WAVELETS: A DIFFERENT WAY TO LOOK AT SUBBAND CODING.

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"WAVELETS" and all that ....

"Wavelets"

$\llcorner\!\rightarrow$ technique to cut up $\left\{\begin{array}{l}\text{data}\\ \text{functions}\\ \text{operators}\end{array}\right\}$

into different frequency components, and to study each component with a resolution matched to their scale.

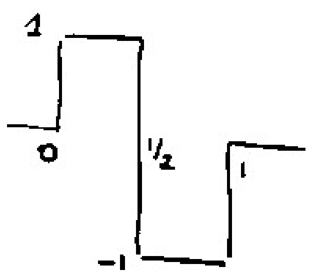
This technique was "invented" independently in several different fields

- pure mathematics. harmonic analysis (Calderón)
- quantum mechanics: coherent states (Aslaksen - Klauder)
- engineering: signal analysis (QMF filters - Esteban & Galland, Smith & Barnwell, Jean Morlet)

Recently (last three years): synthesis between different approaches $\longrightarrow$ very fertile for all branches.

ORTHONORMAL BASES OF WAVELETS.

Old example: Haar basis.



$\longrightarrow \psi(x)$

$$\psi_{jk}(x) = 2^{-j/2}\,\psi(2^{-j}x - k)$$

$$j, k \in \mathbb{Z}$$

$\downarrow$

orthonormal basis for $L^2(\mathbb{R})$

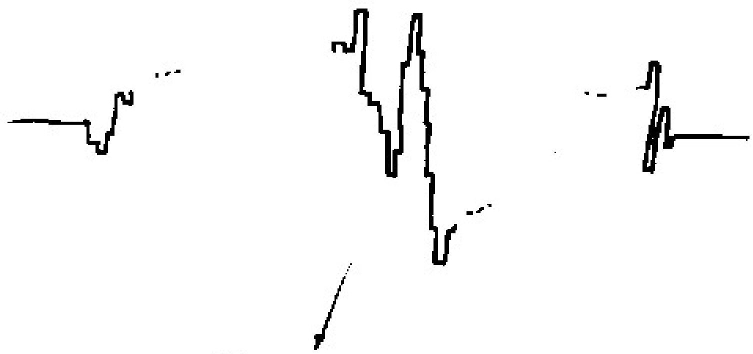
$|\text{support } \psi| = 1$

$|\text{support } \psi_{jk}| = 2^j$

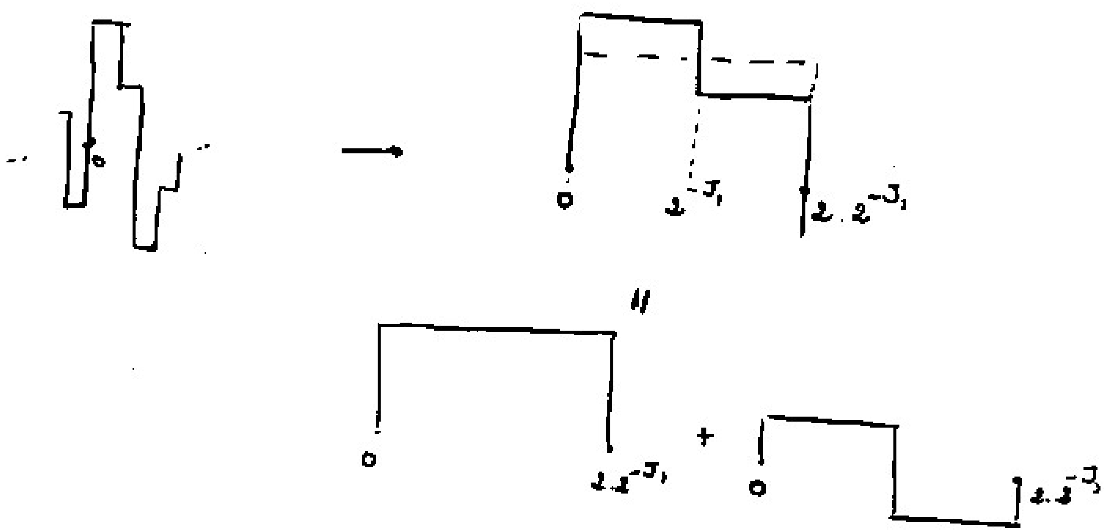
center of support ψ_{jk} : $(k + 1/2)\,2^j$

Proof that ψ_{jk} constitute orthonormal basis?

Sufficient to show that functions with support $\subset [-2^{J_0}, 2^{J_0}]$, piecewise constant on intervals $[k2^{-J_1}, (k+1)2^{-J_1}[$, can be written as combination of $\psi_{j\ell}$.

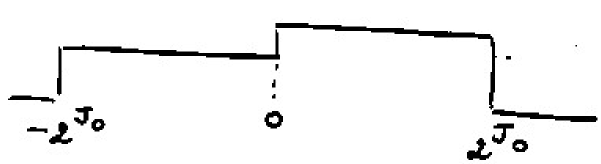


blowup near zero

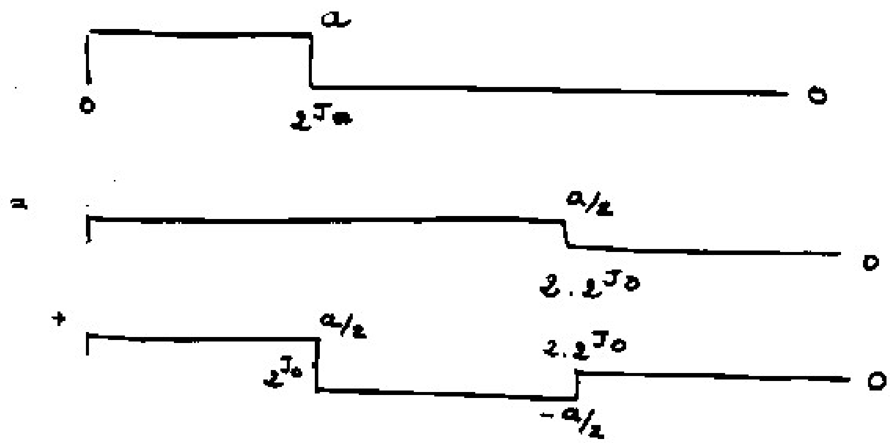


$\Rightarrow \quad f = \sum c_{J_1-1,\ell}\; \psi_{J_1-1,\ell}$ + fct. piecewise constant on $[k2^{-(J_1-1)}, (k+1)2^{-(J_1-1)}[$

$= \dots$

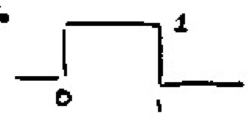
$= \sum_{j=-J_1-J_0}^{J_1-1} c_{j\ell}\, \psi_{j\ell}$ + 

keep going!



$$= \sum_{k=1}^{K} 2^{-k} a\, \psi(2^{-J_0-k} x) + 2^{-K} a\, \phi(2^{-J_0-K} x)$$

where ϕ is



But $\| 2^{-K} a\, \phi(2^{-J_0-K} \cdot) \|_{L^2}^2 = 2^{-2K} |a|^2 \, 2^{J_0+K}$

$\longrightarrow 0$ for $K \to \infty$.

$\Rightarrow$ done!

<u>In fact</u>: proof uses <u>multiresolution analysis</u>.

- introduces "averaging" function ϕ
- space V_j spanned by $\phi(2^{-j} x - k)$

  $V_j \subset V_{j-1}$
- $\mathrm{Proj}_{V_{j-1}} f = \mathrm{Proj}_{V_j} f$ + expansion in the ψ_{jk}.

- Orthonormal bases of wavelets.

For very special ψ :

$$2^{-m/2}\,\psi(2^{-m}t-n) = \psi_{mn}(t)$$

are orthonormal basis.

<u>NB.</u> $b_0 = 1$ $\Rightarrow$ not really a restriction

$a_0 = 2$: computationally easy.

other a_0 also possible. (in fact all rational values are allowed)

These are associated to a beautiful mathematical construction:

Multiresolution analysis. (S. Mallat, Y. Meyer)

- ladder of spaces

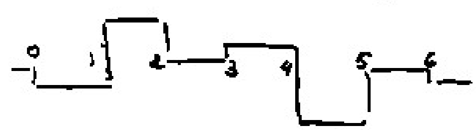
$$\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$$

$$\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$$

$$\bigcup_{m\in\mathbb{Z}} V_m = L^2(\mathbb{R}).$$

V_j : describes functions in which all scales finer than 2^j are left out.

<u>Ex.</u> V_0



⇒ general framework for construction of orthonormal wavelet bases.

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$$

- $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.
- $f \in V_j \iff f(2^j \cdot) \in V_0$
- $\exists\, \phi \in V_0$ so that ϕ_{0k} are o.n. basis for V_0

$$\phi_{0k}(x) = \phi(x-k).$$

MULTIRESOLUTION ANALYSIS.

Then: $\exists$ associated orthonormal wavelet basis.

- W_0 : orthogonal complement in V_{-1} of V_0

$$V_0 \oplus W_0 = V_{-1} \quad , \quad W_0 \perp V_0 .$$

- $\exists\, \psi$ in W_0 so that ψ_{0k} are orthonormal basis for W_0

$\Rightarrow$ $\mathrm{Proj}_{V_{-1}} = \mathrm{Proj}_{V_0}$ + expansion in ψ_{0k}.

- $$\begin{array}{ccccccc} V_1 & \subset & V_0 & \subset & V_{-1} & \subset & V_{-2} \subset \\ & \cup & & \cup & & \cup & \;\;\cup \\ W_1 & & W_0 & & W_{-1} & & W_{-2} \end{array}$$

$f \in W_j \iff f(2^j x) \in W_0$.

$\Rightarrow$ W_j all orthogonal, and $\bigoplus W_j = L^2(\mathbb{R})$

- W_j dilated version on W_0

$$\downarrow$$

o.n. basis $\psi(x-k)$

$\Rightarrow \psi_{jk}(x) = 2^{-j/2}\,\psi(2^{-j}x-k) \qquad k\in\mathbb{Z}$

o.n. basis in W_j

- $\Rightarrow \{\psi_{jk}\,;\, j,k\in\mathbb{Z}\}$ o.n. basis for $L^2(\mathbb{R})$

- recipe for ψ:

  - $\phi\in V_0\subset V_{-1} \leftarrow$ o.n. basis $\phi_{-1\,k}$

$$\phi(x) = \sum_n h_n\,\phi_{-1,n}(x)$$
$$= \sqrt{2}\sum_n h_n\,\phi(2x-n)$$
$$h_n = \langle\phi,\phi_{-1n}\rangle$$

  - $\psi(x) = \sqrt{2}\sum_n(-1)^n h_{-n+1}\,\phi(2x-n)$

To prove existence + recipe for ψ, analyze in detail what W_0 really represents.

A crucial role in this analysis is played by the trigonometric polynomial

$$m_0(\xi) = \frac{1}{\sqrt{2}}\sum_n h_n\,e^{-2\pi in\xi}.$$

$$|m_0(\xi)|^2 + |m_0(\xi+1/2)|^2 = 1.$$

to generalize Haar basis:

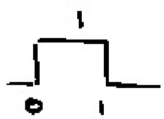
generalize the associated multiresolution analysis.

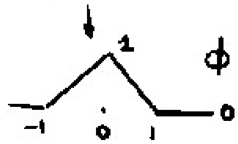
Two paths.

generalize "piecewise constant"

↓

linear
quadratic
cubic
⋮
(splines)





But ϕ_{0k} not orthonormal!

↳ orthonormalization trick.

$$\hat{\tilde{\phi}}(\xi) = \frac{\hat{\phi}(\xi)}{\left(\sum_{\ell} |\hat{\phi}(\xi+\ell)|^2\right)^{1/2}}$$

$\tilde{\phi}_{0k}$ orthonormal
span same space as ϕ_{0k}

→ can be used to construct ψ.

orthonormalization trick loses ...

compact support wanted
arbitrarily high regularity.

only finite # of h_n allowed.

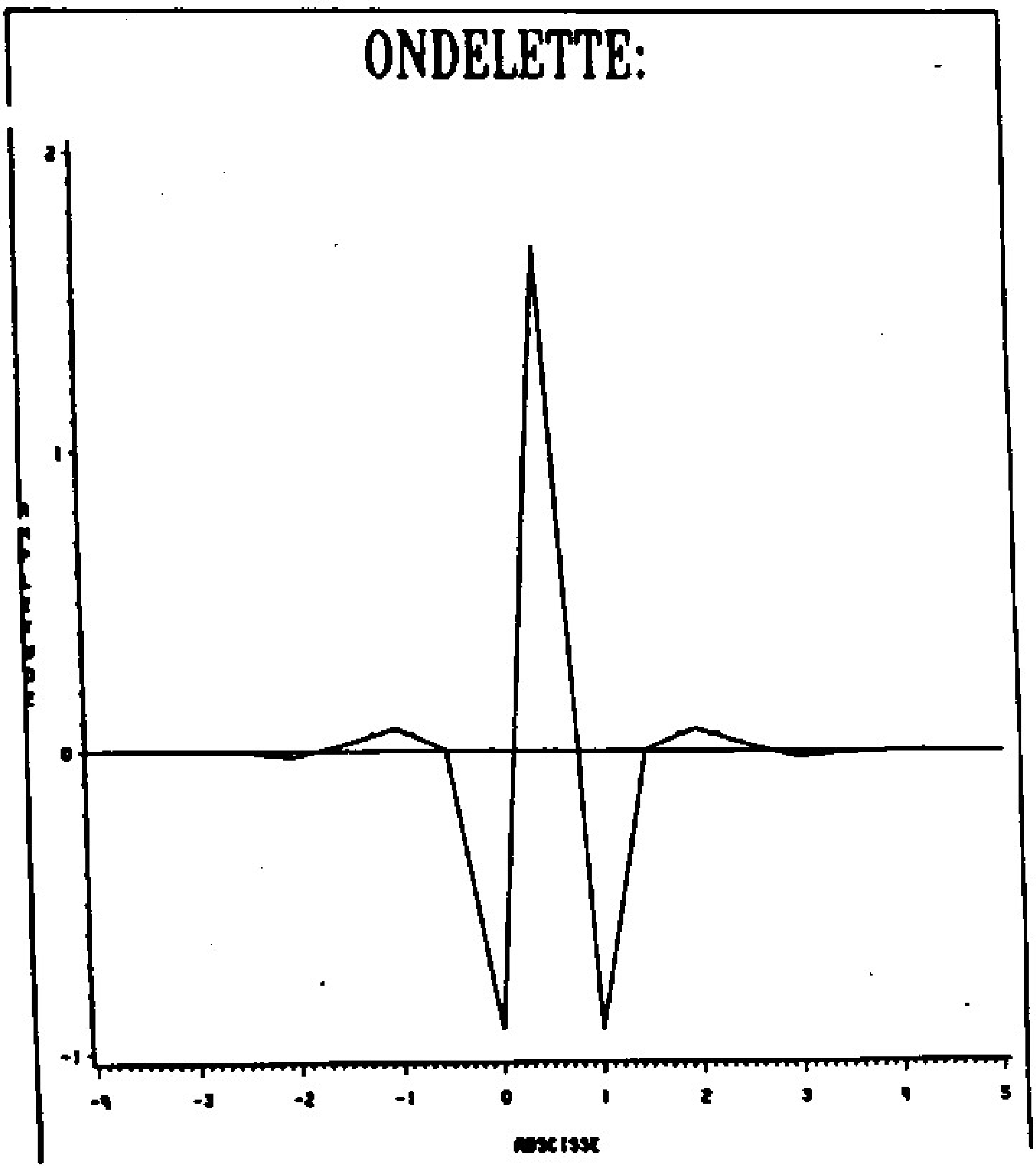
$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_n h_n \, e^{-2\pi i n \xi}$$

$$|m_0(\xi)|^2 + |m_0(\xi + 1/2)|^2 = 1$$

$$\hat{\phi}(\xi) = m_0(\xi/2)\, \hat{\phi}(\xi/2) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$$

. strategy to construct m_0 so that infinite product has decay

. check that strategy works!


ONDELETTE:
2
1
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-1
-4
-3
-2
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0
1
2
3
4
5
ABSCISSE

- <u>Wanted</u>: $m_0(\xi) = \sum_{n=0}^{\cancel{\infty}\, N} \frac{1}{\sqrt{2}} h_n \, e^{-2\pi i n \xi}$

$$|m_0(\xi)|^2 + |m_0(\xi + 1/2)|^2 = 1$$

$$\prod_{j=1}^{\infty} m_0(2^{-j}\xi) \quad \text{decays for } |\xi| \to \infty.$$

- $m_0(\xi) = \left(\dfrac{1 + e^{-2\pi i \xi}}{2}\right)^L \mathcal{F}(\xi)$

$$\frac{1 + e^{-2\pi i \xi}}{2} = e^{-\pi i \xi} \quad \cos \pi \xi$$

$$\prod_{j=1}^{\infty} \cos(2^{-j}\pi\xi) = \frac{\sin \pi \xi}{\pi \xi}$$

$\Rightarrow$ if sufficient control over $\prod_{j=1}^{\infty} \mathcal{F}(2^{-j}\xi)$,
then ϕ will have good decay.

- $\Rightarrow \quad (\cos \pi\xi)^{2L} \underbrace{|\mathcal{F}(\xi)|^2}_{\substack{\text{polynomial in } \cos 2\pi\xi \\ \to \text{ polynomial in } \sin^2 \pi\xi}} + (\sin \pi \xi)^{2L} |\mathcal{F}(\xi + 1/2)|^2 = 1$

$$(1-y)^L P(y) + y^L P(1-y) = 1.$$

$$P(y) = \frac{1}{(1-y)^L} + O(y^L)$$

$$= \sum_{\ell=0}^{L-1} \binom{L-1+\ell}{\ell} y^\ell + O(y^L)$$

$$\Rightarrow \quad |\mathcal{F}(\xi)|^2 = \sum_{\ell=0}^{L-1} \binom{L-1+\ell}{\ell} (\sin \pi\xi)^{2\ell}$$

- use a lemma by Riesz to "extract square root"

$$\longrightarrow \quad \mathcal{F}(\xi) = \sum_{\ell=0}^{L-1} f_\ell \, e^{-2\pi i \xi \ell}$$

f_ℓ real.

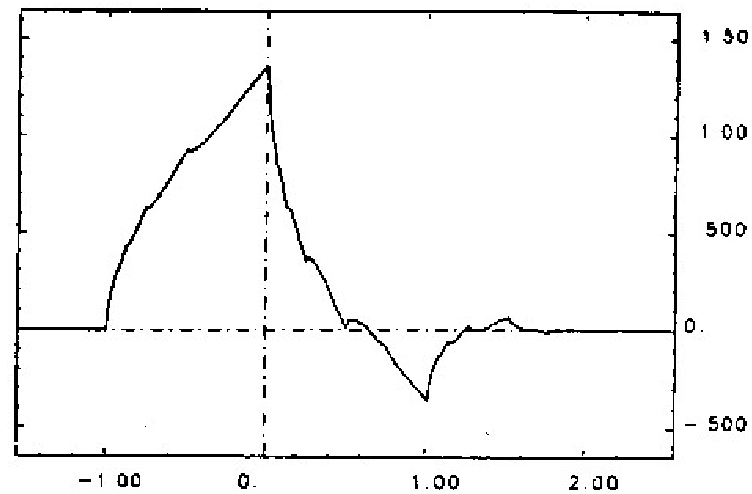
- $$\prod_{j=1}^{\infty} |\mathcal{F}(2^{-j}\xi)| \leq C(1+|\xi|)^{\mu L}$$

$\mu \sim .81$

$$\Rightarrow \quad |\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-\nu L}$$

$\nu \sim .19$

$\Rightarrow$ arbitrarily high regularity!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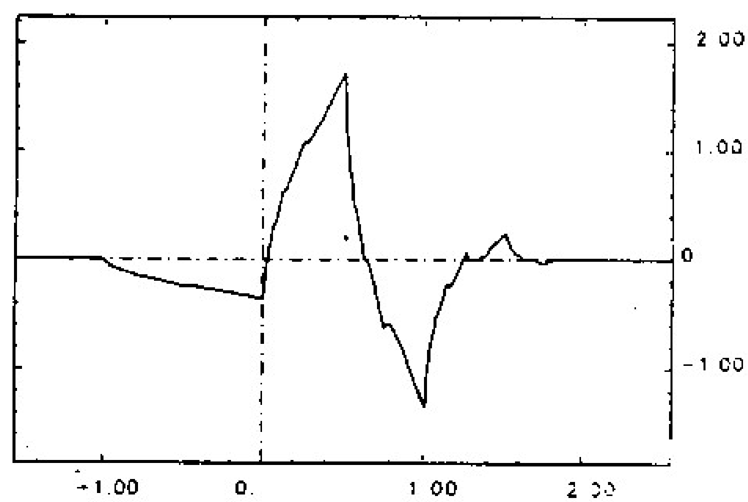
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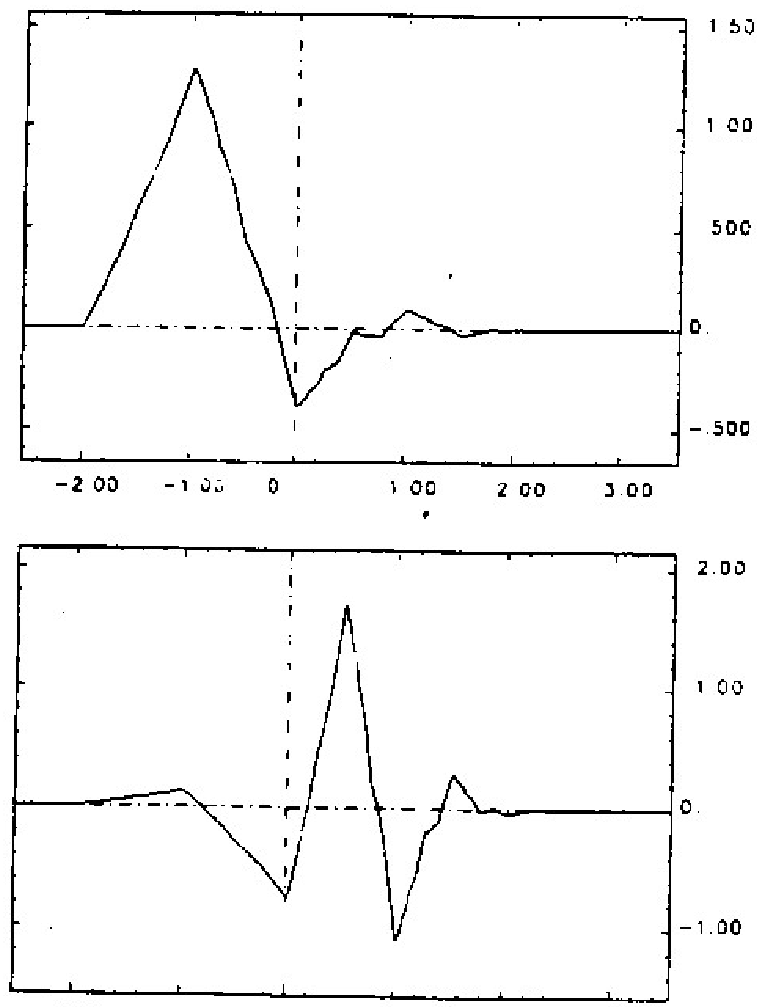
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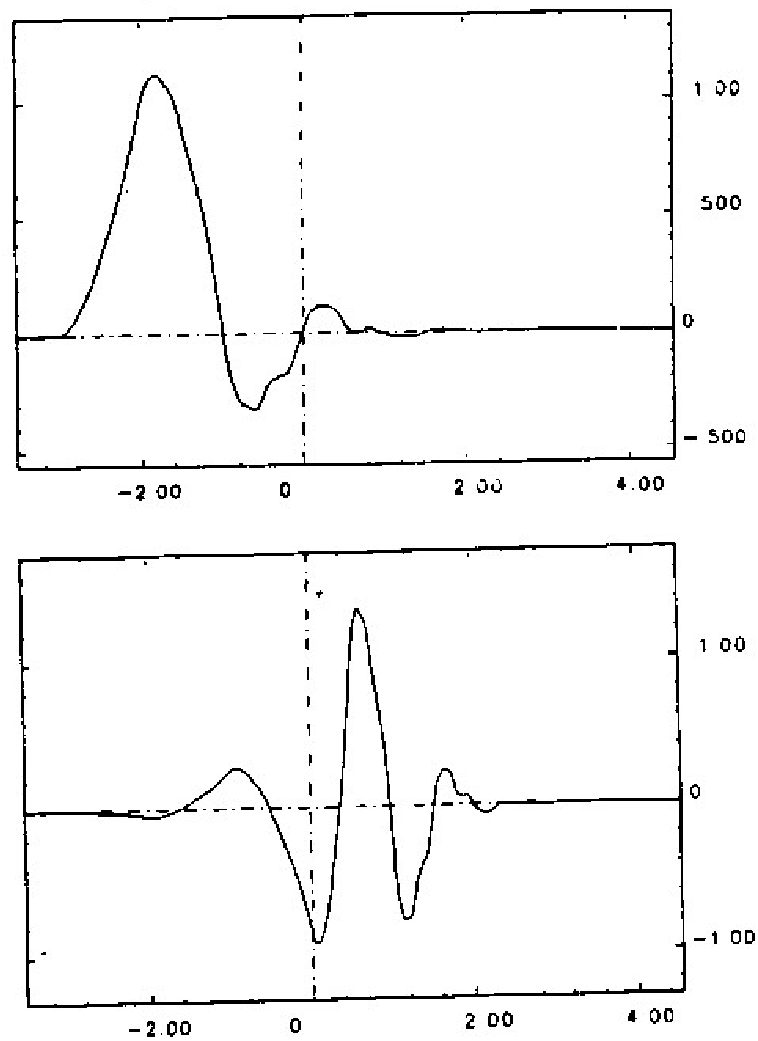
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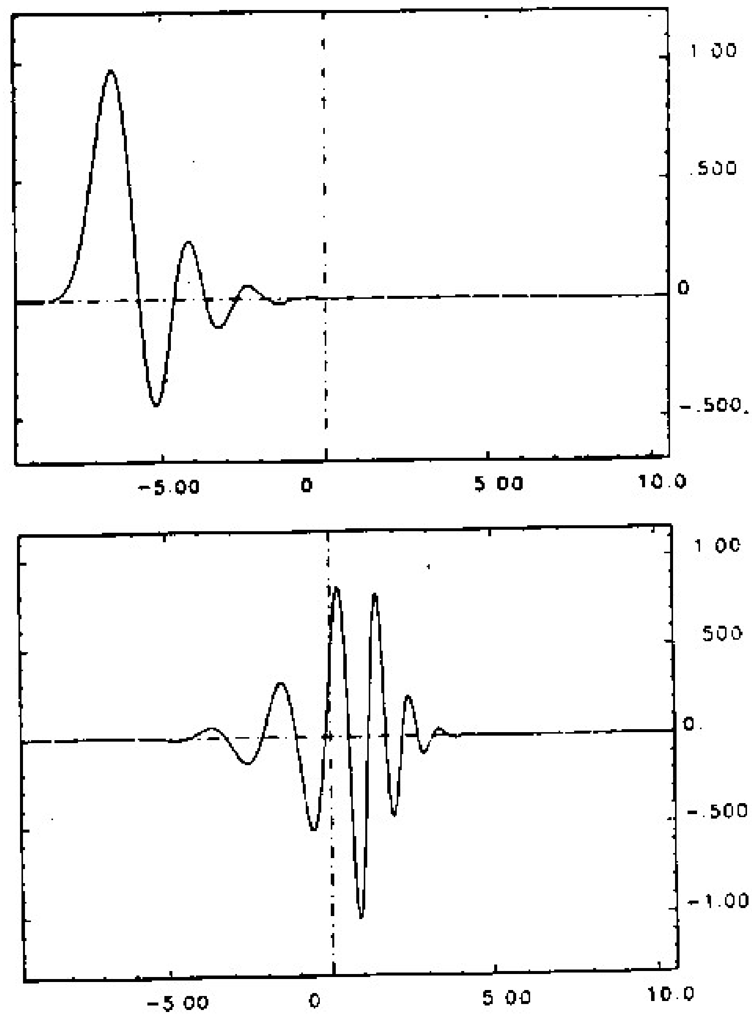
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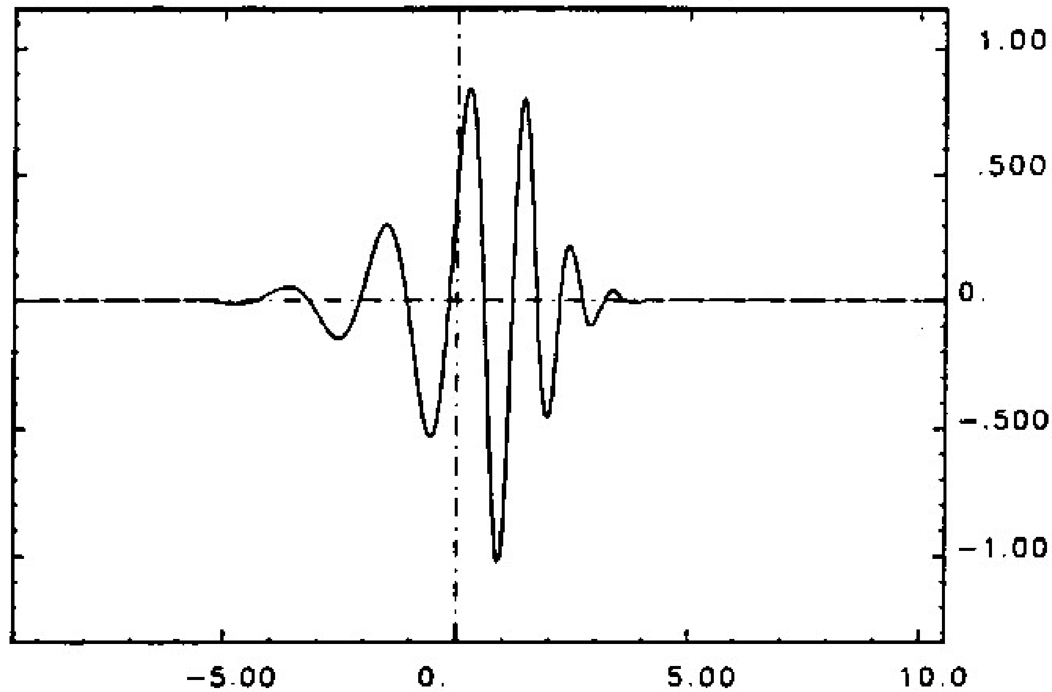
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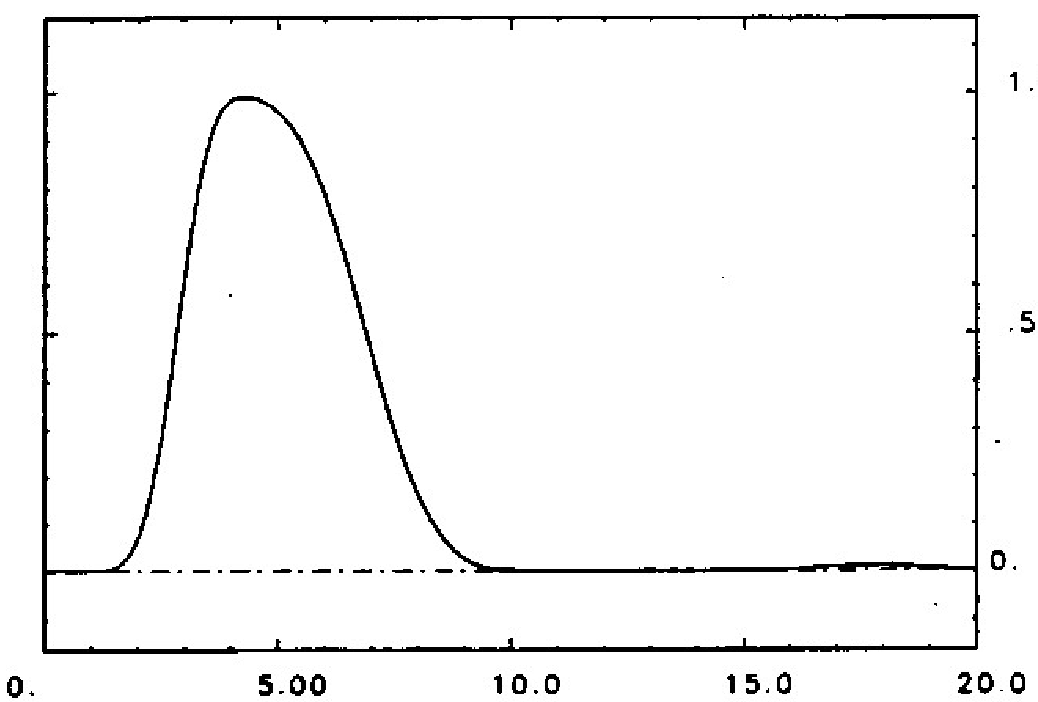
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What does all this have to do with subband coding?

$$f \in V_0 = V_1 \oplus W_1 \qquad f = \sum_n f_n \, \phi_{0n}$$

$$f = s + d \qquad s = \sum_k s_k \, \phi_{1k}$$

$$d = \sum_k d_k \, \psi_{1k}$$

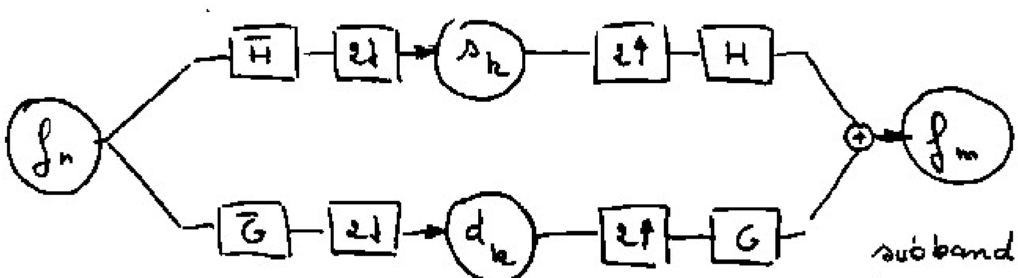
$$\boxed{\begin{aligned} \phi_{1k} &= \sum_n h_{n-2k} \, \phi_{0n} \\ \psi_{1k} &= \sum_n g_{n-2k} \, \phi_{0n} \end{aligned}} \qquad g_n = (-1)^n h_{-n+1}$$

$$\Rightarrow \qquad s_k = \sum_n h_{n-2k} \, f_n$$

$$d_k = \sum_n g_{n-2k} \, f_n$$

Inverse transform: uses transposed matrix

$$f_m = \sum_k \left[h_{m-2k} \, s_k + g_{m-2k} \, d_k \right]$$



subband coding with exact reconstruction.

⇛ the "filter coefficients" h_n, g_n coming from an orthonormal basis of wavelets correspond exactly to the filters in an exact reconstruction subband coding scheme.

(→ compact support for ϕ, ψ important! leads to FIR filters).

What role does regularity play?

Suppose you have the decomposition

$$\begin{tikzcd}
f \arrow[r] \arrow[dr] & f_L \arrow[r] \arrow[dr] & f_{LL} \arrow[r] \arrow[dr] & f_{LLL} & \cdots \\
 & f_H & f_{LH} & f_{LLH} &
\end{tikzcd}$$

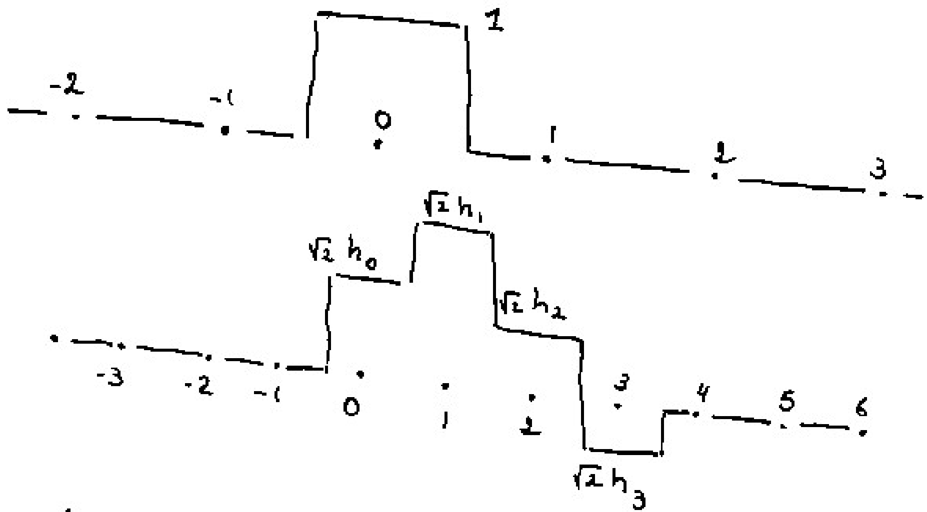
with reconstruction

$$\begin{tikzcd}
f \; \oplus & f_L \; \oplus \arrow[l] & f_{LL} \; \oplus \arrow[l] & f_{LLL} \arrow[l] & \cdots \\
f_H \arrow[u] & f_{LH} \arrow[u] & f_{LLH} \arrow[u] & &
\end{tikzcd}$$

What does a sequence

$$f_{LLL} = \ldots \; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \; \ldots$$

correspond to?



$$(f_0)_n = \delta_{n0}$$

$$(f_j)_k = \sqrt{2} \sum_n h_{n-2k} (f_{j-1})_n$$

$F_j(x)$: piecewise constant on $[\,\ell 2^{-j}, (\ell+1) 2^{-j}[$

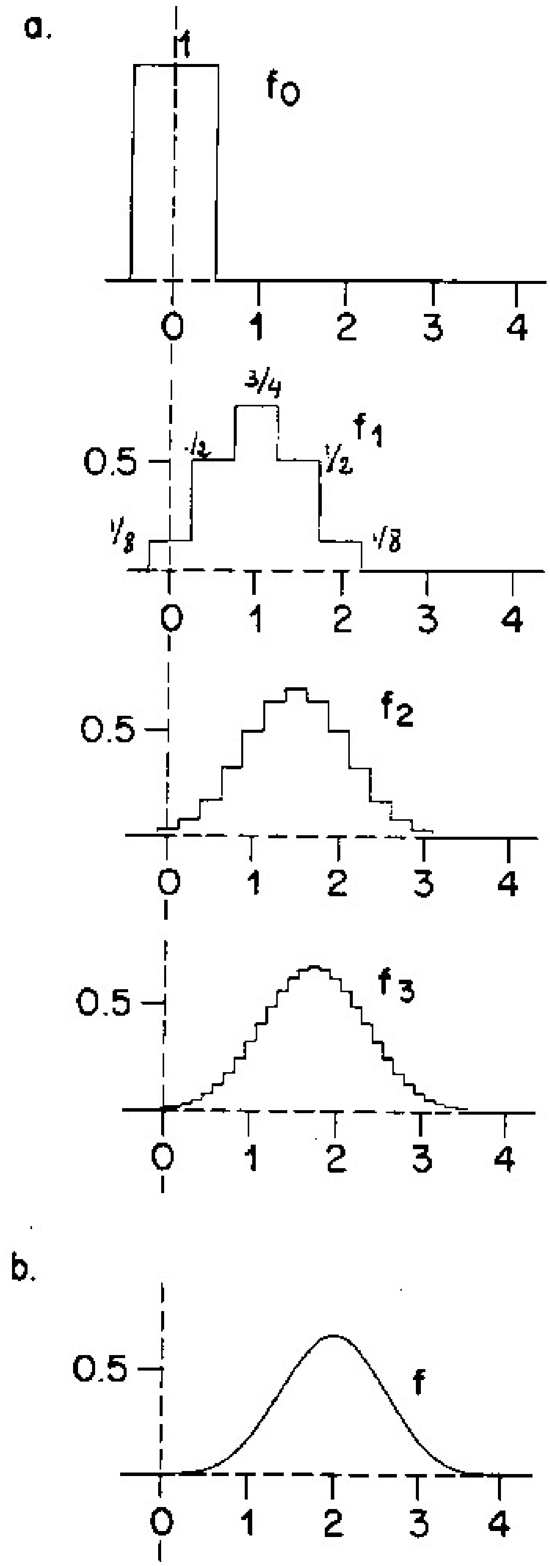
$$F_j(x) = \sqrt{2} \sum_n h_n F_{j-1}(2x-n)$$

$F_j \xrightarrow[j\to\infty]{}$ fixed point of T

$$(TF)(x) = \sqrt{2} \sum_n h_n F(2x-n).$$

But this fixed point is ϕ!

$$\phi(x) = \sqrt{2} \sum_n h_n \phi(2x-n)$$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b.
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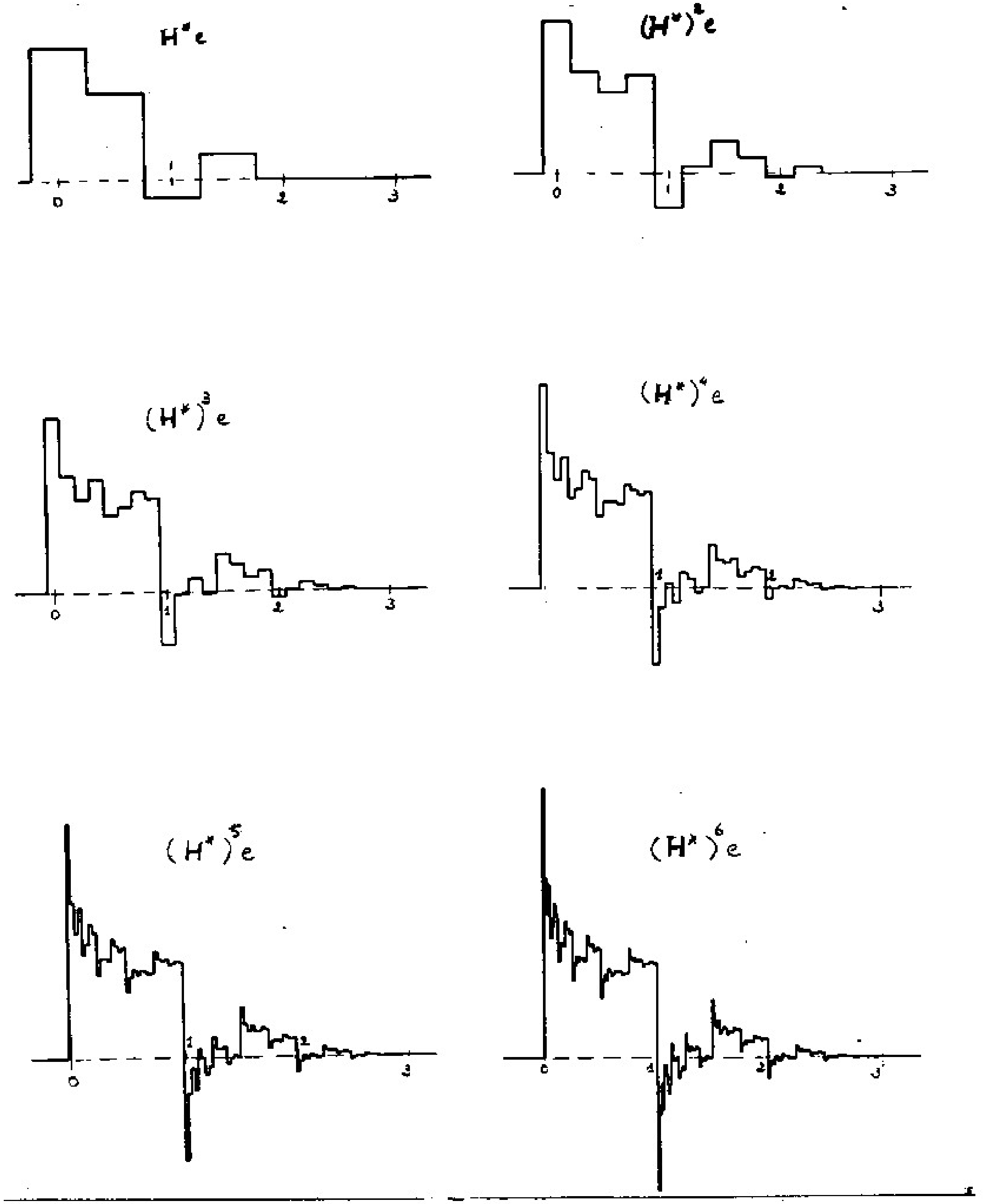

H*e
(H*)²e
(H*)³e
(H*)⁴e
(H*)⁵e
(H*)⁶e

$\Rightarrow$ Regularity is a good idea.

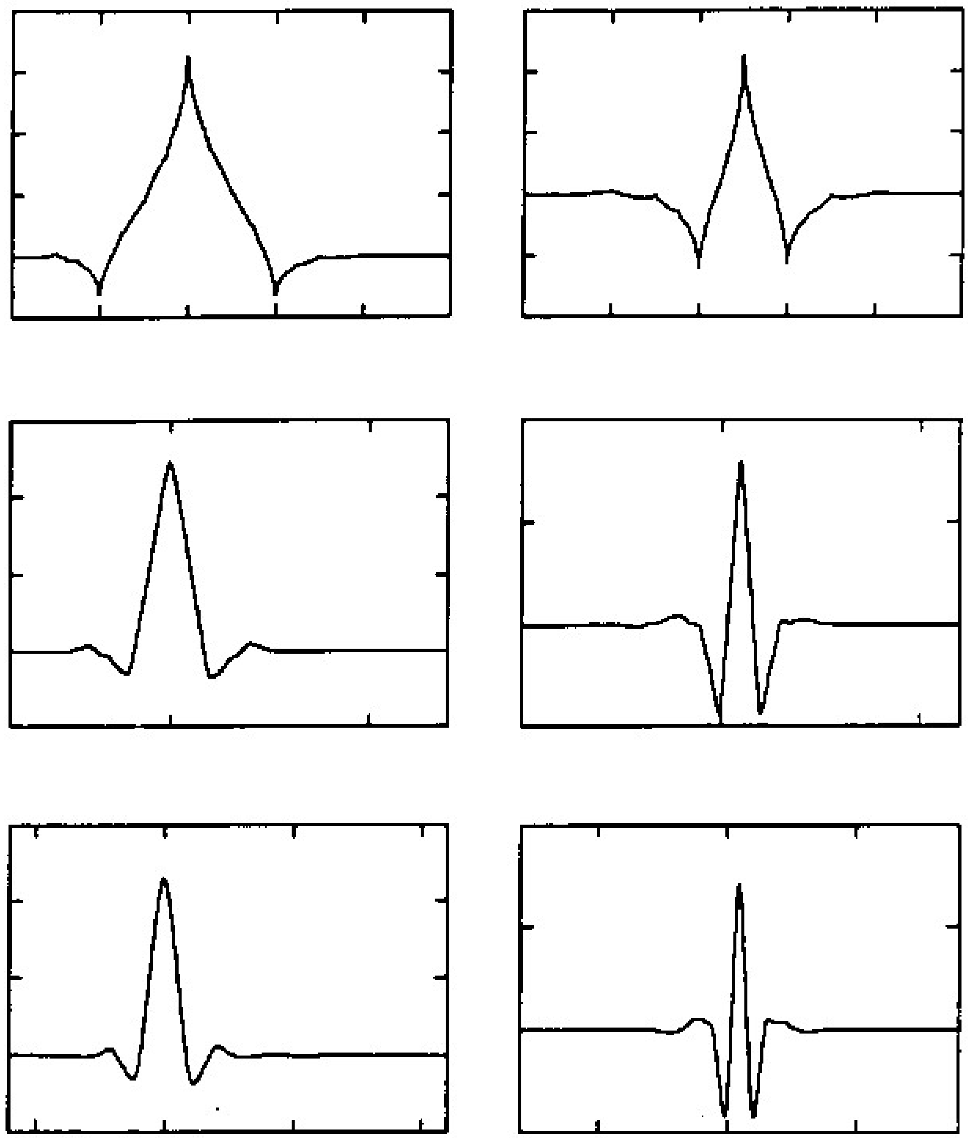
Regularity forces constraints on filters.

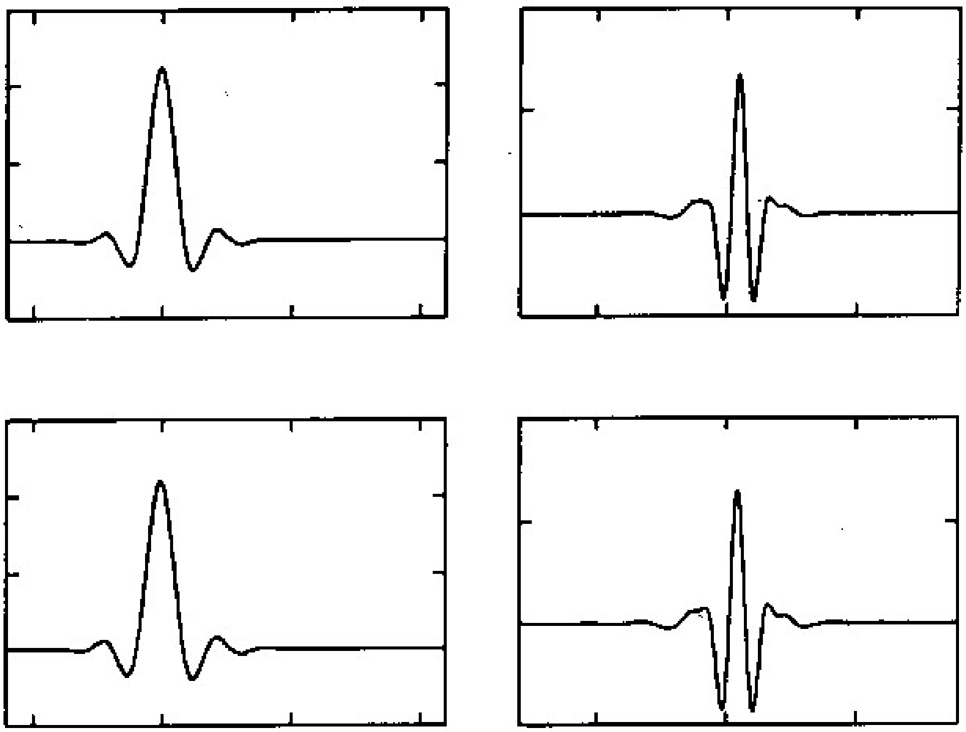
$$\phi, \psi \in C^k$$

$$\Rightarrow \quad m_0(\xi) = \left(\frac{1 + e^{-2\pi i \xi}}{2} \right)^k \cdot \mathcal{J}(\xi).$$

low pass filter has zero of order k at $\xi = 1/2$.

Same is true for generalizations of orthonormal wavelet bases to higher dimensions, or to other dilation factors than 2.





Biorthogonal wavelet bases.

- $f = \sum_{j,k} \langle f, \psi_{jk} \rangle \tilde{\psi}_{jk}$

  correspond to analysis filters
  & synthesis filters

- symmetric $\psi, \tilde{\psi}$ possible ($\longleftrightarrow$ linear phase filters!)

- regularity constraints:

  $$\psi \in C^k \Rightarrow \int dx\, x^\ell\, \tilde{\psi}(x) = 0 \qquad \ell = 0, \dots k-1$$

  $$\Rightarrow m_0(\xi) = \left(\frac{1 + e^{-2\pi i \xi}}{2}\right)^k \mathcal{F}(\xi)$$

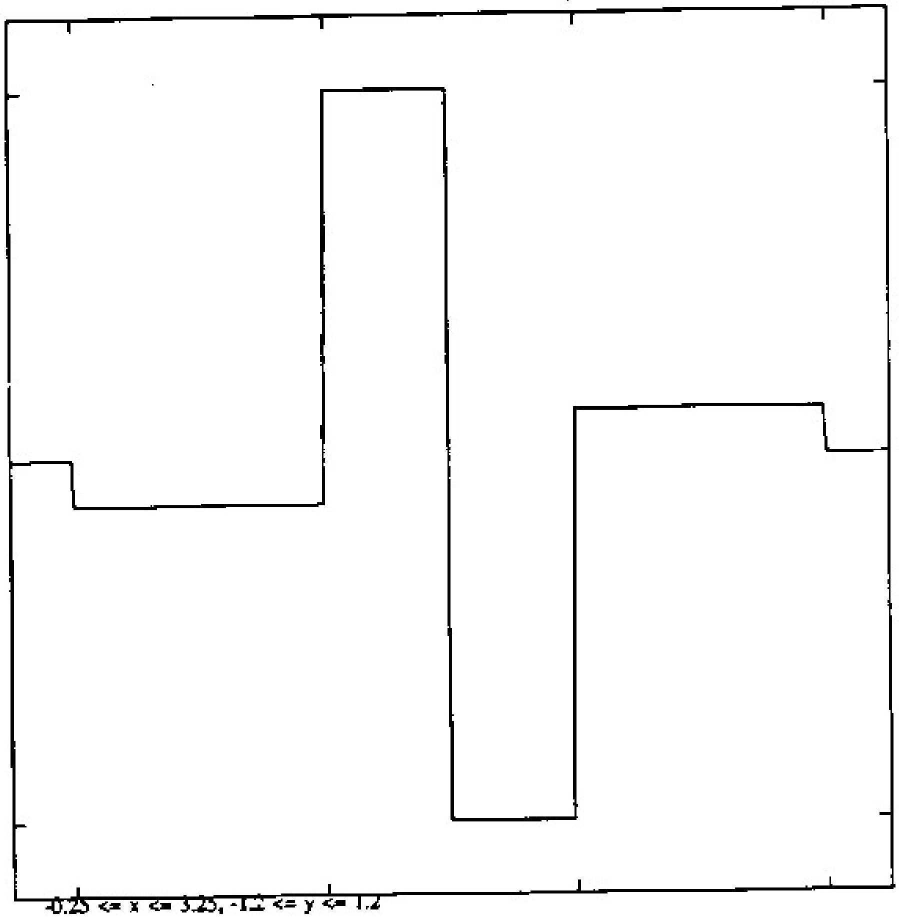
  regularity on both $\psi, \tilde{\psi}$

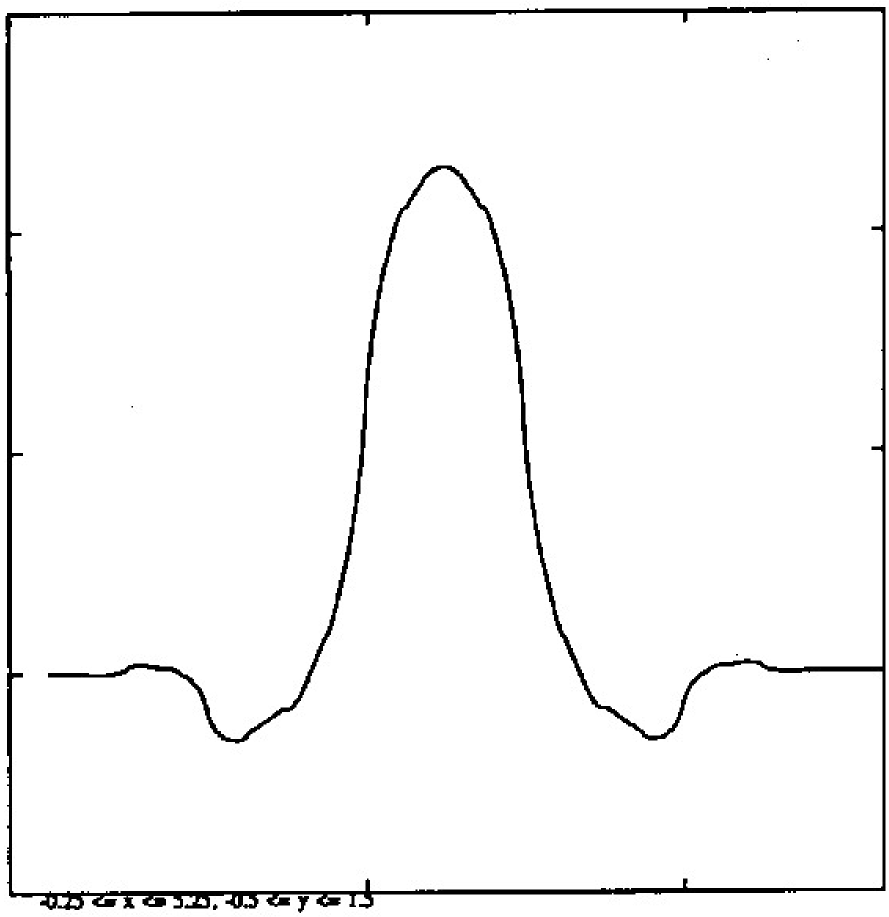
  $\Rightarrow$ both $m_0, \tilde{m}_0$ need factorization of this kind.

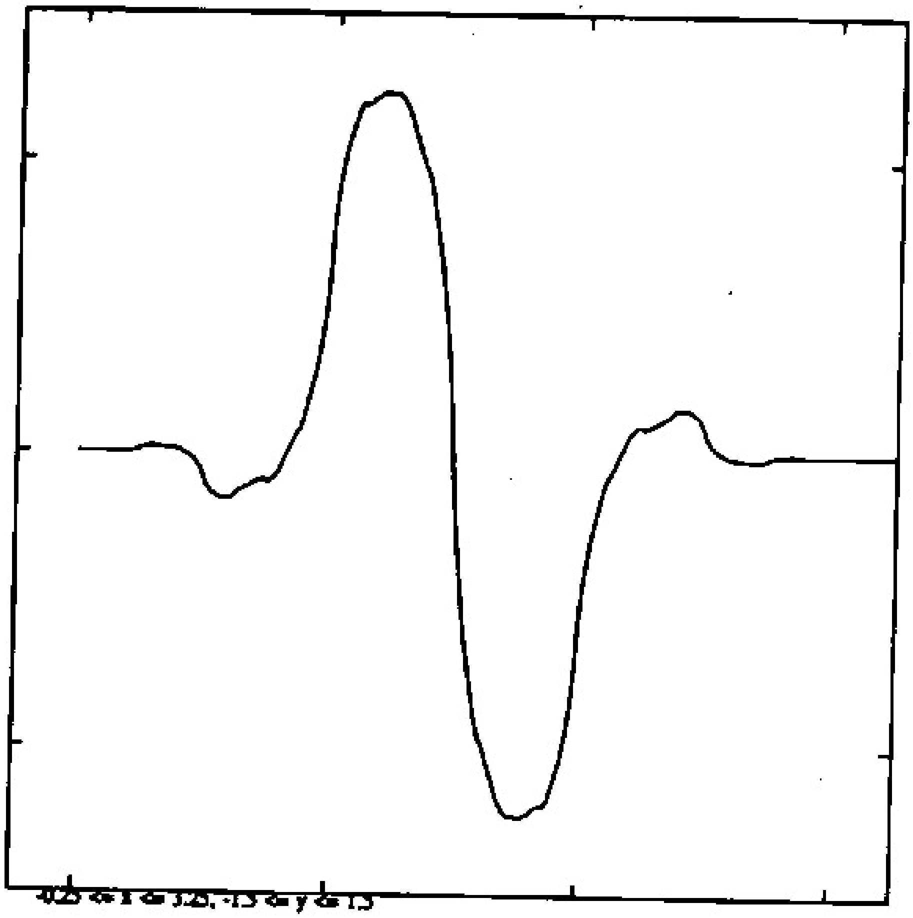
- Examples:
  - m_0 binomial ($\Rightarrow \phi$ B-spline)

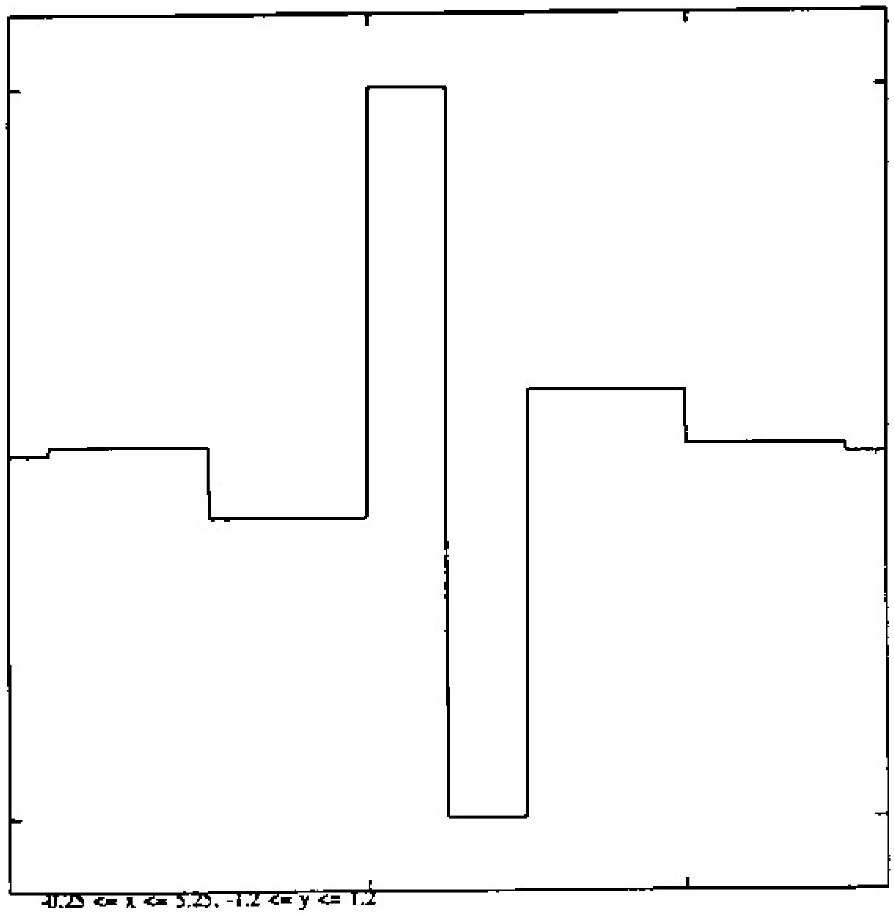
    family of more and more regular $\tilde{m}_0$
  - rearrange previous examples (cf Vetterli)
  - $m_0, \tilde{m}_0$ both very close to orthonormal case

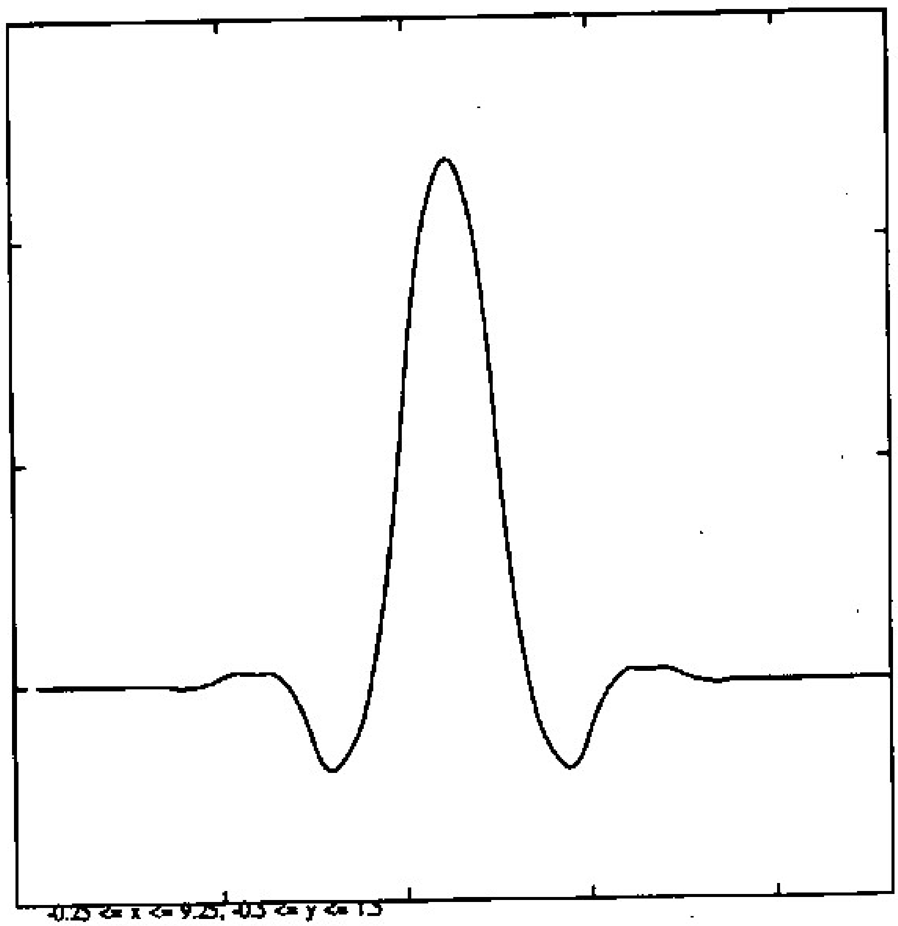
    ex $m_0(\xi) = -.05\, e^{-4\pi i \xi} + .25\, e^{-2\pi i \xi} + .6 + .25\, e^{2\pi i \xi} - .05\, e^{4\pi i \xi}$

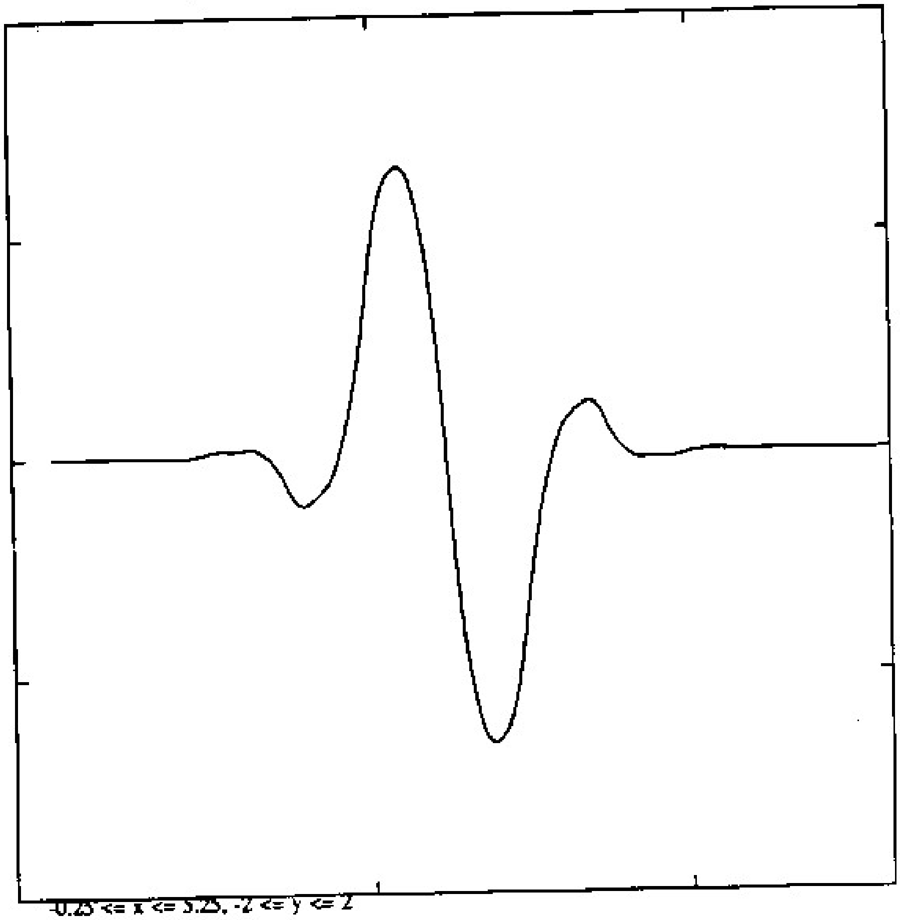

-0.25 <= x <= 3.25, -1.2 <= y <= 1.2


-0.25 <= x <= 5.25, -0.5 <= y <= 1.5


-0.25 <= x <= 3.25, -1.5 <= y <= 1.5


-0.25 <= x <= 5.25, -1.2 <= y <= 1.2


-0.25 <= x <= 9.25, -0.5 <= y <= 1.5


-0.25 <= x <= 5.25, -2 <= y <= 2

"Wavelets" : more than just orthonormal bases!

$\boxed{\text{Continuous case}}$

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right)$$

$\int dx\, \psi(x) = 0$ $\qquad \psi$ symmetric

$$f = C_\psi^{-1} \int_0^\infty \frac{da}{a^2} \int_{-\infty}^{\infty} db \, \langle f, \psi_{ab} \rangle \psi_{ab}$$

$$\langle f, \psi_{ab} \rangle = \int dt \, f(t) \, \psi_{ab}(t)$$

$\boxed{\text{Frames.}}$

$a = a_0^m \qquad m \in \mathbb{Z} \quad (a_0 > 1 \text{ fixed})$

$b = n b_0 a_0^m \qquad n \in \mathbb{Z} \quad (b_0 > 0 \text{ fixed})$

$$\psi_{mn}(x) = a_0^{-m/2} \psi(a_0^{-m} x - n b_0)$$

2 dual points of view:

- characterize f by $\langle f, \psi_{mn} \rangle$
- find α_{mn} so that $f = \sum_{m,n} \alpha_{mn} \psi_{mn}$.

CONCLUSION.

- Subband coding with exact reconstruction

  ⟷ orthonormal wavelet bases

  biorthogonal wavelet bases

  Regularity !

- More to wavelets than subband coding.