# On the Capacity of WDM Passive Optical Networks

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***Abstract*—To lower costs, WDM PONs may share the usage of wavelengths, transmitters, and receivers among ONUs, instead of dedicating the transmission between OLT and each ONU with one individual wavelength and one individual pair of transmitter and receiver. The specific sharing strategy depends on the network architecture and the wavelength supports of transmitters and receivers. Different sharing strategies may yield different achievable data rates of ONUs. In this paper, we make three main contributions in investigating the impact of the sharing strategy on the capacity region of a WDM PON. First, we abstract the data transmission processes in WDM PONs into directed graphs, where arcs represent wavelengths, transmitters, and receivers, as well as relations between them. Second, we apply Ford and Fulkerson's Max-flow, Min-cut Theorem to derive the upper bound of the capacity region of a WDM PON, and prove that the upper bound is achievable. Third, in light of these analytical results, we discuss and compare capacities of some WDM PON architectures.**

***Index Terms*—WDM, passive optical networks, capacity analysis, tunable transceivers, graph theory.**

## I. INTRODUCTION

BY exploring the potential capacity of optical fibers, Wavelength Division Multiplexing (WDM) Passive Optical Networks (PONs) are becoming promising future-proof access network technologies to meet rapidly increasing traffic demands caused by the popularization of Internet and spouting of bandwidth-demanding applications [1], [2]. To provision multiple wavelengths, transmitters, receivers, and other involved optical components need to support proper wavelengths in WDM PONs.

In a typical WDM PON with $n$ Optical Network Units (ONUs) [3], the upstream/downstream data transmission between Optical Line Terminal (OLT) and each ONU is dedicated by a transmitter, a receiver, and a wavelength channel between the transmitter and the receiver. From the MAC layer's perspective, this typical WDM PON architecture is equivalent to $2n$ independent point to point systems, each of which corresponds to the upstream or downstream traffic transmission between one ONU and OLT. The upstream or downstream traffic rate of an ONU can be achieved if it does not exceed the transmitting rate of the transmitter, the receiving rate of the receiver, and the capacity of the wavelength channel. However, this typical WDM PON architecture is cost-prohibitive for two major reasons. First, the network requires $2n$ transmitters to generate $2n$ respective wavelengths. Transmitters can be wavelength-tunable or wavelength-specific. Wavelength-tunable transmitters are costly, and the colored property of wavelength-specific transmitters introduces high operation, administration, and maintenance cost. Second, prices of optical devices such as lasers at transmitters, optical filters at receivers, and Arrayed Waveguide Gratings (AWGs) at remote nodes, usually increase significantly with the growth of their supporting wavelengths. Consequently, the network is unscalable in the number of ONUs.

Many architectures have been proposed to lower the cost of WDM PONs since the first WDM PON architecture was reported [4]. The general approach is to share the usage of resources among ONUs, for example, sharing demultiplexers and optical cables to reduce the cost of the optical distribution network [5], [6]. While reducing the cost, some sharing schemes may have impact on the network capacity. Consider the following wavelength sharing example in a WDM PON with $n$ ONUs. Transmitters in the first $n/2$ ONUs are fixed tuned to one wavelength, while transmitters in the last $n/2$ ONUs are fixed tuned to another wavelength. Because of the wavelength sharing, transmitters of ONUs sharing the same wavelength cannot transmit simultaneously, and hence the sum of the traffic rates of those ONUs cannot exceed the maximum data rate of the transmitter. As compared to the aforementioned typical WDM PON, its capacity region is significantly reduced.

Besides the above wavelength sharing scheme, transmitters and receivers can be shared among multiple ONUs to save the network cost. Transmitter sharing can be easily employed in the downstream transmission, and receiver sharing is usually employed in the upstream data transmission, since transmitters and receivers are located in OLT in these two cases, respectively. In addition, many approaches have been proposed to share transmitters in upstream transmission to save the expensive cost of transmitter at ONUs, which dominates the overall system cost [7], [8].

Different sharing schemes may affect capacities of WDM PONs in different degrees. Network capacity analysis plays an important role in the design of proper networks to accommodate given network traffic. However, it may not be easy to characterize capacities of WDM PONs with all of these wavelength sharing, transmitter sharing, and receiver sharing strategies.

Formerly, Weichenberg *et al.*[9] analyzed capacities of three optical transport network architectures: optical packet switching, optical flow switching, and optical burst switching. However, to the best of our knowledge, the first attempt to analyze the capacities of WDM PONs was our previous work [10]. In [10], we employed bipartite graphs to describe the relation between tunable transmitters and wavelengths, and the

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relation between tunable receivers and wavelengths, respectively. Based on these results, we discussed some strategies of selecting tunable transmitters in designing WDM PONs [11]. The bipartite-graph based approach is useful in characterizing the capacity regions of networks with one or two types of shared objects. However, when networks have three or more types of shared objects, it is difficult for the bipartite graph to fully characterize the relations among these kinds of shared objects.

An approach is desired to consider the entire effect of these multiple types of shared objects and sharing methods on the capacity region. To this end, we propose to model the upstream and downstream data transmission processes in WDM PONs by using directed graphs, where arcs represent wavelength channels, transmitters, and receivers, as well as relations between them. Constraints on achievable rates imposed by these diversified sharing schemes are mapped into rate limits on graph arcs, from which Ford and Fulkerson's Max-flow, Min-cut Theorem can be applied to obtain the upper bound of the achievable rate region, referred to as the cut-set bound. In order to prove that the cut-set bound is achievable, we show that, for the abstracted directed graph of a WDM PON, traffic rates of ONUs are achievable if they fall into the convex hull of the union over all possible routings of the stable set polytope of the conflict graph of flows, and this convex hull is further equivalent to the cut-set bound of the abstracted directed graph of the WDM PON. With these theoretical results, capacities of given WDM PON architectures can be easily obtained, and thus different WDM PON architectures can be compared. Also, in light of these results, given a capacity region to be achieved, network designers can build a WDM PON by properly selecting optical devices.

Note that this paper considers the network capacity from the MAC layer's perspective. The physical layer performances of optical links are important, and it is indeed challenging to derive the optical link capacity with consideration of the physical layer parameters [12]. Considering constraints in both physical layer and MAC layer will further exacerbate the complexity of the problem, and is thus beyond the scope of this article.

The rest of the paper is organized as follows. Section II discusses various sharing strategies in WDM PONs. Section III describes the system model, where the upstream and downstream data transmission processes are abstracted by directed graphs. Section IV discusses the scheme of deriving capacity region of WDM PONs. We prove that rates in WDM PONs are achievable if they satisfy certain constraints on arcs. Section V discusses and compares some architectures with the same capacity. Section VI presents concluding remarks.

## II. Sharing Strategies in WDM PONs and Their Impact on the Capacity

Generally, the communication between OLT and ONUs in WDM PONs undergoes the following process: optical signals are first generated by transmitters, routed and transmitted to the receiver through optical distribution network (ODN), and finally received by receivers. Transmitters, wavelength channels, and receivers constitute three key elements in the communication process, and also three major types of shared objects.

Wavelength-sharing implies that multiple ONUs share one or more wavelengths in upstream or downstream transmission. This has been commonly used in various WDM PON architectures [3], [13], [14]. A typical example is the upstream transmission in candidate architectures of next-generation PON stage 1 proposed by FSAN [15], in which ONUs are equipped with wavelength specific transmitters, and 8, 16, or 32 ONUs share one upstream wavelength.

Transmitter-sharing implies that multiple ONUs share a set of transmitters for their upstream or downstream traffic transmission to save the transmitter cost. It can be easily realized in downstream transmission since transmitters are centralized at OLT. For upstream transmission, many schemes have been proposed to place transmitters at OLT to generate the seeding light, and transmit the seeding light to ONUs for their upstream transmission. Such schemes can realize colorless ONUs, and in turn lower the network cost. This approach has received intensive research attention in the past years [7], [16], [17]. It is worth noting that transmitters at OLT may be shared among upstream and downstream data transmission of all ONUs in this case.

In a receiver-sharing scheme, receivers are shared among data transmission of multiple ONUs. Receiver sharing is usually used for upstream transmission. For example, in RITENET [16], although each ONU has one dedicated wavelength and one dedicated transmitter for the upstream data transmission, ONUs still cannot transmit traffic simultaneously owing to the sharing of receivers at OLT.

Consider wavelengths, transmitters, and receivers as three types of shared objects. Some WDM PON architectures may share only one type of these objects, e.g., the upstream transmission in CPON [3] and LARNET [18], and the downstream transmission in SW-PON [14]. Some architectures may have two types of shared objects. For example, in the downstream transmission in SWE-PON [13], an array of tunable lasers at OLT generates multiple wavelengths, and these wavelengths are further shared among ONUs. In this case, the joint transmitter and wavelength allocation is needed to meet given downstream traffic demands. Some WDM PON architectures may share the usage of all these three or more types of objects. For example, in the upstream transmission of SUCCESS [7], a set of tunable transmitters located at OLT generates multiple wavelength optical signals, which will be further wavelength routed to ONUs, where some ONUs may share the same wavelength. These signals will be modulated and reflected back to OLT, and finally received by a set of tunable receivers at OLT. Tunable transmitters at OLT, tunable receivers at OLT, and wavelength channels are shared among ONUs. A joint wavelength, transmitter, and receiver allocation is required to achieve given upstream traffic rates. When transmitters and receivers support different wavelengths, the resource allocation scheme needs to consider the individual wavelength support of each transmitter and each receiver.

Therefore, it may not be easy to find a proper resource allocation scheme to achieve a given set of traffic rates of ONUs. Even the problem of determining whether a given set of rates is achievable or not can be difficult, and this is the main focus of this paper. We next try to characterize the capacity region of a WDM PON, which contains all achievable traffic rates of ONUs.

## III. System Modeling

To investigate the capacity regions of WDM PONs, we model the data transmission processes in WDM PONs by using directed graphs. Owing to the passive nature of PONs, we assume static wavelength routing between transmitters and receivers in PONs. We do not model the internal structure of network nodes in ODN and other possible optical components in between, but only consider their impacts on the destination receivers of optical signals. The main focus of the modeling is to describe the properties and relations of transmitters, receivers, and wavelength channels, which will affect the capacity of a WDM PON.

We use two directed graphs to abstract the upstream and downstream data transmission processes, respectively. The abstraction is as follows.

- Create a directed arc for each transmitter, each wavelength channel, and each receiver. Note that two same wavelengths in different fibers are considered as two separate wavelengths channels. For notational convenience, we name these arcs as $tx$ *arc*, $wc$ *arc*, and $rx$ *arc*, respectively, in the rest of the paper.
- Connect $tx$ *arcs* to $wc$ *arcs* as follows: if one transmitter can access one wavelength channel, connect the end vertex of the $tx$ *arc* with the start vertex of the $wc$ arc.
- Connect $wc$ *arcs* to $rx$ *arcs* as follows: if one wavelength can be received by one receiver, connect the end vertex of the $wc$ *arc* with the start vertex of the $rx$ *arc*.
- Create source vertices, each of which is connected with the start vertex of one $tx$ *arc*.
- Create a destination vertex, and connect the destination vertex with the end vertices of all $rx$ *arcs*.

The direction in the graph conforms with the transmission direction of the optical signal. Denote the abstracted graph for a WDM PON as $\mathcal{G}$ in the rest of the paper.

Here are two examples. The first example illustrates the modeling of the upstream transmission in the network shown in Fig. 1(a). In the architecture, each ONU has one wavelength-tunable transmitter. Signals from ONUs are multiplexed onto one fiber, and then sent to OLT. OLT receives these optical signals by using an array of receivers. Assume that there are eight ONUs and four wavelengths, tunable transmitters at these eight ONUs can tune to all of these four wavelengths, and each receiver at OLT can receive one respective wavelength. Note that the network architecture contains other components to realize downstream transmission. For simplicity, we only list the components involved in the upstream transmission. Similar upstream transmissions can be found in architectures presented in [19], [20]. Fig. 1(c) illustrates the abstracted directed graph. The graph contains eight $tx$ *arcs*, four $wc$ *arcs*, and *four* $rx$ *arcs*. Each $tx$ *arc* is connected with four $wc$ *arcs*, and each $wc$ *arc* is further connected with one respective $rx$ *arc*.

Different from those in the network shown in Fig. 1(a), in the network shown in Fig. 1(b), ONUs use the seeding light generated at OLT to fulfill their upstream data transmission. A set of transmitters are equipped at OLT. Optical signals generated from these transmitters are multiplexed onto one fiber and sent to ONUs. In this example, two ONUs share the usage of one wavelength. Each ONU has one receiver to receive its downstream traffic, and one modulator to reflect the upstream signal. The upstream signal is reflected back and received by an array of receivers at OLT. Similar architectures can be found in [7], [8]. Assume there are sixteen ONUs, eight wavelengths, four tunable transmitters, and four tunable receivers. All these tunable transmitters and receivers support all eight wavelengths. Fig. 1(d) and Fig. 1(e) illustrate the abstracted directed graphs for the upstream and downstream transmissions, respectively. Note that the connections between $tx$ arcs and $wc$ arcs, and those between $wc$ arcs and $rx$ arcs can be arbitrary depending on the specific wavelength supports of transmitters and receivers. For a lower cost, wavelength-specific transmitters and limited-range tunable transmitters are preferred over full-wavelength-tunable transmitters[21], [22].

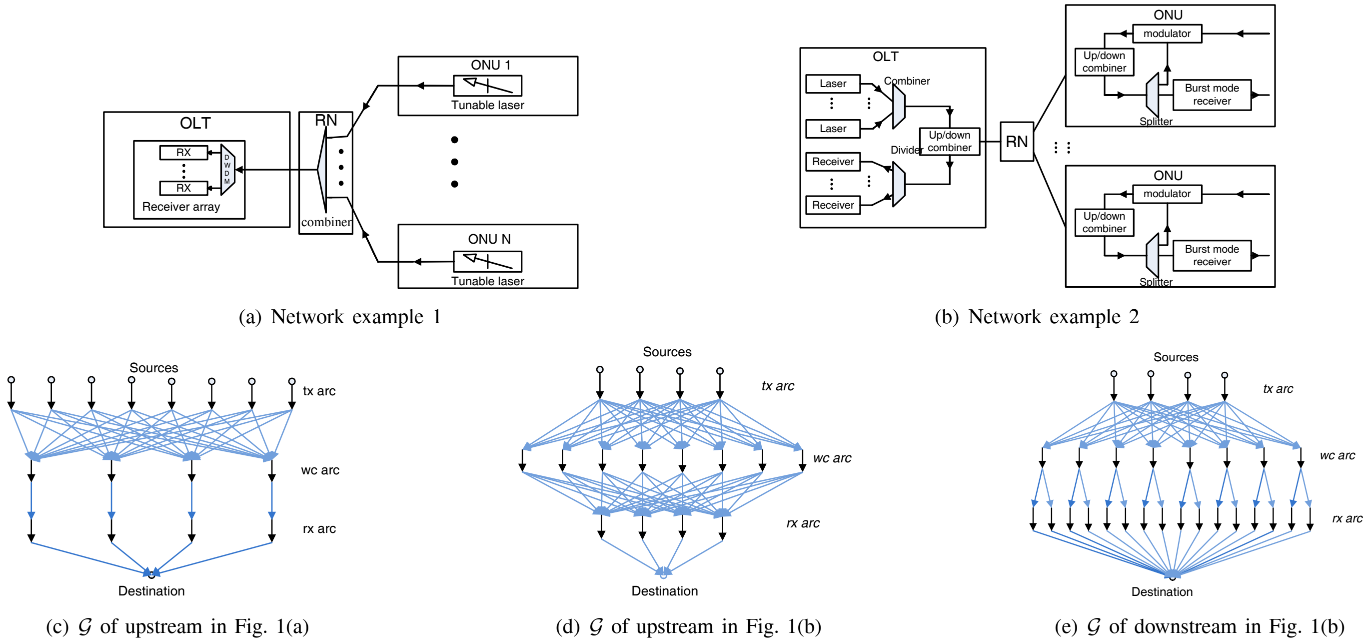
In a WDM PON, any successful traffic transmission is associated with one transmitter, one wavelength channel, and one receiver. A set consisting of a transmitter, a wavelength channel, and a receiver can be considered as one uni-cast route from one source to the destination in the abstracted graph.

Generally, a traffic flow from any source to the destination has to satisfy the following constraints to fulfill their data transmissions.

- $tx$ *arc constraint*: Since transmitters have maximum data transmission rates, the sum of traffic rates of all flows going through a $tx$ arc cannot exceed the maximum data rate of the transmitter.
- $wc$ *arc constraint*: Since two transmitters cannot transmit on the same wavelength channel simultaneously, by sharing the usage of the wavelength channel in the TDM fashion, the total traffic flowing through each $wc$ *arc* cannot exceed the maximum data rate of the transmitter.
- $rx$ *arc constraint*: Since one receiver cannot receive signals in two wavelengths simultaneously, by sharing the usage of the receiver in the TDM fashion, the total traffic flowing through each $rx$ *arc* cannot exceed the maximum data rate of the transmitter.
- *connection arc constraint*: For a given connection arc, the traffic flowing into the arc should be equal to the traffic flowing out of the arc.
- *continuous optical flow constraint*: A successful upstream or downstream data transmission needs one transmitter, one wavelength channel, and one receiver so that a continuous optical flow can go from the source to the destination. Since one transmitter, one wavelength channel, and one receiver can be used for the transmission of only one optical flow at a time, two optical flows cannot transmit simultaneously if they share the same arc in $\mathcal{G}$.

Constraints on $tx$ *arcs*, $wc$ *arcs*, $rx$ *arcs*, and connection arcs state that resources on each arc cannot be overexploited. The *continuous optical flow constraint* states that two optical flows cannot transmit simultaneously if they share the same arc in the abstracted graph $\mathcal{G}$.

For any WDM PON, traffic rates of ONUs can be mapped




(a) Network example 1

(b) Network example 2

(c) $\mathcal{G}$ of upstream in Fig. 1(a)

(d) $\mathcal{G}$ of upstream in Fig. 1(b)

(e) $\mathcal{G}$ of downstream in Fig. 1(b)

Fig. 1. Abstracted graphs of two network examples.

into rates on arcs in graph $\mathcal{G}$. Consider the three scenarios shown in Fig. 1(a) and Fig. 1(b).

- For the upstream transmission shown in Fig. 1(a), resources received by an ONU for its upstream transmission are equivalent to the sum of resources received by flows flowing through the corresponding $tx$ *arc* of the ONU shown in Fig. 1(c). Traffic rates from ONUs can be achieved if the same rates on $tx$ *arcs* can be achieved.
- For the upstream transmission shown in Fig. 1(b), two ONUs share the usage of one wavelength. Resources received by two ONUs sharing the same wavelength equal to the sum of resources received by flows going through the corresponding $wc$ *arc* of these two ONUs shown in Fig. 1(d). Let the rate of a $wc$ *arc* be the sum of traffic rates of ONUs using this wavelength. Then, upstream traffic rates of ONUs are achievable if rates on $wc$ *arcs* can be achieved.
- In Fig. 1(b), downstream traffic rates of an ONU equal to rates of flows flowing through the corresponding $rx$ *arc* of the ONU in Fig. 1(e). Traffic rates from ONUs can be achieved if the same rates on $rx$ *arcs* can be achieved. Therefore, we investigate the achievable rates on $tx$ *arcs*, $wc$ *arcs*, or $rx$ *arcs*, in the above three cases, respectively.

Because of the mapping, we shall next obtain the capacity of a WDM PON by deriving the achievable traffic rates on $tx$ *arcs*, $wc$ *arcs*, and $rx$ *arcs*.

## IV. Capacity Analysis

In this section, we first define achievable traffic rates and the capacity region. Then, conflict graphs are introduced to model the relations between optical flows. Afterwards, we show that traffic rates of optical flows are achievable if they fall into the independent set polytope of the corresponding conflict graph. We further show that, for the conflict graph of any flows in a WDM PON, its independent set polytope equals to its clique inequality region, and vertices form a clique if and only if their corresponding flows go through the same arc. Finally, we derive the capacity region of a WDM PON by applying Ford and Fulkerson's Max-flow, Min-cut Theorem.

We shall first review some terminologies of graph theory.

- An **independent set** contains a set of vertices which are not connected with each other.
- A **clique** contains a set of vertices, any two of which are connected by an edge.
- The **chromatic number** of a graph refers to the minimum number of colors to color the graph.
- A **perfect graph** represents the graph whose chromatic number equals to the size of the largest clique for each of its induced subgraphs.
- A cycle in a graph is **chordless** if no proper subset of the vertices of the cycle forms a cycle.
- **Odd hole** represents a chordless cycle of length of at least four, and contains an odd number of edges.

### A. Definitions of Achievable Traffic Rate and Capacity Region

First, we define the achievable traffic rates and the capacity region.

*Definition 1:* A set of traffic rates of ONUs is achievable if there exists a resource allocation scheme to satisfy the above stated $tx$ *arcs* constraints, $wc$ *arcs* constraints, $rx$ *arcs* constraints, *connection arcs* constraints, and *continuous optical flow constraints*.

*Definition 2:* The capacity region of a WDM contains all sets of achievable traffic rates of ONUs.

Our main idea in deriving the capacity of a WDM PON is to first obtain the achievable rates with respect to one particular resource allocation instance, and then calculate the union of achievable rates over all possible resource allocation instances.

Here, one resource allocation instance refers to a number of resource sets, each of which contains one transmitter, one wavelength channel, and one receiver. Each set is to fulfill the data transmission of one optical flow. The achievable rates with respect to one resource allocation instance refer to the achievable rates of these optical flows.

### *B. The Conflict Graph Model*

Owing to the continuous optical flow constraints, we characterize the relations between different traffic flows by utilizing conflict graphs, which have been employed to address routing problems in all-optical networks [23], optical flow switch without core buffering [9], and multi-hop wireless networks with interference consideration [24].

Denote the conflict graph with respect to a set of flows $\mathbf{f} =< f_1, f_2, ... >$ in the abstracted graph $\mathcal{G}$ as $C(\mathbf{f}, \mathcal{G})$. In $C(\mathbf{f}, \mathcal{G})$, vertices represent optical flows $f_1, f_2, ...$ in graph $\mathcal{G}$, and two vertices are connected if they share the same arc in graph $\mathcal{G}$. For notational convenience, we denote a flow $f$ by a three-tuple: $\{tx\ arc, wc\ arc, rx\ arc\}$.

Consider the upstream transmission in a network with four transmitters at four ONUs, three wavelengths, and two receivers at OLT. Transmitters at the first and second ONU is fixed tuned to wavelength 1, the transmitter at the third ONU can be tunable to wavelength 2 and 3, and the transmitter at the fourth ONU can be tuned to wavelength 3. Fig. 2 (a) illustrates the abstracted directed graph of a WDM PON. We consider four flows $f_1 : \{e_1, e_5, e_8\}$, $f_2 : \{e_2, e_5, e_8\}$, $f_3 : \{e_3, e_7, e_8\}$, and $f_4 : \{e_4, e_7, e_8\}$ as shown in Fig. 2 (b). Fig. 2 (c) describes the conflict graph of these four flows.

With the conflict graph, the continuous optical flow constraints are transformed into the constraints that flows cannot transmit simultaneously if their corresponding vertices are connected in the conflict graph. Flows can transmit simultaneously only if their corresponding vertices are within the same independent set. Then, the following property about achievable traffic rates can be derived.

*Property 1:* Traffic rates of flows are achievable if they fall into the independent set polytope of conflict graph $C(\mathbf{f}, \mathcal{G})$.

### *C. Clique Inequality Region*

For any general graph, its independent set polytope is dominated by the region limited by clique inequalities [9], [24], referred to as clique inequality region. The two regions are equivalent when the graph is perfect. Without loss of generality, we derive achievable traffic rates on $tx$ *arcs*. The same conclusions can be drawn for rates on $wc$ *arcs* and $rx$ *arcs*. We next show that, for the conflict graph $C(\mathbf{f}, \mathcal{G})$ where each flow in $\mathbf{f}$ originates from one *tx arc*, its independent set polytope equals to its clique inequality region by using Lemma 1 and Theorem 1.

*Lemma 1:* If a set of flows $\mathbf{f}$, each of which originates from one *tx arc*, forms a chordless cycle in the conflict graph, the largest size of the chordless cycle in conflict graph $C(\mathbf{f}, \mathcal{G})$ is 4.

*Proof:* Assume five flows $f_1$, $f_2$, $f_3$, $f_4$, and $f_5$ on five $tx$ *arcs* form a chordless cycle. Because of the chordless property, no more than two of them flow through the same arc. Denote these five flows as $\{e^1_1, e^1_2, e^1_3\}$, $\{e^2_1, e^2_2, e^2_3\}$, $\{e^3_1, e^3_2, e^3_3\}$, $\{e^4_1, e^4_2, e^4_3\}$, and $\{e^5_1, e^5_2, e^5_3\}$, respectively. Without loss of generality, assume the connection edges in the cycle of these five flows are between $f_1$ and $f_2$, $f_2$ and $f_3$, $f_3$ and $f_4$, $f_4$ and $f_5$, and $f_5$ and $f_1$, respectively. Then, the following conditions have to be satisfied.

$$\begin{cases} e^1_2 = e^2_2 \text{ or } e^1_3 = e^2_3 \\ e^2_2 = e^3_2 \text{ or } e^2_3 = e^3_3 \\ e^3_2 = e^4_2 \text{ or } e^3_3 = e^4_3 \\ e^4_2 = e^5_2 \text{ or } e^4_3 = e^5_3 \\ e^5_2 = e^1_2 \text{ or } e^5_3 = e^1_3 \end{cases} \quad (1)$$

After checking all possibilities, it can be obtained that these constraints cannot be satisfied without letting three of them be equal to each other. Thus, there does not exist a chordless cycle with size five, and the largest size of a chordless cycle is 4. ■

*Theorem 1:* For flows $\mathbf{f}$, each of which going through one *tx arc*, the stable set polytope of conflict graph $C(\mathbf{f}, \mathcal{G})$ equals to the region limited by clique inequalities in graph $C(\mathbf{f}, \mathcal{G})$.

*Proof:* The stable set polytope of a graph is equivalent to the region limited by its clique inequalities when the graph is a perfect graph. It was proved that graphs are perfect if and only if they are Berge graphs. A graph $G$ is a Berge graph if and only if neither $G$ nor $\bar{G}$ contains an odd hole. Based on Lemma 1, there is no cordless cycle with size greater than four. Thus, the graph $C(\mathbf{f}, \mathcal{G})$ is a perfect graph. ■

Then, rates of flows in an abstracted graph $\mathcal{G}$ of a WDM PON can be achieved if they satisfy clique inequalities in its corresponding conflict graph $C(\mathbf{f}, \mathcal{G})$. The next problem is to find cliques in graph $C(\mathbf{f}, \mathcal{G})$.

### *D. Formation of Cliques*

For conflict graph $C(\mathbf{f}, G)$ of a general graph $G$, clique may be formed by vertices whose corresponding flows do not go through the same arc in $G$, and finding whether there is a clique of a given size is NP-complete. However, in terms of the conflict graph $C(\mathbf{f}, \mathcal{G})$ of the abstracted graph $\mathcal{G}$ of a WDM PON, we will show that vertices form a clique only if their corresponding flows go through the same arc in graph $\mathcal{G}$.

*Lemma 2:* In conflict graph $C(\mathbf{f}, \mathcal{G})$, flows form a clique only if they flow through the same arc in graph $\mathcal{G}$.

*Proof:* Assume two flows form a clique. These two flows must share one arc since any two vertices are connected in $C(\mathbf{f}, \mathcal{G})$ only if their corresponding flows share one arc in $\mathcal{G}$.

Assume three flows $f_1$, $f_2$, and $f_3$ on three $tx$ *arcs* form a clique, and they do not go through the same arc. Denote these three flows as $\{e^1_1, e^1_2, e^1_3\}$, $\{e^2_1, e^2_2, e^2_3\}$, and $\{e^3_1, e^3_2, e^3_3\}$, respectively. Then, $e^1_1 \neq e^1_2$, $e^1_1 \neq e^1_3$, and $e^1_2 \neq e^1_3$. In order to let these three flows form a clique, the following conditions have to be satisfied.

$$\begin{cases} e^2_2 = e^1_2 \text{ or } e^2_3 = e^1_3 \\ e^2_2 = e^3_2 \text{ or } e^2_3 = e^3_3 \\ e^3_2 = e^1_2 \text{ or } e^3_3 = e^1_3 \end{cases} \quad (2)$$

To satisfy these constraints, either $e^1_2 = e^2_2 = e^3_2$ or $e^1_3 = e^2_3 = e^3_3$. Under both conditions, these three flows have to share the same arc. Thus, flows cannot form a clique if they do not go through the same arc.

Assume $n$ flows $f_1$, $f_2$, ..., $f_n$ on $n$ respective $tx$ *arcs* form a clique, and they do not go through the same arc. Any three of them must flow through the same arc. Without

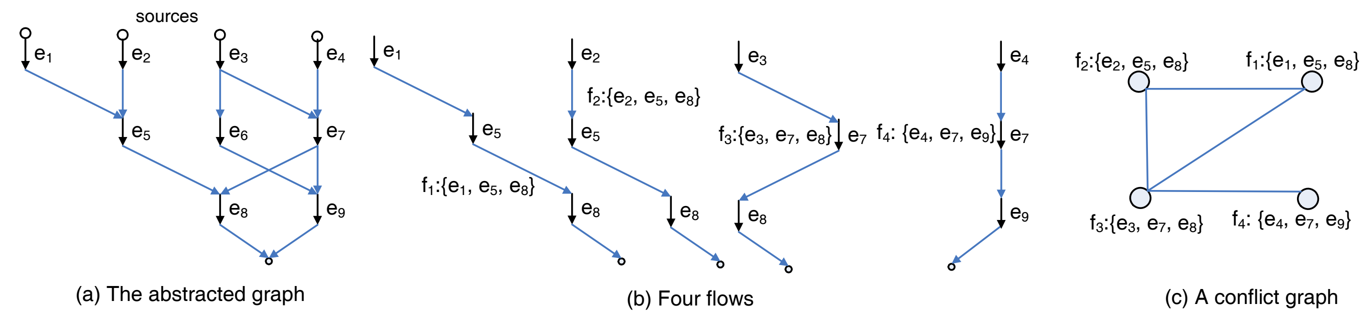



Fig. 2. The conflict graph of four flows in an abstracted graph.

loss of generality, assume flows $f_1$, $f_2$, and $f_3$ share $e_2^1$, i.e., $e_2^1 = e_2^2 = e_2^3$. In order to form a clique, any other flow $f_i$ must share one arc with $f_1$, $f_2$, and $f_3$, respectively. To achieve this, either $e_2^i = e_2^1$ or $e_3^i = e_3^1, e_3^i = e_3^2, e_3^i = e_3^3$. If $e_2^i = e_2^1$, flows $f_i$, $f_1$, $f_2$, and $f_3$ share the same arc $e_2^1$. If $e_3^i = e_3^1, e_3^i = e_3^2, e_3^i = e_3^3$, then, $e_3^i = e_3^1 = e_3^2 = e_3^3$ and flows $f_i$, $f_1$, $f_2$, and $f_3$ share the same arc. Thus, flows $f_1$, $f_2$, ..., $f_n$ must share the same arc in order to form a clique, and we have thus proved the lemma. ■

With Lemma 2, Property 1, and Theorem 1, we can derive the following property regarding to the capacity of a WDM PON.

*Theorem 2:* For any given traffic rates on $tx$ *arcs*, they are achievable if they satisfy constraints on $tx$ *arcs*, $wc$ *arcs*, $rx$ *arcs*, and connection arcs.

*Proof:* According to Property 1 and Theorem 1, traffic rates are achievable if they fall into the clique inequality region. Based on Lemma 2, vertices form a clique in graph $C(\mathbf{f}, \mathcal{G})$ only if their corresponding flows share the same arc in graph $\mathcal{G}$. Then, clique inequalities are satisfied if the traffic rates satisfy constraints on $tx$ *arcs*, $wc$ *arcs*, $rx$ *arcs*, and connection arcs. Therefore, traffic rates are achievable if they satisfy constraints on $tx$ *arcs*, $wc$ *arcs*, $rx$ *arcs*, and connection arcs. ■

### *E. Capacity Region Derivation*

The above shows that, for a particular resource allocation instance, traffic rates of optical flows are achievable if they satisfy constraints on $tx$ *arcs*, $wc$ *arcs*, $rx$ *arcs*, and connection arcs. Then, for given traffic rates on $tx$ *arcs*, we can check all possible resource allocation instances, and see whether their induced optical flows satisfy those constraints. However, it is computationally intensive.

To avoid the checking of all possible resource allocation instances, we can consider the traffic rates on $tx$ *arcs* as the traffic rates of their corresponding sources, and obtain the capacity region by applying the Ford and Fulkerson's Max-flow, Min-cut theorem. The max-flow min-cut theorem states that, in a network, the maximum traffic passing from the source to the sink is equal to the minimum capacity that needs to be removed from the network so that no flow can pass from the source to the sink.

Based on the Ford and Fulkerson's Max-flow, Min-cut theorem, we can derive the following theorem regarding achievable rates on $tx$ *arcs*.

*Theorem 3:* Given a set of traffic rates on *tx arcs*, they are achievable if, for any subset of the traffic rates, the sum of the rates is not greater than the sum of capacities on arcs in its corresponding minimum cut.

Theorem 3 states that the capacity region of a WDM PON is equal to the cut-set bound of the abstracted directed graph of the WDM PON.

Note that Lemma 1, Lemma 2, Theorem 1, and Theorem 2 hold for traffic rates on *wc arcs* and *rx arcs* as well. Hence, we should be able to derive theorems similar to Theorem 3 for them. However, *wc arcs* and *rx arcs* are not directly connected to sources in the abstracted graph. In order to relate traffic rates on these arcs with traffic rates on some sources such that the Ford and Fulkerson's Max-flow, Min-cut theorem can be applied, we make the following changes of the abstracted graph.

To obtain the achievable traffic rates on *wc arcs*, we divide graph $\mathcal{G}$ into two graphs. The first subgraph contains *tx arcs*, *wc arcs*, and the connection arcs between them. The directions of arcs in the subgraph are the reverse directions of arcs in the original graph $\mathcal{G}$. Each *wc arc* is associated with one source. The second subgraph contains *wc arcs*, *rx arcs*, and the connection arcs between them. The directions in the subgraph follow those in the original graph $\mathcal{G}$. Each *wc arcs* is associated with one source. Then, the achievable rates on *wc arcs* can be obtained by applying the Ford and Fulkerson's Max-flow, Min-cut theorem on the two subgraphs.

To obtain the achievable traffic rates on *rx arcs*, we reverse directions on all arcs, and associate each *rx arc* with one source. Then, the Ford and Fulkerson's Max-flow, Min-cut theorem can be applied to obtain the capacity region.

## V. Illustrative Examples of Some WDM PONs

Consider the directed graph as shown in Fig. 2(a) for example. By applying Theorem 3, we obtain its capacity region as follows:

$$\begin{cases} r_1, r_2, r_3, r_4 \leq C \\ r_1 + r_2 \leq C \\ r_1 + r_3, r_1 + r_4, r_2 + r_3, r_2 + r_4, r_3 + r_4 \leq 2C \\ r_1 + r_2 + r_3, r_1 + r_3 + r_4, r_1 + r_2 + r_4, r_2 + r_3 + r_4 \leq 2C \\ r_1 + r_2 + r_3 + r_4 \leq 2C \end{cases} \tag{3}$$

where $r_i$ refers to the traffic rate on the $i$th $tx$ *arc*. The numbering is from left to right. Each of these constraints is associated with one cut for a set of $tx$ *arcs*. There are totally $2^4 - 1 = 15$ nonzero subsets for the total of four $tx$ *arcs*.

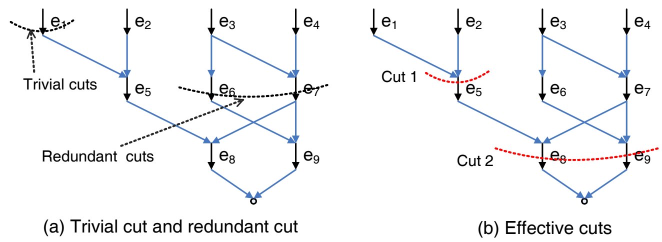



Fig. 3. Trivial, redundant, and effective cuts of a graph.

Hence, the total number of constraints is 15. Among these 15 constraints, some are not necessary in delineating the capacity region. For example, $r_1 + r_3$ must not exceed $2C$ if both $r_1 \leq C$ and $r_3 \leq C$. Hence, the constraint $r_1 + r_3 \leq 2C$ is not necessary. After removing unnecessary constraints, the capacity region of the WDM PON can be described as follows:

$$\begin{cases} r_1 + r_2 \leq C \\ r_1 + r_2 + r_3 + r_4 \leq 2C \\ r_1, r_2, r_3, r_4 \leq C \end{cases} \quad (4)$$

Among these necessary constraints, constraints $r_1, r_2, r_3, r_4 \leq C$ always need to be satisfied regardless of the WDM PON constituents. We refer these constraints to as *trivial constraints*, and their corresponding cuts to as *trivial cuts*. For cuts whose corresponding constraints are unnecessary, we refer them to as *redundant cuts*. Cuts which are neither trivial nor redundant are referred to as *effective cuts*. The capacity region contains rates which satisfy constraints associated with trivial cuts and effective cuts. For the directed graph as shown in Fig. 2 (a), Cut 1 and Cut 2 are effective cuts as shown in Fig. 3 (b).

It is worth noting that removing some arcs from the abstracted graph does not change its *effective cuts*. For example, in the directed graph as shown in Fig. 2(a), by removing the arc connecting the end vertex of arc $e_3$ and the start vertex of arc $e_7$, effective cuts still contain Cut 1 and Cut 2 only. This implies that changing the transmitter at ONU 3 from a tunable transmitter into a wavelength-fixed transmitter does not shrink the network capacity region. These arcs are referred to as *redundant arcs*. A graph without redundant arcs are referred to as a *condense graph*. For graphs as shown in Fig. 1(c), Fig. 1(d), and Fig. 1(e), their corresponding condense graphs are shown in Fig. 4(a), Fig. 4(b), and Fig. 4(c), respectively.

The comparisons of the three cases are as follows:

- In Fig. 4(a), transmitters at the first four ONUs are wavelength-specific ones, whereas all transmitters in Fig. 1(c) are full-wavelength tunable ones.
- In Fig. 4(b), each transmitter and each receiver only need to support five wavelengths, whereas transmitters and receivers in Fig. 1(d) have to be full-range tunable to all wavelengths.
- In Fig. 4(c), transmitters only need to support five wavelengths, whereas transmitters in Fig. 1(e) have to be full-range tunable to all wavelengths.

If the cost of tunable transmitters is low with a small number of support wavelengths, architectures as shown in Fig. 4(a), Fig. 4(b), and Fig. 4(c) cost less than those as shown in Fig. 1(c), Fig. 1(d), and Fig. 1(e), respectively.

Therefore, instead of using full wavelength-tunable transmitters and receivers, proper selection of limited wavelength-tunable transmitters and receivers can achieve the same capacity region with cost reduction.

## VI. Conclusion

In this paper, we have analyzed the capacities of WDM PONs with any general sharing of transmitters, receivers, and wavelength channels. We first describe the upstream/downstream data transmission process in WDM PONs by using a multi-source single-destination directed graph, where arcs represent transmitters, receivers, wavelength channels, as well as their relations. Any successful traffic transmission can be mapped into some routings from a source to the destination in the abstracted directed graph. Then, we construct conflict graphs for flows in the abstracted directed graph, and prove that the conflict graph is a perfect graph, and flows form a clique if and only if they go through the same arc in the abstracted graph. Attributed to these two properties, we can obtain the capacity region of a WDM PON by applying the Ford and Fulkerson's Max-flow, Min-cut theorem to the abstracted directed graph of the WDM PON. With these results, we continue to discuss some WDM PONs with the same capacity region, and introduce the concept of condense graphs. A WDM PON whose abstracted graph is a condense graph may incur lower cost, but can still achieve the same capacity with those with redundant arcs.

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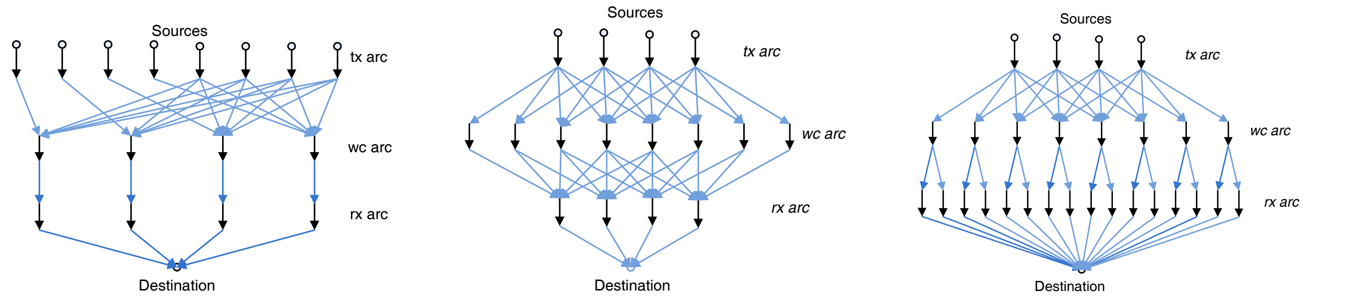


(a) One condense graph for Fig. 1(c)

(b) One condense graph for Fig. 1(d)

(c) One condense graph for Fig. 1(e)

Fig. 4. Condense graphs.

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