roots of a transcendental function by locating the singularities of its reciprocal function. In this section, we follow the derivation steps detailed in [9] for explicit solutions of such functions in order to better explain its implementation that is of our interest as emphasized in the following section.

Cauchy's theorem states that if a function is analytic in a simple connected region containing the closed curve C, the path integral of the function around the curve C is zero. On the other hand, if a function, $f(z)$, contains a singularity at $z = z_0$ somewhere inside C but analytic elsewhere in the region, then the singularity can be removed by multiplying $f(z)$ with $(z - z_0)$, i.e. a pole-zero cancellation. Cauchy's theorem implies that the path integral of the new function

$(z - z_0)\, f(z)$ around C must be zero

$$\oint_C (z - z_0)\, f(z) dz = 0. \tag{42}$$

The evaluation of the integral yields a first-order polynomial in z_0 with constant coefficients, and its solution for z_0 provides the location of the singularity [9]

$$z_0 = \frac{\oint_C z f(z) dz}{\oint_C f(z) dz}. \tag{43}$$

This is an explicit expression for the singularity of the function $f(z)$. Now, a root finding problem is restated as a singularity at the root. Note that (43) gives the location of the desired root and it can be evaluated for any closed path by employing either an analytical or a numerical technique. Luck and Stevens in [9] suggested a simple method for evaluation of (43) that results in a particularly easy calculation by using a circle in the complex plane that circumscribes the root. The closed curve C is described as a circle in the complex plane with its center h and radius R, expressed as

$$\begin{aligned} z &= h + Re^{j\theta}, \\ dz &= jRe^{j\theta} d\theta, \end{aligned} \tag{44}$$

where $0 \leq \theta \leq 2\pi$, $h \in \mathbb{R}$, and $R \in \mathbb{R}$. Values of h and R do not matter as long as the circle circumscribes the root z_0. Cauchy's argument principle [28] or graphical methods may be used to determine the number of roots enclosed by the path C. Let

$$w(\theta) = \left. f(z) \right|_{z = h + Re^{j\theta}} = f(h + Re^{j\theta}). \tag{45}$$

Then (43) becomes [9]

$$z_0 = h + R \left[\frac{\int_0^{2\pi} w(\theta) e^{j2\theta} d\theta}{\int_0^{2\pi} w(\theta) e^{j\theta} d\theta} \right]. \tag{46}$$

One can easily evaluate (46) by employing Fourier analysis since the n*th* Fourier series coefficient for any $x(t)$ is calculated as

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} x(t) e^{jnt} dt. \tag{47}$$

It is observed that the term in brackets in (46) is equal to the ratio of the second Fourier series coefficient over the first one for the function $w(\theta)$. Fourier series coefficients can be easily calculated numerically by using discrete Fourier transform (DFT) or by using its fast implementation, i.e. fast Fourier transform (FFT). Note that given $f(z)$ is analytic at h, multiplying $f(z)$ by a factor $(z - h) = Re^{j\theta}$ does not change the location of the singularities of $f(z)$. It means that for a given singularity the term in brackets is also equal to any ratio of the $(m+1)th$ to the mth Fourier series coefficients of $w(\theta)$ for $m \geq 1$ [9]. Therefore, it is very easy to implement as we discuss it further in the following section.

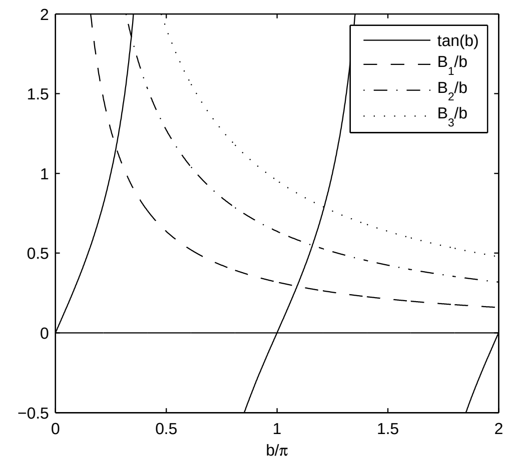

Figure 1. Functions $\tan(b)$ and B/b for various values of B where $B_1 = 1$, $B_2 = 2$, and $B_3 = 3$.

VI. A Simple Method for Explicit KLT Kernel of AR(1) Discrete Process

In this section, we highlight the theory behind the proposed KLT kernel derivation method for AR(1) discrete process by utilizing classical research on continuous random process with exponential autocorrelation function. Moreover, we summarize and present a step-by-step implementation of the novel technique reported herein for the explicit expression of the kernel.

A. Continuous Random Process with Exponential Autocorrelation

Now, we will derive an explicit expression for the roots of the transcendental equation that are required in the definition of the continuous characteristic function depicted in (33) and (35). We will focus on (33) stating that the discussion is similar for (35). Let $\alpha = B$ and $T = 2$ in (33) and rewrite it as follows

$$b \tan(b) = B. \tag{48}$$

One must calculate the positive roots of (48), $b_m > 0$, in order to determine the even indexed characteristic values and functions given in (37) and (39), respectively. Fig. 1 displays functions $\tan(b)$ and B/b for various values of B. It is apparent from the figure that for the m*th* root, a suitable choice for the closed path C is a circle of radius $R = \pi/4$ centered at $h_m = (m - 3/4)\,\pi$. A straightforward way to configure the equation given in (48) to provide a singularity is to simply use the inverse of rearranged (48) as follows [9]

$$f(b) = \frac{1}{b \sin(b) - B \cos(b)}. \tag{49}$$

Applying (46) to (49) results in an explicit expression for the m*th* root. This expression can be evaluated by calculating a pair of adjacently indexed coefficients of size-L DFT (coefficients of two adjacent harmonics) as described in Sec. V or by using a numerical integration method. Therefore, $w_m(\theta)$ of (45) for this case is defined as

$$w_m(\theta) = f\left(h_m + Re^{j\theta}\right) = \frac{1}{\left(h_m + Re^{j\theta}\right)\sin\left(h_m + Re^{j\theta}\right) - B\cos\left(h_m + Re^{j\theta}\right)}, \tag{50}$$

where $0 \leq \theta \leq 2\pi$. Hence, the m*th* root is located at

$$b_m = h_m + R\left[\frac{\int_0^{2\pi} w_m(\theta)e^{j2\theta}d\theta}{\int_0^{2\pi} w_m(\theta)e^{j\theta}d\theta}\right]. \tag{51}$$

The MATLAB™ code given in Alg. 1 shows the simplicity of this root finding method to solve transcendental equations. The computational cost of deriving explicit KLT kernel for an AR(1) discrete process expressed in (25) is shown to be quite attractive and emphasized in the next subsection. It is observed from (51) and Alg. 1 (last line) that we do not need all DFT (FFT) coefficients to solve the problem since it requires only two Fourier series coefficients. Therefore, it is possible to further improve the computational cost of the root finding method displayed in Alg. 1 by employing a discrete summation operator to implement (47) numerically. Hence, it will have a computational complexity of $O(N)$ instead of $O(NlogN)$ required for FFT algorithms.

B. AR(1) Discrete Process

In order to derive an explicit expression for the roots of the transcendental equation that are required in the definition of the discrete KLT kernel given in (24), we need to calculate the first $N/2$ positive roots of two transcendental equations as given [5]

$$\tan\left(\omega\frac{N}{2}\right) = -\gamma\tan\left(\frac{\omega}{2}\right) \tag{52}$$

$$\tan\left(\omega\frac{N}{2}\right) = \frac{1}{\gamma}\cot\left(\frac{\omega}{2}\right), \tag{53}$$

where N is the transform size, $\gamma = (1+\rho)/(1-\rho)$, and ρ is the first-order correlation coefficient for AR(1) discrete process. Derivations of (52) and (53) are provided in the Appendix for readers of interest. Roots of (52) and (53) correspond to the odd and even indexed eigenvalues and eigenvectors, respectively. Fig. 2 displays functions $\tan(\omega N/2)$ and $-\gamma\tan(\omega/2)$ for $N = 8$ and various values of ρ. It is apparent from the figure that for the m*th* root of (52), a suitable choice for the closed path C in (43) is a circle of radius

$$R_m = \begin{cases} \pi/2N & m \leq 2 \\ \pi/N & m > 2 \end{cases}, \tag{54}$$

centered at $h_m = (m - 1/4)(2\pi/N)$ where $1 \leq m \leq N/2$. Similar to the continuous case, we reconfigure (52) and rather look for the poles of its reciprocal function

$$g(\omega) = \frac{1}{\tan(\omega N/2) + \gamma\tan(\omega/2)}. \tag{55}$$

The function $w(\theta)$ of (45) for this case is defined as

$$w_m(\theta) = g\left(h_m + R_m e^{j\theta}\right) = \frac{1}{\tan\left[\left(h_m + R_m e^{j\theta}\right)\frac{N}{2}\right] + \gamma\tan\left[\left(h_m + R_m e^{j\theta}\right)\frac{1}{2}\right]}, \tag{56}$$

where $0 \leq \theta \leq 2\pi$. Hence, the m*th* root is located at

$$\omega_m = h_m + R_m\left[\frac{\int_0^{2\pi} w_m(\theta)e^{j2\theta}d\theta}{\int_0^{2\pi} w_m(\theta)e^{j\theta}d\theta}\right]. \tag{57}$$

The procedure is the same for deriving the roots of (53) with the exceptions that (56) must be modified as follows

$$w_m(\theta) = \frac{1}{\tan\left[\left(h_m + R_m e^{j\theta}\right)\frac{N}{2}\right] - \frac{1}{\gamma}\cot\left[\left(h_m + R_m e^{j\theta}\right)\frac{1}{2}\right]}, \tag{58}$$

and a suitable choice for the closed path C is a circle of radius $R_m = \pi/N$ centered at

$$h_m = \begin{cases} (m-1/2)(2\pi/N) & m \leq 2 \\ (m-1)(2\pi/N) & m > 2 \end{cases}, \tag{59}$$

that can be determined by generating a plot similar to the ones in Figs. 1 and 2.

Finally, we summarize the steps of the proposed method to derive an explicit KLT kernel of dimension N expressed in

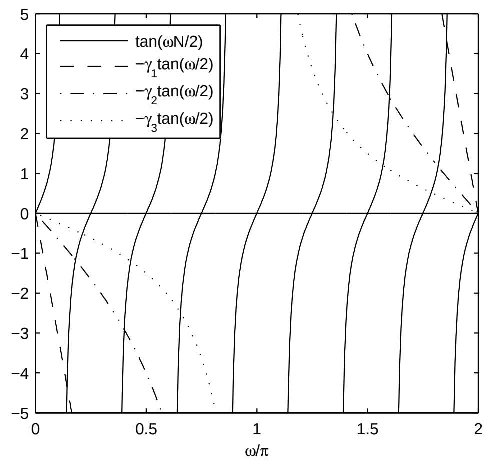

Figure 2. Functions $\tan(\omega N/2)$ and $-\gamma\tan(\omega/2)$ for $N = 8$ and various values of ρ with $\rho_1 = 0.9$, $\rho_2 = 0.6$, and $\rho_3 = 0.2$ where $\gamma_i = (1+\rho_i)/(1-\rho_i)$, $i = 1, 2, 3$.

Algorithm 1 MATLAB™ code of the method to calculate roots of transcendental equation given in (48). For $B = 2$, first root is calculated as 1.076873986311804 using this function.

```
L = 128;                              % FFT size. See (51)
B = 2;                                          % See (48)
m = 1;                              % Root index. See (51)
h = (m-3/4)*pi; % Center of the circle in b. See (44)
R = pi/4;       % Radius of the circle in b. See (44)
th = linspace(0, 2*pi*(1-1/L), L);  % Theta. See (44)
b = h+R*exp(1i*th); % Points on the circle. See (44)
w = 1./(b.*sin(b)-B*cos(b));                    % See (50)
W = fft(conj(w),L);                             % See (51)
b_m = h + R*W(3)/W(2)               % mth root. See (51)
```

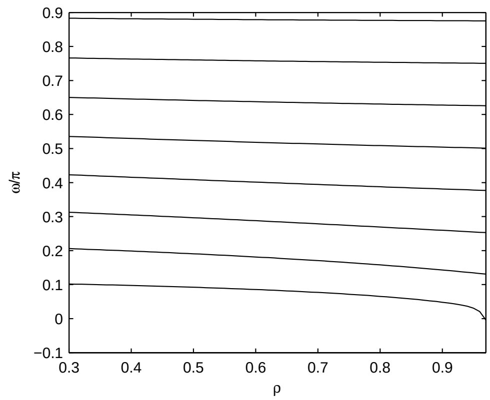

Figure 3. The roots of the transcendental tangent equation, $\{\omega_k\}$, as a function of ρ for $N = 8$.

(25) for an arbitrary discrete data set by employing an AR(1) approximation as follows.

1) Estimate the first-order correlation coefficient $\rho = R_{xx}(1)/R_{xx}(0) = E\{x(n)x(n+1)\}/E\{x(n)x(n)\}$ of AR(1) model for the given data set $\{x(n)\}$ where n is the index of random variables (or discrete-time) and $-1 < \rho < 1$.
2) Calculate the positive roots $\{\omega_k\}$ of the polynomial given in (24) by substituting (56) and (58) into (57) for odd and even values of k, respectively, and use the following indexing

$$m = \begin{cases} k/2+1 & k \text{ even} \\ (k+1)/2 & k \text{ odd} \end{cases}. \tag{60}$$

3) Plug in the values of ρ and $\{\omega_k\}$ in (22) and (25) to calculate the eigenvalues λ_k and eigenvectors defining the KLT matrix $\mathbf{A}_{KLT}$, respectively.

Remark 1. The computational cost of the proposed method to derive KLT matrix of size $N \times N$ for an arbitrary signal source has two distinct components. Namely, the calculation of the first-order correlation coefficient ρ for the given signal set, and the calculation of the roots $\{\omega_k\}$ of (24) that are plugged in (25) in order to generate the resulting transform matrix $\mathbf{A}_{KLT}$. The roots $\{\omega_k\}$ of the transcendental tangent equation, calculated by using (57), as a function of ρ and for $N = 8$ are displayed in Fig. 3. Similarly, the values of $\{\omega_k\}$ for $\rho = 0.95$ and $N = 4,\ 8,\ 16$ are tabulated in Table I.

Remark 2. Other processes like higher order AR, autoregressive moving average (ARMA), and moving average (MA) can also be approximated by using AR(1) modeling [10]. Therefore, the proposed method to derive explicit KLT kernel for AR(1) discrete process may also be beneficial for other stochastic processes of interest.

Remark 3. It was reported in the literature that the signal independent DCT kernel is identical to the KLT kernel of AR(1) discrete process in the limit when $\rho \to 1$ [29].

VII. Performance Comparison

We present comparative performance of the proposed KLT kernel derivation method and competing techniques. The comparison is twofold. First, we compare energy compaction performance of KLT derived by the new technique and DCT for the monochrome test image LENA. This metric has been widely used in source coding [2]. Second, we compare KLT kernels obtained by the new derivation method and the popular numerical technique called divide and conquer (D&Q) [3, 30]. This comparison includes their respective computational cost and discrepancies between the two kernels for the same statistics.

Figure 4. Monochrome LENA image of size 256×256 pixels used as the test signal.

Table II
Gain of transform coding, G_{TC}^N, of separable 2D KLT and DCT transforms for LENA image with block sizes $N = 4, 8, 16, 32, 64$ where estimated $\rho_h = 0.93$ and $\rho_v = 0.97$, calculated from the entire image, are used to generate $\mathbf{A}_{KLT}^h$ and $\mathbf{A}_{KLT}^v$ for all values of N.

N	4	8	16	32	64
DCT	36.91	67.16	90.68	105.53	114.03
KLT	37.11	67.77	91.59	106.48	114.80

A. Energy Compaction

Although there are various performance metrics including decorrelation efficiency, the gain of transform coding over pulse code modulation (PCM) of an $N \times N$ unitary transform for a given input correlation is particularly significant and widely utilized in transform theory as defined [2]

$$G_{TC}^N = \frac{\frac{1}{N}\sum_{k=0}^{N-1}\sigma_k^2}{\left(\prod_{k=0}^{N-1}\sigma_k^2\right)^{\frac{1}{N}}}, \tag{61}$$

where σ_k^2 is the variance of the k*th* transform coefficient. Note that KLT is the unique block transform that optimally repacks the signal energy among the perfectly decorrelated (pairwise uncorrelated) coefficients for the given autocorrelation matrix $\mathbf{R}_x$. Fixed transform DCT has been shown to be an attractive approximation to KLT particularly for highly correlated processes due to its affordable computational cost with acceptable performance [1, 2]. In this section, we highlight the energy compaction performance of KLT generated based on an AR(1) approximation to signal correlations and DCT for a still frame image. Note that this comparison uses first-order correlation coefficients for horizontal and vertical dimensions that are estimated once and for the entire image.

We calculate the horizontal and vertical first-order correlation coefficients, and define the resulting 2D AR(1) correlation model [2], respectively, for size 256×256 LENA image

Table I
THE VALUES OF $\{\omega_k\}$ FOR $\rho = 0.95$ AND $N = 4,\ 8,\ 16$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$N = 4$	0.157	0.815	1.584	2.362												
$N = 8$	0.109	0.423	0.801	1.188	1.577	1.968	2.359	2.750								
$N = 16$	0.075	0.224	0.408	0.599	0.793	0.988	1.183	1.378	1.574	1.770	1.966	2.162	2.358	2.554	2.750	2.946

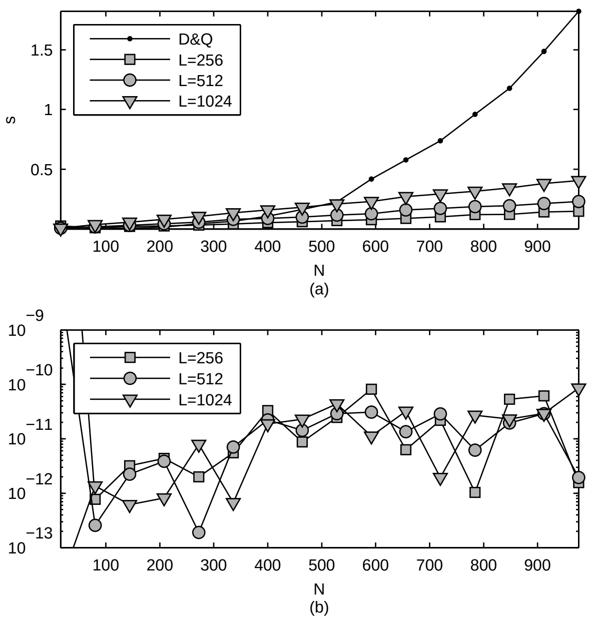

Figure 5. (a) Computation time, in seconds, to calculate $\mathbf{A}_{KLT,DQ}$ and $\mathbf{A}_{KLT,E}$ (with $L = 256, 512, 1024$) for $\rho = 0.95$ and $16 \leqslant N \leq 1024$, (b) Corresponding distances, d_N, measured with (65) for different N and L.

displayed in Fig. 4 as follows [2]

$$R_{xx}(m,n) = \sigma_x^2 \rho_h^{|m|} \rho_v^{|n|}, \tag{62}$$

where $m, n = 0, \pm1, \pm2, \ldots, \pm255$ and

$$\sigma_x^2 = \frac{1}{(1-\rho_h^2)(1-\rho_v^2)}. \tag{63}$$

Then, we employ the proposed method to generate the corresponding $\mathbf{A}_{KLT}^h$ and $\mathbf{A}_{KLT}^v$ matrices of size $N \times N$ by using (25). Now, we perform separable 2D transforms of various block sizes on LENA image by using $\mathbf{A}_{KLT}^h$ and $\mathbf{A}_{KLT}^v$ in horizontal and vertical dimensions, respectively. Similarly, we use $\mathbf{A}_{DCT}$ matrix in both dimensions for the DCT case generated from the fixed kernel [1]

$$\mathbf{A}_{DCT} = [A(k,n)] = \frac{1}{c_k}\cos\left[\frac{(2n+1)k\pi}{2N}\right]$$
$$c_k = \begin{cases} \sqrt{N} & k = 0 \\ \sqrt{N/2} & k \neq 0 \end{cases},\ 0 \leq k,n \leq N-1. \tag{64}$$

Now, we calculate G_{TC}^N from (61) for these two separable 2D transforms of various sizes for LENA image as tabulated in Table II. It is observed from Table II that KLT slightly outperforms DCT as expected [2, 29]. An adaptive KLT method that mimics variations of signal correlations may provide improved energy compaction with additional implementation cost.

B. Kernel Derivation Efficiency

Herein, we compare the computational cost of generating KLT kernel for the given statistics by employing D&Q [3, 30] and the proposed method expressed in (25). Moreover, we measure the distance between the kernels generated by the two competing derivation methods. The distance metric between the two kernels is defined as follows

$$d_N = \left\| \mathbf{A}_{KLT,DQ}^{\mathrm{T}} \mathbf{A}_{KLT,DQ} - \mathbf{A}_{KLT,E}^{\mathrm{T}} \mathbf{A}_{KLT,E} \right\|_2, \tag{65}$$

where $\|\cdot\|_2$ is the norm-2, $\mathbf{A}_{KLT,DQ}$ and $\mathbf{A}_{KLT,E}$ are the $N \times N$ KLT matrices obtained by using D&Q and the proposed derivation method for explicit kernel (25), respectively. Note that the performance of the proposed derivation method in terms of precision and computational speed highly depends on the FFT size L, used in evaluating (57). Therefore, the distance metric, d_N, of (65) and the time it takes to calculate the kernel by using (25) are affected by the value of L. Computation times (in seconds) to generate $\mathbf{A}_{KLT,DQ}$ and $\mathbf{A}_{KLT,E}$ (using the values of $L = 256,\ 512\,1024$) for the case of $\rho = 0.95$ and $16 \leqslant N \leqslant 1024$ are displayed in Fig. 5.a. Both computations are performed by using one thread on a single processor. The machine used for the simulations has Intel® Core™ i5-520M CPU and 8 GB of RAM. It is observed from Fig. 5.a that the proposed method significantly outperforms the D&Q algorithm for larger values of N. Moreover, corresponding distances, d_N, measured with (65) for various N and L are displayed in Fig. 5.b. They show that the proposed method is significantly faster than the currently used numerical methods with negligible discrepancy between the two kernels. It is highlighted that the time complexity of D&Q algorithm is $O(N^3)$ [30] where the proposed method requires $O(LN)$ for the FFT size of L in root finding, and the transform size of N. Furthermore, the proposed KLT kernel derivation algorithm has the so-called *embarrassingly parallel* nature. Hence, it can be easily implemented to be run on multiple threads and processors for any k. Therefore, its implementation speed can be improved to make it faster than the one displayed in Fig. 5.a.

VIII. CONCLUSIONS

We introduced a simple method to derive explicit KLT kernel for AR(1) discrete process. The mathematical foundations of the proposed method and its ease of implementation were detailed in the paper. The merit of the proposed kernel derivation method was highlighted by performance comparisons with DCT, and the numerical D&Q algorithm. The proposed method can be easily implemented on high performance computing devices with highly parallel architectures such as field programmable gate array (FPGA) and graphics processing unit (GPU) for data intensive and real-time applications.

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APPENDIX

We provide the proof of (22), (23), and (25) that were first reported in [5] without detailed derivations. For a discrete random signal, $x(n)$, discrete Karhunen-Loève (K-L) series expansion is written as

$$\sum_{m=0}^{N-1} R_{xx}(n,m)\phi_k(m) = \lambda_k \phi_k(n), \tag{A.1}$$

where m and n are the independent discrete random variables. $R_{xx}(n,m) = E\{x(n)x(n+m)\}$, $m, n = 0, 1, \ldots, N-1$, is the autocorrelation function of the random signal, λ_k is the k*th* eigenvalue, and $\phi_k(n)$ is the corresponding k*th* eigenfunction. For AR(1) discrete process [2]

$$R_x(n,m) = R_x(n-m) = \rho^{|n-m|}, \tag{A.2}$$

Hence, the discrete K-L series expansion for an AR(1) process from (A.1) and (A.2) is stated as follows

$$\sum_{m=0}^{N-1} \rho^{|n-m|}\phi_k(m) = \lambda_k \phi_k(n). \tag{A.3}$$

In order to eliminate the magnitude operator, (A.3) can be rewritten in the form

$$\sum_{m=0}^{n} \rho^{n-m}\phi_k(m) + \sum_{m=n+1}^{N-1} \rho^{m-n}\phi_k(m) = \lambda_k \phi_k(n). \tag{A.4}$$

From the continuous case, (32), we have the solution for the k*th* eigenvector [8, 18]

$$\phi_k(t) = c_1 e^{j\omega_k t} + c_2 e^{-j\omega_k t}, \tag{A.5}$$

where c_1 and c_2 are arbitrary constants, t is the independent continuous variable, $-T/2 \le t \le T/2$, and $\omega_k = b_k$. Let us shift this eigenfunction by $T/2$ and discretize at the sampling grid $t_n = nT_s$, $0 \le n \le N-1$ where $T_s = T/(N-1)$. Accordingly, sampled eigenfunction is written as

$$\phi_k(n) = c_1 e^{j\omega_k (n - \frac{N-1}{2})} + c_2 e^{-j\omega_k (n - \frac{N-1}{2})}. \tag{A.6}$$

We consider the case where $c_1 = c_2$ noting that the case for $c_1 = -c_2$ is similar. Now, we rewrite (A.6) for $c_1 = c_2$ as

$$\phi_k(n) = c_1 \cos\left[\omega_k \left(n - \frac{N-1}{2}\right)\right]. \tag{A.7}$$

Now, we substitute (A.7) in (A.4) and define a new independent discrete variable $p = m - (N-1)/2$ in order to rewrite (A.3) as follows

$$\begin{aligned} &\sum_{p=-\frac{N-1}{2}}^{n-\frac{N-1}{2}} \rho^{n-p-\frac{N-1}{2}} \cos(\omega_k p) \\ &+ \sum_{p=n+1-\frac{N-1}{2}}^{\frac{N-1}{2}} \rho^{p+\frac{N-1}{2}-n} \cos(\omega_k p) \\ &= \lambda_k \cos\left[\omega_k \left(n - \frac{N-1}{2}\right)\right]. \end{aligned} \tag{A.8}$$

We focus on the first summation on the left in (A.8) stating that the result for the second summation on the left is similar. We rewrite the first summation as

$$\frac{1}{2}\rho^{n-\frac{N-1}{2}}\left(\sum_{p=-\frac{N-1}{2}}^{n-\frac{N-1}{2}}\left(\rho^{-1}e^{j\omega_k}\right)^p+\sum_{p=-\frac{N-1}{2}}^{n-\frac{N-1}{2}}\left(\rho^{-1}e^{-j\omega_k}\right)^p\right). \tag{A.9}$$

Using the fact that

$$\sum_{n=N_1}^{N_2}\beta^n=\frac{\beta^{N_1}-\beta^{N_2+1}}{1-\beta}, \tag{A.10}$$

and following simple steps, it is shown that (A.9). Hence, the first summation on the left side of (A.8), is equal to

$$\frac{\rho^{n+2}\cos(\omega_1)-\rho\cos(\omega_2)-\rho^{n+1}\cos(\omega_3)+\cos(\omega_4)}{1-2\rho\cos(\omega_k)+\rho^2}. \tag{A.11}$$

Similarly, the second summation on the left side of (A.8) is equal to

$$\frac{\rho^{N-n+1}\cos(\omega_1)+\rho\cos(\omega_2)-\rho^{N-n}\cos(\omega_3)-\rho^2\cos(\omega_4)}{1-2\rho\cos(\omega_k)+\rho^2}, \tag{A.12}$$

where

$$\begin{aligned}\omega_1&=\omega_k\left[(N-1)/2\right]\\ \omega_2&=\omega_k\left[n-(N-1)/2+1\right]\\ \omega_3&=\omega_k\left[(N-1)/2+1\right]\\ \omega_4&=\omega_k\left[n-(N-1)/2\right]\end{aligned} \tag{A.13}$$

for both (A.11) and (A.12). Now, we turn our attention to (A.8) and solve for λ_k. Discrete K-L expansion given in (A.3) can also be written in the frequency domain by using Fourier transform as follows

$$S_x(e^{j\omega})\Phi_k(e^{j\omega})=\lambda_k\Phi_k(e^{j\omega}). \tag{A.14}$$

where $S_x(e^{j\omega})$ is the power spectral density (PSD) of an AR(1) discrete process, and expressed as

$$S_x(e^{j\omega})=\mathcal{F}\left\{\rho^{|n-m|}\right\}=\frac{1-\rho^2}{1-2\rho\cos(\omega)+\rho^2}. \tag{A.15}$$

where $\mathcal{F}\{\cdot\}$ is the Fourier transform operator. Fourier transform of the eigenfunction in (A.7) is calculated as

$$\begin{aligned}\Phi_k(e^{j\omega})&=\mathcal{F}\{\phi_k(n)\}\\ &=c_1e^{-j\omega_k\frac{N-1}{2}}\left[\delta(\omega-\omega_k)+\delta(\omega+\omega_k)\right],\end{aligned} \tag{A.16}$$

where $\delta(\omega-\omega_0)$ is an impulse function of frequency located at ω_0. By substituting (A.15) and (A.16) into (A.14) we derive the expression for eigenvalues as follows

$$\lambda_k=\frac{1-\rho^2}{1-2\rho\cos(\omega_k)+\rho^2}. \tag{A.17}$$

(A.17) shows that the eigenvalues are the samples of the power spectral density and equal to (22). Substituting (A.11), (A.12), and (A.17) in (A.4) one can show that

$$\rho=\frac{\cos(\omega_kN/2+\omega_k/2)}{\cos(\omega_kN/2-\omega_k/2)}. \tag{A.18}$$

Using trigonometric identities, the relationship between ω_k and ρ in (A.18) is rewritten as follows

$$\tan\left(\omega_k\frac{N}{2}\right)=\left(\frac{1-\rho}{1+\rho}\right)\cot\left(\frac{\omega_k}{2}\right). \tag{A.19}$$

Similarly, for the case of $c_1=-c_2$, following the same procedure, the relationship between ω_k and ρ is shown to be

$$\tan\left(\omega_k\frac{N}{2}\right)=-\left(\frac{1+\rho}{1-\rho}\right)\tan\left(\frac{\omega_k}{2}\right). \tag{A.20}$$

Finally, from (A.19) and (A.20), it is observed that ω_k are the positive roots of the equation

$$\left[\tan\left(\omega\frac{N}{2}\right)+\gamma\tan\left(\frac{\omega}{2}\right)\right]\left[\tan\left(\omega\frac{N}{2}\right)-\frac{1}{\gamma}\cot\left(\frac{\omega}{2}\right)\right]=0, \tag{A.21}$$

where $\gamma=(1+\rho)/(1-\rho)$. Using trigonometric identities (A.21) can be rewritten as

$$\tan(N\omega)=-\frac{(1-\rho^2)\sin(\omega)}{\cos(\omega)-2\rho+\rho^2\cos(\omega)}, \tag{A.22}$$

that is the same transcendental equation expressed in (23).

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