The Cantor Set and Its Applications in Nonlinear Dynamics and Chaos

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Abstract

The Cantor set discovered and introduced by Henry Smith and Georg Cantor has many interesting properties and consequences in the fields of Set Theory, Topology, and Fractal Theory. However, its applications in Dynamical Systems is that which will be discussed. Various systems including the logistic equation are analyzed using tools such as horseshoe maps which form Cantor sets, and consequently have the same property as Cantor sets within a certain domain.

Keywords: Cantor set, horseshoe map, logistic map

1 Cantor Sets

A Cantor set is any set homeomorphic to $C = \Pi_{n=1}^{\infty} F_n$, where each F_n is the two-point space {0,1}. Kuperberg [2]. This is however a complicated definition, and requires a knowledge of topology. Instead of defining it topologically it is usefull to construct the set and present pictoral examples.

The Cantor set is constructed from the set of Real numbers in the unit interval $[0, 1]$. Lets call the intial set S_0. For the first iteration a fraction α is taken away from S_0 such that S_1 contains two disconnected sets of Real numbers $[0, \frac{1}{2}(1 - \alpha)]$ and $[1 - \frac{1}{2}(1 - \alpha), 1]$. Lets $a := \frac{1}{2}(1 - \alpha)$ and $b := 1 - \frac{1}{2}(1 - \alpha)$, and call $[0, a]$ S_{1a} and $[b, 1]$ S_{1b}. For the second iteration take away the fraction α from S_{1a} and S_{1b} such that S_2 contains four disconnected sets of Real numbers $[0, \frac{1}{2}(a-\alpha a)]$, $[a-\frac{1}{2}(a-\alpha a), a]$, $[b, b+\frac{1}{2}(a-\alpha a)]$, and $[1-\frac{1}{2}(a-\alpha a), 1]$. This is continued until S_∞ is reached, and S_∞ is called the Cantor set; more specifically the middle - α Cantor set. Two illustrations are shown in Figure 1.

Figure 1: Example of set S_7. Wikipedia [3].

For the sake of mathematical rigor a closed form formula for each iteration is needed. It is easy to see $S_1 = \frac{1}{2}(1-\alpha)S_0 \cup [\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)S_0]$. The formula for S_2, S_3, etc may seem somewhat difficult to derive, but with a little inspection and some computations a repeating pattern is seen. Notice $S_2 = \frac{1}{2}(1-\alpha)S_1 \cup [\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)S_1]$ and $S_3 = \frac{1}{2}(1-\alpha)S_2 \cup [\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)S_2]$. We may assume the formula for the n^{th} case follows this pattern.

$$S_n = \frac{1}{2}(1-\alpha)S_{n-1} \cup [\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)S_{n-1}] \qquad (1)$$
$$0 < \alpha < 1, S_0 = [0,1], \qquad n \neq \infty$$

Proof. The proof of n arbitrarily large is shown, however for $n = \infty$ a more rigorous proof is required.

By definition

$$S_0 = [0,1].$$

It is easy to see

$$S_1 = \frac{1}{2}(1-\alpha)S_0 \cup [\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)S_0] = [0, \frac{1}{2}(1-\alpha)] \cup [1-\frac{1}{2}(1-\alpha), 1],$$

is true.

Assume

$$S_n = \frac{1}{2}(1-\alpha)S_{n-1} \cup [\frac{1}{2}(1+\alpha)+\frac{1}{2}(1-\alpha)S_{n-1}],$$

is true.

It can be shown

$$S_{n+1} = \frac{1}{2}(1-\alpha)S_n \cup [\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_n].$$

Let

$$\beta_n = \frac{1}{2}(1-\alpha)\beta_{n-1} \cup [\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)\beta_{n-1}], \tag{2}$$

and

$$\beta_{n-1} = S_n = \frac{1}{2}(1-\alpha)S_{n-1} \cup [\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_{n-1}] \tag{3}$$

Plugging Equation 3 into Equation 2 gives,

$$\beta_n = \frac{1}{2}(1-\alpha)S_n \cup [\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_n] = S_{n+1}$$

By induction

$$S_n = \frac{1}{2}(1-\alpha)S_{n-1} \cup [\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_{n-1}]$$
$$S_0 = [0,1], \qquad n \neq \infty$$

thereby completing the proof. //

1.1 Some Important Properties of Cantor Sets

1. Cantor sets are self-similar fractals.
   - Cantor sets look the same no matter the level at which they are seen. All $n+1$ sections of S_n looks the same as S_0 when magnified.
2. Cantor sets are completely disconnected.
   - There are no intervals within a Cantor set (i.e. it has a topological dimension of zero, however it has a nonzero fractal dimension).
3. Cantor sets have a measure of zero. Strogatz [1].

- The length of S_n is α^{∞}. When $n = \infty$ the length of S_n is zero, because $0 < \alpha < 1$.
- Another way to find the measure is to subtract the measure of the complement of the Cantor set from the total length of [0,1]. The length of the complement of the Cantor set is

$$\sum_{n=0}^{\infty}(\frac{1}{\alpha}-1)^n\alpha^{n+1} = \alpha\sum_{n=0}^{\infty}(\frac{1}{\alpha}-1)^n\alpha^n = \alpha\sum_{n=0}^{\infty}(1-\alpha)^n = \frac{\alpha}{\alpha} = 1 \tag{4}$$

4. Cantor sets are uncountable. Strogatz [1].

   - This can be shown by using Cantor's diagonal argument. Strogatz [1].

2 Horseshoe Maps

Although no simple definition is available, a constructive definition of horseshoe maps can be formulated. Imagine a blob of points. This is now flattened in one direction and stretched in the other, then folded back such that most of it is bounded in the unit square. This is S_1. For S_2 to S_n we stretch each previous iteration and fold it back such that all the straight portions of the map is contained within the unit square, forming a Cantor set within the unit square when $n = \infty$. The first two iteration and their inverse are illustrated in Figure 2.

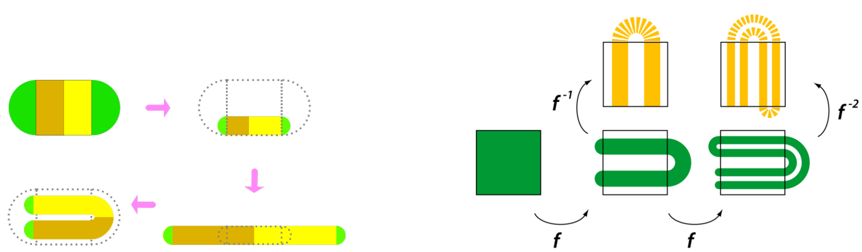

Figure 2: First iteration of a pill shaped region under horseshoe mapping (left) and first two iterations of the unit square under horseshoe mapping (right, green) and the inverse horseshoe mapping (right, yellow). Wikipedia [4].

The intersection of the horseshoe map and the inverse horseshoe map forms an invariant set. The two maps also form two way Cantor sets, a Cantor dust as in Figure 3.

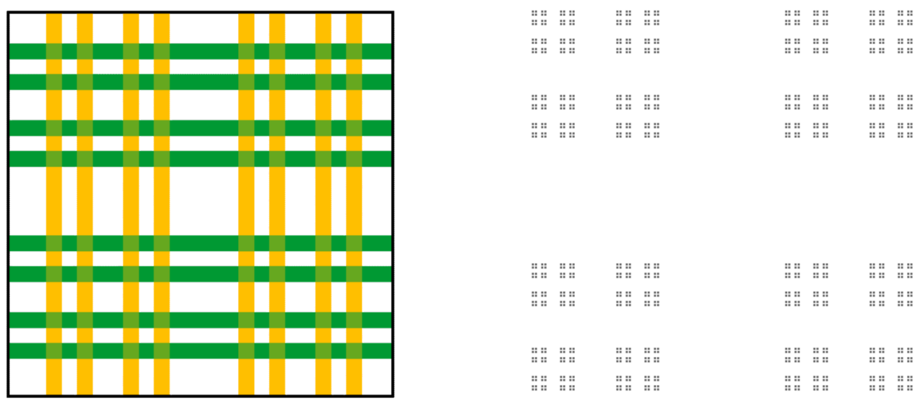

Figure 3: Intersection of horseshoe map and inverse horseshoe map (left) and example of the first few iterations of a Cantor dust (right). Wikipedia [3].

3 The Logistic Map

Using the logistic map, properties of the Cantor set and horseshoe maps may be easily demonstrated. The iterates of the mapping

$$\begin{aligned} x_{n+1} &= \lambda x_n(1 - x_n) \\ \lambda &= 4.1, \end{aligned} \tag{5}$$

demonstrate properties of the horseshoe map. Doing some numerical computation or following the orbits on a cobweb plot it can be shown when the initial iterate starts on the Cantor set all the iterates will lie on the Cantor set, and consequently stay within the unit square.

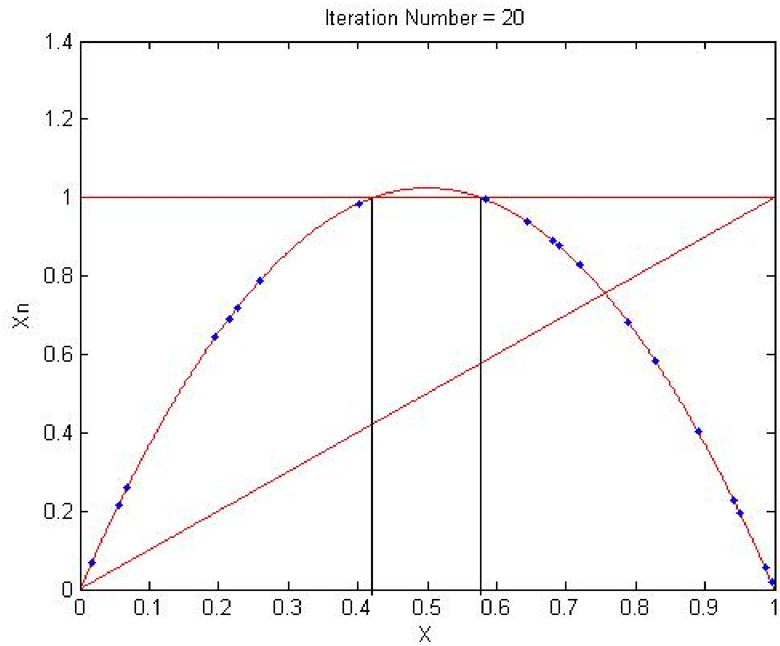

Figure 4: Iterates of the logistic map for $\lambda = 4.1$.

The logistic map has been analyzed thoroughly in texts such as [1] therefore it would not make much sense if more analysis was done.

4 Another Simple Discrete Map

A discrete system similar to the model of the Set - Reset Flip - Flop circuit, Blackmore et al. [7] may be used to demonstrate properties of the horseshoe map.

$$x_{n+1} = x[\lambda(1 - x_n) + y] \tag{6}$$
$$\lambda = 4.1.$$

When y is small the system acts like the logistic map. The iterates that stay in for all time start out on the same or similar Cantor set as that of the logistic map for the same value for λ shown in Figure 5. However, when the value of y is increased the map starts to change. The map gets wider, which does nothing to the structure of the Cantor set of the map other than

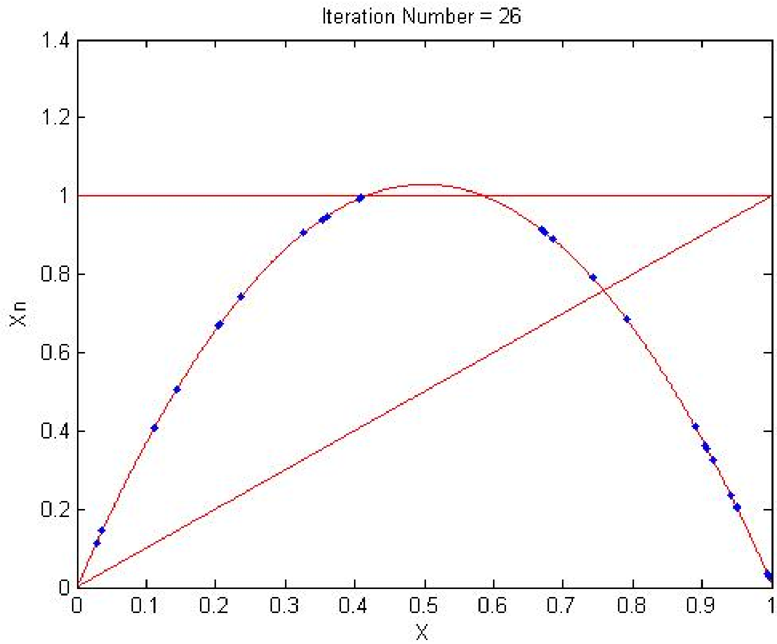

Figure 5: Iterates of Equation 6. $y = .02$, $x_0 \approx .999$

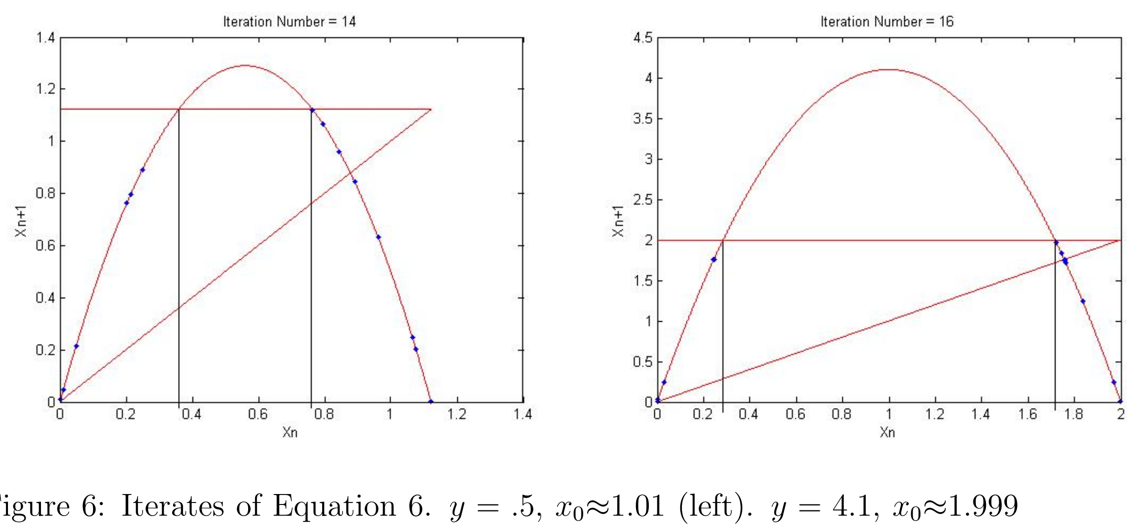

Figure 6: Iterates of Equation 6. $y = .5$, $x_0 \approx 1.01$ (left). $y = 4.1$, $x_0 \approx 1.999$ (right).

changing the domain, refer to Figure 6.

An interesting thing happens with this map that does not happen with the logistic map. If λ is fixed and y is allowed to increase the height increases. The same thing happens to the logistic map when λ is increased. However, unlike the logistic map although the height is increasing so is the width. Therefore, there is a larger window for finding evidence of periodic orbits and chaos through numerical simulations. The middle α that is removed is not as large as that of the logistic map of the same height. Also, more iterates may be taken with less tweeking in this map than the logistic map.

5 Concluding Remarks

The Cantor set is a useful tool in analyzing discrete dynamical systems. Utilizing Cantor sets one may avoid assigning a value to the initial variable on one of the removed portions of the Cantor set, which would throw the next iteration out of the square. In all the maps shown a value close to the edge of the domain was chosen. This is where the Cantor set is the most dense so the probability of choosing a good value is far greater, and less tweaking is necesary to find an adequate value.

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