Local stability of random exponential marking

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Abstract: Random exponential marking (REM) is an attractive adaptive queue management algorithm. It uses the quantity known as 'price' to measure the congestion in a network. REM can achieve high utilisation, small queue length, and low buffer overflow probability. Many works have used control theory to provide the stable condition of REM without considering the feedback delay. Recently, sufficient conditions for local stability of REM have been provided when the sources have a uniform one- or two-step feedback delay. Nevertheless, no work has been done for the case of arbitrary uniform delay. The authors propose a continuous time model to generalise the local stable condition for REM in a multilink and multisource network with arbitrary uniform feedback delay.

1 Introduction

TCP is one of the major transport protocols over the current Internet. TCP provides end-to-end congestion control by dynamically adjusting the transmitting rate based on the congestion feedback. In the network, packets are either dropped or marked when congestion happens. Before random early detection (RED) [1] was proposed, the major congestion control scheme was DropTail, which suffers from synchronisation of the senders and oscillation of buffer occupation. RED tries to overcome the shortcomings of DropTail by dropping packets in a probabilistic manner before the buffer overflows. Explicit congestion notification (ECN) [2] notifies users of congestion by marking the packets probabilistically instead of dropping them. The users react to the marked packets as if packet loss is detected.

To achieve the desired properties, such as maintaining fairness among users and stabilising queue length in the network, packet marking or dropping schemes should be designed very carefully. Adaptive queue management (AQM) schemes [3–7] have been proposed to mark or drop packets intelligently. One of the major challenges faced by AQM is how to achieve a stable system. Recent works [8, 9] have discussed the dynamic behaviour of TCP-AQM within the framework of a feedback control system. It has been shown [8] that TCP-RED can exhibit chaotic behaviour when the RED parameters fall into a certain region. It has also been demonstrated [9] that instability of TCP-RED is the inevitable result of the scheme itself.

REM [7] is a very attractive AQM scheme in terms of achieving high link utilisation, stable queue length and low packet loss. REM distinguishes itself from other AQM schemes by introducing the concept of 'price'. The 'price' is the measurement of congestion at each link. Unlike in RED, the 'price' is decoupled from performance measures such as loss, queue length or delay. At each link, REM continuously updates the value of the price (the update scheme is explained in Section 2) and marks packets with exponential probability. Readers are referred to [7] for more details.

Like other AQM algorithms, REM has to address the stability problem. Extensive simulations have shown that REM exhibits highly stable behaviour in a wide range of network configurations. In [7], the local stable condition was studied under the discrete time model without considering feedback delay. In [10] and [11], global stability was proved for zero feedback delay in continuous and discrete time models, respectively. In [12], for the first time, the local stability of REM with feedback delay was investigated. Nevertheless, Yin and Low [12] used the discrete time model and only presented the analytical result of one- and two-step uniform feedback delay.

The major contribution of this paper is derivation of the local stable conditions of REM with any value of uniform feedback delay in a multilink and multisource network by using the continuous time model. To the best of our knowledge, this problem results has not been addressed before.

2 Background and problem formulation

Consider a network with L links, each with capacity C_i ($i = 1, 2, \ldots L$). Assume there exist S TCP sources, which share the L links. The routing policy is expressed by an $L \times S$ matrix A with elements a_{ij} defined as

$$a_{ij} = \begin{cases} 1 & \text{if source } j \text{ uses link } i \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The following notation is adopted:

- $p_i(t)$ non-negative price for link i at time t. The corresponding vector form is $\boldsymbol{p}(t) = (p_1(t), p_2(t), \ldots p_L(t))$
- $q_i(t)$ queue length of link i at time t. The corresponding vector form is $\boldsymbol{q}(t) = (q_1(t), q_2(t), \ldots q_L(t))$
- $x_j(t)$ rate of source j at time t, $(j = 1, 2 \ldots, S)$
- $r_i(t)$ total rate of all sources crossing link i
- $y_j(t)$ total price of all links used by source j

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For the sake of brevity, the variable t in the above notation is omitted in subsequent discussions.

From the above notation, we have

$$r_i = \sum_{j=1}^{S} a_{ij} x_j \tag{2}$$

and

$$y_i = \sum_{i=1}^{L} a_{ij} p_i \tag{3}$$

According to [10], the TCP source j adjusts its sending rate by maximising $U_j(x_j) - p_j x_j$; here, $U_j(x_j)$ is the utility function of source j and is strictly concave increasing. The utility function of TCP Reno is [9]

$$U_j(x_j) = \frac{\sqrt{2}}{d_j} \arctan\left(\frac{d_j x_j}{\sqrt{2}}\right) \tag{4}$$

and for TCP Vegas [9].

$$U_j(x_j) = \log(x_j) \tag{5}$$

In both versions of TCP, source j can be modelled to adjust its transmission rate in a smoothed version of the following adjustment [7, 13, 14]:

$$\begin{aligned} x_j(t) =& [U_j']^{-1}(y_j(t - d_j)) \\ =& [U_j']^{-1}\left(\sum_{i=1}^{L} a_{ij} p_i(t - d_j)\right) \end{aligned} \tag{6}$$

where $[U_j']^{-1}$ is the inverse function of the derivative of utility function U_j' (it exists since U_j' is strictly concave-increasing). As in [7], we assume that the forward delay is zero and the backward delay is d_j. To simplify our analysis, we only consider the case of homogeneous delay, implying that $d_j = d$ for all j. Thus, (6) is reduced to

$$x_j(t) = [U_j']^{-1}\left(\sum_{i=1}^{L} a_{ij} p_i(t - d)\right) \tag{7}$$

We adopt the continuous time model [10] to describe the dynamics of REM

$$\frac{dq_i(t)}{dt} = \begin{cases} r_i - C_i & \text{if } q_i(t) > 0 \\ [r_i - C_i]^+ & \text{if } q_i(t) = 0 \end{cases} \tag{8}$$

$$\frac{dp_i(t)}{dt} = \begin{cases} \gamma(\alpha q_i + r_i - C_i) & \text{if } p_i(t) > 0 \\ \gamma[\alpha q_i + r_i - C_i]^+ & \text{if } p_i(t) = 0 \end{cases} \tag{9}$$

where $[z]^+ \triangleq \max(0, z)$, and γ and α are positive constants. Equations (7)–(9) can be interpreted as a gradient projection algorithm used to solve an optimisation problem. There is a trade-off between the selection of large α and small α: a small α leads to a high link utilisation at the cost of a large transient queue length; on the other hand, a large α allows a small transient queue length and a low link utilisation. γ corresponds to the step size of the gradient algorithm. Readers are referred to [7, 10, 12] for detailed discussion on γ and α. It is difficult to analyse the above system due to the nonlinear terms $[\cdot]^+$. In the next Section, we will linearise them so that the analysis becomes tractable.

3 Cost model and performance evaluation

Equations (8) and (9) define a nonlinear dynamic system. Assume $rank(\boldsymbol{A}) = L$. Let $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ be the equilibrium of the original system. It can be shown [7, 11, 12] that

$$\begin{aligned} x_j^* =& [U_j']^{-1}(y_j^*) = [U_j']^{-1}\left(\sum_{i=1}^{L} a_{ij} p_i^*\right), \\ & r_j^* = C_i \quad \text{and} \quad q_i^* = 0 \end{aligned} \tag{10}$$

at the bottleneck link. To simplify our analysis, we develop a linearised version of the original system. With the above properties of the equilibrium, (8) and (9) can be rewritten as

$$\frac{d(q_l(t) - q_l^*)}{dt} = r_l(t) - C_l \tag{11}$$

$$\frac{d(p_l(t) - p_l^*)}{dt} = \gamma[\alpha(q_l(t) - q_l^*) + r_l(t) - r_l^*] \tag{12}$$

Here, we only keep the linear terms in the above equations. Following the similar procedure in [7], we use the first-order Taylor expansion around (p^*, q^*) to further simplify the system. $[U_j']^{-1}(y_j)$ can be expressed as

$$\begin{aligned} [U_j']^{-1}(y_j) =& [U_j']^{-1}(y_j^*) + [[U_j']^{-1}]'(y_j^*)(y_j - y_j^*) \\ =& [U_j']^{-1}(y_j^*) + \frac{1}{U_j''(x_j^*)}(y_j - y_j^*) \end{aligned} \tag{13}$$

Thus, we have

$$\begin{aligned} r_i(t) =& \sum_{j=1}^{S} a_{ij} x_j(t) \\ =& \sum_{j=1}^{S} a_{ij} [U_j']^{-1} y_j(t - d) \\ =& \sum_{j=1}^{S} a_{ij} [U_j']^{-1}(y_j^*) \\ &+ \sum_{j=1}^{S} a_{ij} \frac{1}{U_j''(x_j^*)}(y_i(t - d) - y_i^*) \end{aligned} \tag{14}$$

According to the property $r_i^* = C_i$ at the bottleneck link, we have

$$C_i = \sum_{j=1}^{S} a_{ij} x_j^* = \sum_{j=1}^{S} a_{ij} [U_j']^{-1}(y_j^*) \tag{15}$$

Combining (14) and (15),

$$r_i(t) - C_i = \sum_{i=1}^{S} a_{ij} \frac{1}{U_j''(x_j^*)}(y_j(t - d) - y_j^*) \tag{16}$$

According to (3),

$$y_j(t - d) - y_j^* = \sum_{l=1}^{L} a_{lj}(p_l(t - d) - p_l^*) \tag{17}$$

Combining (16) and (17),

$$r_i(t) - C_i = \sum_{l=1}^{L} \sum_{j=1}^{S} a_{ij} a_{lj} \frac{1}{U_j''(x_j^*)}(p_l(t - d) - p_l^*) \tag{18}$$

Denote $\eta_j = -1/U_j''(x_j^*)$ and define the diagonal matrix

$$\boldsymbol{K} = \text{diag}\{\eta_1, \eta_2 \ldots, \eta_S\} \tag{19}$$

and the following new variables:

$$\bar{q}_i(t) = q_i(t) - q_i^* \tag{20}$$

$$\bar{p}_i(t) = p_i(t) - p_i^* \tag{21}$$

Using (17)–(21), (11) and (12) can be rewritten in the following matrix forms:

$$\frac{d\bar{\boldsymbol{q}}(t)}{dt} = -\boldsymbol{AKA}^T\bar{\boldsymbol{p}}(t-d) \tag{22}$$

$$\frac{d\bar{\boldsymbol{p}}(t)}{dt} = \gamma[\alpha\bar{\boldsymbol{q}}(t) - \boldsymbol{AKA}^T\bar{\boldsymbol{p}}(t-d)] \tag{23}$$

Taking the Laplace transforms of (22) and (23)

$$s\bar{\boldsymbol{q}}(s) = -e^{-sd}\boldsymbol{AKA}^T\bar{\boldsymbol{p}}(s) \tag{24}$$

$$s\bar{\boldsymbol{p}}(s) = \gamma[\alpha\bar{\boldsymbol{q}}(s) - e^{-sd}\boldsymbol{AKA}^T\bar{p}(s)] \tag{25}$$

Substituting (24) into (25)

$$s\bar{\boldsymbol{p}}(s) = -\gamma e^{-sd}\left(1+\frac{\alpha}{s}\right)\boldsymbol{AKA}^T\bar{\boldsymbol{p}}(s) \tag{26}$$

or equivalently

$$\left[s\boldsymbol{I} + \gamma e^{-sd}\left(1+\frac{\alpha}{s}\right)\boldsymbol{AKA}^T\right]\bar{\boldsymbol{p}}(s) = 0 \tag{27}$$

To have a nontrivial solution of $\bar{p}(s)$, it is required that

$$\det\left[s\boldsymbol{I} + \gamma e^{-sd}\left(1+\frac{\alpha}{s}\right)\boldsymbol{AKA}^T\right] = 0 \tag{28}$$

which is called the characteristic equation. Since $\boldsymbol{A}$ is an $L \times S$ matrix with rank L, and $\boldsymbol{K}$ is an $S \times S$ diagonal matrix, $\boldsymbol{AKA}^T$ is an $L \times L$ positive definite matrix. Therefore, we can conclude that $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_L$, where $\lambda_1, \lambda_2, \ldots, \lambda_L$ are the eigenvalues of $\boldsymbol{AKA}^T$. Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L)$. Equation (28) is equivalent to (theorem 7.23 of [15])

$$\det\left[s\boldsymbol{I} + \gamma e^{-sd}\left(1+\frac{\alpha}{s}\right)\Lambda\right] = 0 \tag{29}$$

or

$$s + \gamma e^{-sd}\left(1+\frac{\alpha}{s}\right)\lambda_k = 0 \quad k = 1, 2 \ldots L \tag{30}$$

To have a stable system, all of the L roots of (30) should be on the left-half plane. Next, we shall derive the stable conditions for the above system.

Theorem 1: If the feedback delay d is zero, the system is always stable.

Proof: If d is zero, from (30) we have

$$s^2 + \gamma s + \gamma\alpha\lambda_k = 0 \tag{31}$$

Solving this equation, we obtain

$$s = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\gamma\alpha\lambda_k}}{2} \tag{32}$$

It can be seen that s is always on the left-half plane regardless of the value of γ. This completes the proof.

Theorem 2: If the feedback delay d satisfies $d < D$, the system is stable. Here

$$D = \frac{\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\theta}\right)}{\theta}$$

$$\theta = \sqrt{\frac{\gamma^2\lambda^2 + \sqrt{\gamma^4\lambda^4 + 4\gamma^2\alpha^2\lambda^2}}{2}}$$

$$\lambda = \max_k\{\lambda_k\}$$

Proof: For a fixed value of k, let $\tilde{d}_k$ be the smallest number such that the root for (30) stays right on the imaginary axis. According to theorem 1, when d is zero, the roots of (30) are all on the left-half plane. Therefore, if $d < \min_k \tilde{d}_k$, then all the roots of (30) are on the left-half plane. Next, following the similar procedure of the proof of theorem 2 in [6], we first determine $\tilde{d}_k$.

Let the root of (30) be $s_k = \mathrm{j}\theta_k$; here, j is the unit of the imaginary number instead of the index used before. Since the roots on the imaginary axis are complementary, we only consider $\theta_k > 0$. Hence, (30) becomes

$$\frac{\gamma e^{-\mathrm{j}\theta\tilde{d}_k}\left(1+\frac{\alpha}{\mathrm{j}\theta}\right)\lambda_k}{\mathrm{j}\theta} = -1 \tag{33}$$

There are two conditions on the magnitude and angle, respectively

$$\left|\frac{\gamma e^{-\mathrm{j}\theta\tilde{d}_k}\left(1+\frac{\alpha}{\mathrm{j}\theta_k}\right)\lambda_k}{\mathrm{j}\theta_k}\right| = 1 \tag{34}$$

and

$$\angle\frac{\gamma e^{-\mathrm{j}\theta\tilde{d}_k}\left(1+\frac{\alpha}{\mathrm{j}\theta_k}\right)\lambda_k}{\mathrm{j}\theta_k} = (2n+1)\pi, \quad n = 0, \pm 1, \pm 2 \ldots \tag{35}$$

The condition on the magnitude in (34) leads to

$$\theta_k = \sqrt{\frac{\gamma^2\lambda_k^2 + \sqrt{\gamma^4\lambda_k^4 + 4\gamma^2\alpha^2\lambda_k^2}}{2}} \tag{36}$$

From (35),

$$\theta_k\tilde{d}_k + \arctan\left(\frac{\alpha}{\theta_k}\right) + \frac{\pi}{2} = (2n+1)\pi, \quad n = 1, 2, \ldots \tag{37}$$

Taking $n = 0$ in (37),

$$\tilde{d}_k = \frac{\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\theta_k}\right)}{\theta_k} \tag{38}$$

It can be verified that d_k is an increasing function of λ_k. Therefore

$$\min_k \tilde{d}_k = \min_k\left\{\frac{\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\theta_k}\right)}{\theta_k}\right\} = \frac{\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\theta}\right)}{\theta} \tag{39}$$

where

$$\theta = \sqrt{\frac{\gamma^2\lambda^2 + \sqrt{\gamma^4\lambda^4 + 4\gamma^2\alpha^2\lambda^2}}{2}} \tag{40}$$

Finally,

$$D = \min_k \tilde{d}_k = \min_k\left\{\frac{\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\theta_k}\right)}{\theta_k}\right\} \tag{41}$$

This completes the proof.

4 Discussion of results

According to theorem 2, the system is stable if $d < D$. We can rewrite this condition as

$$\alpha d < \alpha D = \frac{\alpha}{\theta}\left\{\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\theta}\right)\right\}$$
$$= \frac{1}{x}\left\{\frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)\right\} \quad (42)$$

where

$$x = \sqrt{\frac{(\gamma\lambda/\alpha)^2 + \sqrt{(\gamma\lambda/\alpha)^4 + 4(\gamma\lambda/\alpha)^2}}{2}}$$

Figure 1 shows the local stable regions for REM based on (41). The horizontal axis is $\gamma\lambda/\alpha$; the vertical axis is αd. If the system falls into the region below the curve, it is locally stable. Otherwise, it is not stable. We can see that the local stability of REM depends on four parameters: γ, α, d and λ. γ and α are parameters set by the REM algorithm; λ and d are parameters determined by the network topology and the routing policy, which may not be controlled by the REM algorithm. In other words, although some pairs of γ and α lead to local stability under some network conditions, they can cause instability under other network conditions. Therefore, the values of γ and α should be set very carefully.

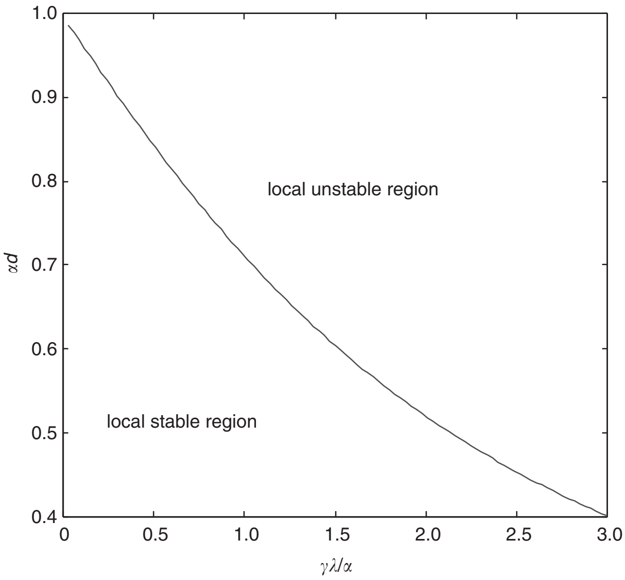

Fig. 1 *Local stable region for parameters of REM*

5 Conclusions

We have used the continuous time model to investigate the local stable conditions for REM in the multilink and multisource network, in which all sources have the same feedback delay. Our study shows that the local stability of REM depends on both the algorithm parameter settings and the network conditions.

There are still several aspects which need to be investigated. First, our approach is based on the linearisation of a nonlinear system, like other research on AQM. However, [8] has demonstrated the important role of the nonlinearity in AQM. Therefore, exploration of the nonlinearity effect is critical to the full understanding of the performance of REM and other AQM schemes. Secondly, our approach assumes the homogeneity of the feedback delay. In the real world, different users can experience different propagation delay and queueing delay. As a result, the feedback delay can exhibit significant heterogeneity. The investigation of this effect will be reported at a later date research.

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7 References

1 Floyd, S., and Jacobson, V.: 'Random early detection gateways for congestion avoidance', *IEEE/ACM Trans. Netw.*, 1993, **10**, (4), pp. 397–413
2 Ramakrishnan, K., Floyd, S., and Black, D.: 'The addition of explicit congestion notification (ECN) to IP'. RFC3168, Sept 2001
3 Kunniyur, S., and Srikant, R.: 'Analysis and design of an adaptive virtual queue (AVQ) algorithm for active queue management'. Proc. ACM SIGCOMM, San Diego, CA, USA, 2001, pp. 123–134
4 Gibbens, R., and Kelly, F.: 'Distributed connection acceptance control for a connectionless network'. Proc. 16th Int. Teletraffic Congr., Edinburgh, UK, June 1999, pp. 941–952
5 Hollot, C., Misra, V., Towsley, D., and Gong, W.: 'On designing improved controllers for AQM routers supporting TCP flows'. Proc. IEEE INFOCOM, Anchorage, AK, USA, Apr. 2001, pp. 1726–1734
6 Katabi, D., Handley, M., and Rohrs, C.: 'Congestion control for high bandwidth-delay product networks'. Proc. ACM SIGCOMM, Pittsburgh, PA, USA, Aug. 2002, pp. 89–102
7 Athuraliya, S., Li, V.H., Low, S.H., and Yin, Q.: 'REM: active queue management', *IEEE Netw.*, 2001, **15**, (3), pp. 48–53
8 Ranjan, P., Abed, E.H., and La, R.J.: 'Nonlinear instabilities in TCP-RED'. Proc. IEEE INFOCOM, New York, NY, USA, June 2002, pp. 248–258
9 Low, S.H., Paganini, F., Wang, J., Adlakha, S., and Doyle, J.C.: 'Dynamics of TCP/AQM and a scalable control'. Proc. IEEE INFOCOM, New York, NY, USA, June 2002, pp. 239–248
10 Paganini, F.: 'On the stability of optimization based flow control'. Proc. American Control Conf., Arlington, VA, USA, June 2001, pp. 4689–4694
11 Yin, Q., and Low, S.H.: 'Convergence of REM flow control at a single link', *IEEE Commun. Lett.*, 2001, **5**, (3), pp. 119–121
12 Yin, Q., and Low, S.H.: 'On stability of REM algorithm with uniform delay'. Proc. IEEE GLOBECOM, Taipei, Taiwan, Nov. 2002, pp. 2649–2653
13 Low, S.H.: 'A duality model of TCP flow control'. Proc. ITC Specialist Seminar on IP traffic management, modeling and management, Monterey, CA, USA, Sept. 2000
14 Low, S.H., Peterson, L., and Wang, L.: 'Understanding Vegas: a duality model', *J. ACM*, 2002, **49**, (2), pp. 207–235
15 Noble, B., and Daniel, J.W.: 'Applied linear algebra' (Prentice Hall, Englewood Cliffs, NJ, 1988, 3rd edn.)