Comparison of Random Sums in some Integral Orderings and Applications[1]

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Abstract

Some new order preservation properties of stopped sums of independent nonnegative random variables, when the stopping variable is independent of the summands, is investigated. We show that such randomly stopped sums preserve the stochastic Laplace as well as the integral harmonic mean residual life orders. For the case of Laplace orders, there is a suitable converse for each of the order preservation results. Exponential distributions are characterized within the class of random sums with geometric stopping times, via simple moment conditions on the summand obeying a suitably weak aging hypothesis.

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1. Introduction and Summary.

Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be two sequences of random variables (r.v.s.), henceforth simply denoted by $\{X_n\}$, $\{Y_n\}$. Let M, N be non-negative integer valued r.v.s. independent of the sequences $\{X_n\}, \{Y_n\}$. Consider the random sums,

$$U = \sum_{n=1}^{M} Y_n, \quad V = \sum_{n=1}^{N} X_n \tag{1.1}$$

with the convention $U = 0$, if $M = 0 (V = 0$, if $N = 0)$. If $<_P$ is a given *stochastic order* (partial ordering among r.v.s) such that $M <_P N$, $Y_n <_P$

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X_n all $n \geq 1$ and possibly some reasonable conditions on the sequences $\{Y_n\}, \{X_n\}$ implies $U <_P V$; we say that randomly stopped sums of the type in (1.1) possess an *external monotonicity property* under the stochastic order $<_P$.

Jeen-Marie and Liu [8] investigated such monotonicity properties of random sums when the underlying summands are arbitrary sequences of real valued r.v.s. In the context of many applications where such randomly stopped sums appear, it is known that $\{X_n\}, \{Y_n\}$ are sequences of nonnegative r.v.s, often with additional structure such as independence, or even the i.i.d. property. In particular, compound geometric distributions which correspond to the r.v. V in (1.1), when X_n are in i.i.d. and N is geometric, occurs among others in the study of many queueing systems, reliability, stress-strength models, and risk and ruin problems. *Stochastic order* properties of the summands X_n which are preserved by the geometric compound V (with geometrically distributed N) have been investigated by several authors (see Szekli [15], Bhattacharjee *et al.* [2]).

Our purpose in this article is to demonstrate two new external monotonicity properties of such random sums. Specifically, it is shown that randomly stopped sums of independent sequences preserve the Laplace order property of the summands when the stopping times are correspondingly ordered, and in fact when the summands constitute i.i.d. sequences, suitable converses also hold. We further show that for nonnegative and independent (but not necessarily i.i.d.) summands, randomly stopped sums preserve the *integral harmonic mean residual life order with equal means*, which extends and includes a result of Szekli [15]. As an application, we deduce some characterizations of exponentiality within the family of geometric compounds - an important subclass of random sums, in terms of moment conditions on the underlying distribution generating the geometric compound.

2. Main results.

Consider the partial ordering $<_{icx}$ (increasing convex order) between real valued r.v.s. S, T with respective distribution functions (d.f.s) F and G, defined by

$$\begin{aligned} S <_{icx} T \quad &\Leftrightarrow \quad \int_x^\infty \{1 - F(u)\} du \leq \int_x^\infty \{1 - G(u)\} du, \text{ all real } x \\ &\Leftrightarrow \quad Eh(S) \leq Eh(T), \quad \text{all convex } \uparrow h \end{aligned} \tag{2.1}$$

and the Laplace domination ordering ($<_L$) between non-negative r.v.s. S, T with $ES < \infty$, defined by

$$S <_L T \Leftrightarrow E(e^{-sT}) \leq E(e^{-sS}), \text{ all } s > 0. \tag{2.2}$$

The convex order $<_{cx}$ is the stochastic order $<_{icx}$ with equal means ($S <_{cx} T \Leftrightarrow S <_{icx} T$, and $ES = ET < \infty$). For $<_{icx}$ ($<_{cx}$) restricted to non-negative r.v.s, we may take $x \geq 0$ and h to be convex and nondecreasing (convex, respectively) on $[0, \infty)$ in (2.1). See Müller and Stoyan [12] for an overview and some ramifications of the above orderings. In particular, note that the HNBUE (hermonic new better than used in expectation) and the class-$\mathcal{L}$ properties, familiar in reliability theory (Klefsjö [9], [10]) are special cases of $<_{icx}$ and $<_L$ orderings (*viz.*, take T to be an exponentially distributed r.v. with $ET = ES < \infty$ in (2.1)-(2.2). Then the corresponding specialized versions of (2.1), (2.2) respectively define the HNBUE and class-$\mathcal{L}$ property of the r.v. S).

Our main order preservation findings, for random sums of the type in (1.1), are the two theorems and corresponding corollaries in section 2. An application of our result and some characterizations of exponentiality among compound geometric distributions are indicated in section 3. A forthcoming paper by Kulik and Szekli (2004) on dependence orderings for some functionals of multivariate point processes, is conceptually related to the context of this paper and may be of interest to readers of the present article.

To set our main results in perspective; it is useful to recall the relevant findings of Jean-Marie and Liu [8], which can be summarized as follows:

Proposition (Jean-Marie and Liu [8]) *Let $<_{P_1}, <_{P_2}$ be stochastic (partial) orderings, among r.v.s, closed under convolution, such that in the context of the r.v.s in* (1.1),

$$M <_{P_1} N, \quad Y_n <_{P_2} X_n \text{ all } n \geq 1$$

hold. Then any of the following conditions is sufficient to imply $U <_{P_2} V$.

(i) $<_{P_1} \Rightarrow <_{st} \Rightarrow <_{P_2}$, $\;Y_n \geq 0$ *w.p.*1, *all* $n \geq 1$.

(ii) $<_{P_1} \Rightarrow <_{st}, <_{icx} \Rightarrow <_{P_2}$, *and either* $EX_n \geq 0$ *for* $n \geq 1$ *or,* $EY_n \geq 0$ *for* $n \geq 1$.

(iii) $<_{P_1} \Rightarrow <_{icx} \Rightarrow <_{P_2}$ *and either* $0 \leq X_n \uparrow_{icx}$, *or,* $0 \leq Y_n \uparrow_{icx}$.

(iv) *same as condition* (iii) *with* $<_{cx}$ *replacing* $<_{icx}$.

(Here, $<_{st}$ denotes the strong stochastic majorization order; *viz.*, $Y <_{st} X$ if $P(Y > x) \leq P(X > x)$, all real x. The notation $X_n \uparrow_{icx}$ stands for $X_n <_{icx} X_{n+1}$, all $n \geq 1$).

In particular, taking $<_{P_1} \equiv <_{P_2} \equiv <_{st}$ in (i), and $<_{P_1} \equiv <_{P_2} \equiv <_{icx}$ in (iii) above implies

Corollary 2.0 (Szekli [15]) (i) *$M <_{st} N$, and $0 \leq Y_n <_{st} X_n$ for all $n \geq 1$ implies $U <_{st} V$.*
(ii) *$M <_{icx} N$, $Y_n <_{icx} X_n$ all $n \geq 1$ and , either, $0 \leq X_n \uparrow_{icx}$, or $0 \leq Y_n \uparrow_{icx}$ implies $U <_{icx} V$.*

Prompted by our interest in the case of many applications, as outlined in section 1, when the summands in (1.1) are independent and non-negative; we are led to ask if similar order preservation results hold for other stochastic orders that are meaningful in applied contexts. Theorem 2.1 provides results of this genre under the Laplace order $<_L$, which is weaker than the convex order $<_{cx}$. Theorem 2.4 similarly proves closure under the integral harmonic mean residual life order with equal means.

Theorem 2.1. *Consider the randomly stopped sums U, V in* (1.1), *with non-negative summand sequences $\{Y_n\}, \{X_n\}$ respectively.*

A. *When the summands are i.i.d. sequences*,
(i) *if $M <_L N$, then $Y_1 <_L X_1$ implies $U <_L V$,*
(ii) *if $N <_L M$, then $U <_L V$ implies $Y_1 <_L X_1$,*
(iii) *if $M \overset{d}{=} N$, then $U <_L V$ if and only if $Y_1 <_L X_1$.*

B. *When both summand sequences are independent*,
(i) *if $Y_n <_L Z <_L X_n$ for some Z, all $n \geq 1$, then $M <_L N$ implies $U <_L V$,*
(ii) *if $X_n <_L Z <_L Y_n$ for some Z, all $n \geq 1$ and $\inf_n P(X_n = 0) = 0$, then $U <_L V$ implies $M <_L N$.*
(iii) *if both sequences are i.i.d., $X_1 \overset{d}{=} Y_1$ with common d.f. continuous at zero, then $U <_L V$ if and only if $M <_L N$.*

C. *For i.i.d. summands*, **B**(i) - (ii) *can be improved as follows,*
(i) *If $Y_1 <_L X_1$, then $M <_L N$ implies $U <_L V$,*

(ii) *If* $X_1 <_L Y_1$ *and* $P(Y_1 = 0) = 0$, *then* $U <_L V$ *if and only if* $M <_L N$.

Proof. Let $\phi_M(z) := E(z^M),\ 0 < z < 1$ be the probability generating function (p.g.f.) of M, and $L_{Y_n}(s) = E(e^{-sY_n}), s \geq 0$, the Laplace transform of the non-negative i.i.d. r.v.s. Y_n, for $n \geq 1$. Let $\phi_N(z)$ and $L_{X_n}(s)$ be defined similarly as the p.g.f. of N and Laplace transform of X_n. Further when the summands are i.i.d., denote $L_{Y_n}(s), L_{X_n}(s)$ by $L_Y(s)$ and $L_X(s)$.
A (i) Since, for the two i.i.d. sequences,

$$X_n <_L Y_n \text{ iff } L_{Y_n}(s) \geq L_{X_n}(s),\ \ s \geq 0 \tag{2.3}$$

and for the stopping variables M and N,

$$\begin{aligned} M <_L N \text{ iff } \phi_M(z) &= E(z^M),\ 0 < z < 1 \\ &= E(e^{(-lns)M}),\ \ s > 0 \\ &\leq E(e^{(-lns)N}) = E(z^N) = \phi_N(z); \end{aligned} \tag{2.4}$$

we have, for all $s \geq 0$,

$$\begin{aligned} E(e^{-sU}) = \sum_{n=0}^{\infty} P(M = n)\{L_Y(s)\}^n &= \phi_M(L_Y(s)) \\ &\geq \phi_N(L_Y(s)),\ \text{ by (2.4)} \\ &\geq \phi_N(L_X(s)),\ \text{ by (2.3)} \\ &= E(e^{-sV}) \end{aligned} \tag{2.5}$$

which proves $U <_L V$.

(ii) Suppose $U <_L V$, i.e., $E(e^{-sU}) \geq E(e^{-sV}), s \geq 0$. Since $N <_L M$ iff $\phi_N(z) \geq \phi_M(z),\ 0 < z < 1$, the two hypotheses together imply,

$$\phi_N(L_Y(s)) \geq \phi_M(L_Y(s)) = E(e^{-sU}) \geq E(e^{-sV}) = \phi_N(L_X(s)),\ \ s \geq 0$$

which in turn implies $L_Y(s) \geq L_X(s)$ for all $s \geq 0$ by monotonicity of the p.g.f. ϕ_N. Thus $Y_1 <_L X_1$.

(iii) Since $M \overset{d}{=} N$ iff $\phi_M = \phi_N$ pointwise; i.e., iff $M <_L N <_L M$, the claim follows by combining (i) and (ii).

B (i) Under the stated hypotheses, we have for $s \geq 0$,

$$E(e^{-sU}) = P(M = 0) + \sum_{n=1}^{\infty} P(M = n) E(e^{-s\sum_{i=1}^{n} Y_i} | M = n) \tag{2.6}$$

$$= P(M=0) + \sum_{n=1}^{\infty} P(M=n) \prod_{i=1}^{n} L_{Y_i}(s)$$
$$\geq P(M=0) + \sum_{n=1}^{\infty} P(M=n) L_Z^n(s)$$
$$= \phi_M(L_Z(s)), \tag{2.7}$$

where $L_Z(\cdot)$ is the Laplace transform of Z; we have used the independence of M and $\{Y_i\}$ in the second step above, and the next step follows from $Y_i <_L Z$ all $i \geq 1$. An analogous computation similarly yields,

$$\begin{aligned} E(e^{-sU}) &= P(N=0) + \sum_{n=1}^{\infty} P(N=n) \prod_{i=1}^{n} L_{X_i}(s) \\ &\leq P(N=0) + \sum_{n=1}^{\infty} P(N=n) L_Z^n(s), \quad \text{since} Z <_L X_i \text{ all } i \geq 1 \\ &= \phi_N(L_Z(s)) \\ &\leq \phi_M(L_Z(s)) \\ &\leq E(e^{-sU}), \text{ using (2.7).} \end{aligned}$$

Note, the second inequality above follows from $M <_L N$ and $L_Z(s) \in (0,1]$ for $s \geq 0$.

(ii) Since $U <_L V$, using (2.6), its counterpart for $E(e^{-sV})$ and the hypothesis $X_n <_L Z <_L Y_n, \ n \geq 1$; we have

$$\begin{aligned} \phi_M(L_Z(s)) &= P(M=0) + \sum_{n=1}^{\infty} P(M=n) L_Z^n(s) \\ &\geq P(M=0) + \sum_{n=1}^{\infty} P(M=n) \prod_{i=1}^{n} L_{Y_i}(s) \\ &= E(e^{-sU}) \\ &\geq E(e^{-sV}) \\ &= P(N=0) + \sum_{n=1}^{\infty} P(N=n) \prod_{i=1}^{n} L_{X_i}(s) \\ &\geq P(N=0) + \sum_{n=1}^{\infty} P(N=n) L_Z^n(s) \\ &= \phi_N(L_Z(s)), \quad \text{for all } s \geq 0. \end{aligned} \tag{2.8}$$

Since $z := L_Z(s) \downarrow$ from 1 to $P(Z=0)$ as $s \uparrow$ on $[0, \infty)$; (2.8) is equivalent to

$$\phi_M(z) \geq \phi_N(z) \text{ for } z \in (P(Z=0), 1].$$

This is equivalent to $M <_L N$, provided $P(Z=0)=0$. To check that such is indeed the case, let $s \to \infty$ in the inequality $E(e^{-sZ}) \leq E(e^{-sX_n}),\ s \geq 0$, which holds in virtue of $X_n <_L Z$, all $n \geq 1$, implies $P(Z=0) \leq \inf_{n\geq 1} P(X_n = 0) = 0$.

(iii) When both $\{X_n\}, \{Y_n\}$ are i.i.d. with $X_1 \stackrel{d}{=} Y_1$, and their common d.f. has no atom at zero; both the hypothesis in **B** (i) - (ii) are satisfied by any r.v. Z such that $Z \stackrel{d}{=} Y_1$. Hence combining (i) and (ii) implies the desired conclusion.

C (i) The claim that $Y_1 <_L X_1$ and $M <_L N$ together imply $U <_L V$, is a restatement of **A** (i). An alternative argument follows by choosing $Z = Y_1 = Y_i <_L X_i$, all $i \geq 1$ in **B** (i).

(ii) Suppose $U <_L V$. Since each summand sequence is i.i.d., we have

$$\phi_M(L_Y(s)) = E(e^{-sU}) \geq E(e^{-sV}) = \phi_N(L_X(s)) \geq \phi_N(L_Y(s)),\ s \geq 0 \quad (2.9)$$

and similarly, extending the inequality for the first term in the other direction,

$$\phi_M(L_X(s)) \geq \phi_M(L_Y(s)) \geq \phi_N(L_X(s)),\ s \geq 0. \quad (2.10)$$

As $s \to 0+$, the inequlaties (2.9) - (2.10) together imply

$$\phi_M(z) \geq \phi_N(z), \quad \text{for } A < z \leq 1 \quad (2.11)$$

where $A := \min(P(X=0), P(Y=0)) = P(Y=0)$, since $X <_L Y$ guarantees

$$P(X=0) = \lim_{s\to 0+} E(e^{-sX}) \geq \lim_{s \to 0+} E(e^{-sY}) = P(Y=0).$$

If $P(Y=0)=0$, then (2.11) implies $M <_L N$. Note, the d.f. of X need not be continuous at zero for the last conclusion, although such continuity would be sufficient to conclude $M <_L N$. □

Corollary 2.2 *For the random sums of nonnegative independent r.v.s. in* (1.1),

(i) *if $Y_n <_{icx} X_n, n \geq 1$; then $M <_L N$ implies $U <_L V$,*

(ii) *if $X_n <_{icx} Y_n, n \geq 1$; then $U <_L V$ implies $M <_L N$,*

(iii) *if* $X_n \overset{d}{=} Y_n, n \geq 1,\ \inf_{n\geq 1} P(X_n = 0) = 0$*; then* $U <_L V$ *if and only if* $M <_L N$.

Proof. (i) - (ii) $Y_n <_{icx} X_n$ implies that there exists a r.v. Z_n such that

$$X_n <_{st} Z_n <_{cx} Y_n$$

by a result of Müller and Stoyan (theorem 1.5.14, p.22 in [12]). Since $<_{st}$ and $<_{cx}$ are both stronger than $<_L$ (viz., $<_{st} \Rightarrow <_{icx} \Rightarrow <_L$, and $<_{cx} \Rightarrow <_{icx}$), we have $X_n <_L Z_n <_L Y_n$. Now apply theorem 2.1 **B** (i). This proves the first claim. An analogous argument applies for claim (ii).

(iii) If $X_n \overset{d}{=} Y_n$, then $Y_n <_{icx} X_n <_{icx} Y_n$. If additionally, $\inf_n P(X_n = 0) = 0$, then the hypotheses in both claims (i) and (ii) hold. □

Definition : If S and T are *non-negative* r.v.s. (such as lifetimes), we say (Shaked and Shanthikumar [14]) that

(i) S is smaller than T in the *mean residual life order* ($S <_{mrl} T$) if,

$$E(S - t|S > t) \leq E(T - t|T > t),\ t \geq 0$$

(ii) S is smaller than T in the *integral harmonic mean residual life order* ($S <_{hmrl} T$) if,

$$\int_0^t \frac{dx}{E(S - x|S > x)} \geq \int_0^t \frac{dx}{E(T - x|T > x)},\quad t \geq 0. \tag{2.12}$$

Clearly, $<_{mrl}$ implies $<_{hmrl}$. Since one usually wants to compare lifetime distributions sharing a common mean, our interest centers on the possibility of $<_{mrl}$ and $<_{hmrl}$ orders being preserved by random sums under the equal means asumption. Let $<^*_{mrl}$ ($<^*_{hmrl}$) be the stochastic orders $<_{mrl}$ ($<_{hmrl}$ respectively) restricted to lifetimes with equal means, i.e.,

$$S <^*_{mrl} (<^*_{hmrl}) T \Leftrightarrow S <_{mrl} (<_{hmrl}) T,\ \text{and } ES = ET < \infty.$$

It is known (theorem 3.A.13, p.91 in [14]) that if $S \geq 0, T \geq 0$ w.p.1, then $S <_{mrl} T \Rightarrow S <_{icx} T$. Hence $S <^*_{mrl} T \Rightarrow S <_{cx} T$. The converse does not in general hold. On the other hand, it turns out that $<^*_{hrml}$ is equivalent to the familiar convex order $<_{cx}$ restricted to r.v.s with d.f.s supported by the half line, a fact that we will exploit to prove theorem 2.4.

Lemma 2.3. *Let $S \geq 0$, and $T \geq 0$ w.p. 1. Then,*

$$S <^*_{hmrl} T \Leftrightarrow S <_{cx} T$$

Proof. Let S_e be a r.v. that has the equilibrium distribution with density $\{P(S > x)/ES\}$ induced by S, and let T_e similarly denote a r.v. that corresponds to the equilibrium distribution induced by T. Since

$$P(S_e > t) = \exp\left\{-\int_0^t \frac{dx}{E(S-x|S>x)}\right\},$$

and an analogous representation holds for the tail of T_e; it follows that $S \geq 0, T \geq 0$ w.p.l implies

$$\begin{aligned} S <^*_{hmrl} T &\Leftrightarrow ES = ET \text{ and } S_e <_{st} T_e \\ &\Leftrightarrow ES = ET \text{ and } \int_t^\infty \{P(T > x) - P(S > x)\}dx \geq 0,\ t > 0 \\ &\Leftrightarrow S <_{cx} T \end{aligned}$$

$$\Leftrightarrow Eh(S) \leq Eh(T), \quad \text{all convex } h \text{ on } [0, \infty). \tag{2.13}$$

The last step follows from a characterization of $<_{cx}$ restricted to non-negative r.v.s ([3], [12]). □

When T is exponential with $ES = ET \equiv \mu < \infty$, (2.12) reduces to the inequality

$$\int_0^t \frac{dx}{E(S-x|S>x)} \geq \frac{t}{\mu}, \quad \text{all } t > 0$$

which is the so called HNBUE (Harmonic New Better then Used in Expectation) property of S, well known in reliability context.

When S, T are both *discrete* on $\{0, 1, 2, \ldots\}$, one can similarly show that

$$S <^*_{hmrl} T \Leftrightarrow \sum_{k=0}^{n-1}\{E(S-k|S \geq k)\}^{-1} \geq \sum_{k=0}^{n-1}\{E(T-k|T \geq k)\}^{-1}, \quad \text{all } n \geq 1$$

$$\text{and } ES = ET \tag{2.14}$$

$$\Leftrightarrow Eh(S) \leq Eh(T), \quad \text{whenever } \{h(n),\ n \geq 0\} \text{ is a convex sequence} \tag{2.15}$$

Note, the last statement in (2.15) follows from lemma 2.3 and [2] (Corollary 2.2). When T is geometric on $\{0, 1, 2, \ldots\}$ with mean μ, (2.14) reduces to the discrete-HNBUE property,

$$\sum_{k=0}^{n-1} \{E(S-k|S \geq k)\}^{-1} \geq \frac{n}{\mu}, \text{ all } n \geq 1.$$

Theorem 2.4. *Randomly stopped sums in* (1.1) *with nonnegative independent summands, preserve the harmonic mean residual life order with equal means, i.e.,*

$$M <_{hmrl}^{*} N, \; Y_i <_{hmrl}^{*} X_i, \text{ all } i \geq 1 \Rightarrow U <_{hmrl}^{*} V.$$

Proof. In view of lemma 2.3, we will be done if $Eh(U) \leq Eh(V)$ for all real valued functions h which are convex on $[0, \infty)$. We prove this by combining two separate claims, viz.,

$$Eh(\sum_{i=1}^{N} Y_i) \leq Eh(V), \quad \text{for convex } h \text{ on } [0, \infty) \tag{2.16}$$

$$Eh(U) \leq Eh(\sum_{i=1}^{N} Y_i), \quad \text{for convex } h \text{ on } [0, \infty) \tag{2.17}$$

To prove (2.16), recall that $<_{cx}$ is closed under convolutions [12]. Hence, for any convex h on the half-line, and $n \geq 0$,

$$\begin{aligned} E\{h(\sum_{i=1}^{N} Y_i)|N=n\} &= Eh(\sum_{i=1}^{n} Y_i) \\ &\leq Eh(\sum_{i=1}^{n} X_i) = E\{h(\sum_{i=1}^{N} X_i)|N=n\}, \end{aligned} \tag{2.18}$$

where the first and the last step uses the fact that N is independent of $\{X_n\}, \{Y_n\}$. (Note, for $n = 0$, (2.18) is trivially true as equality, since both sides $= h(0)$.) Hence,

$$\begin{aligned} Eh(\sum_{i=1}^{N} Y_i) &= E\{E(h(\sum_{i=1}^{N} Y_i)|N)\} \\ &\leq E\{E(h(\sum_{i=1}^{N} X_i)|N)\} \equiv E\{E(h(V)|N\} = Eh(V). \end{aligned}$$

To prove the other claim in (2.17); given any convex h on $[0,\infty)$, set

$$H(n) := Eh(S_n),\ n \geq 0$$

where $S_n := \sum_{i=1}^n Y_i,\ n \geq 1$ and $S_0 = 0$. For the sequence $H(n)$ so defined, we have,

$$\begin{aligned}\Delta H(n) := H(n+1) - H(n) &= E\{h(S_n + Y_{n+1}) - h(S_n)\} \\ &= \int_0^\infty \int_0^\infty \{h(x+y) - h(x)\} dF(y) dF^{*n}(x) \\ &\equiv \int_0^\infty I(x) dF^{*n}(x) \\ &= EI(S_n)\end{aligned}$$

where F^{*n} is the n-fold self-convolution of F, and

$$I(x) := \int_0^\infty \{h(x+y) - h(x)\}\ dF(y),\ x \geq 0.$$

By convexity of h, the integrand is $\uparrow$ in x for each $y \geq 0$. Thus $I(x)$ is $\uparrow$ for $x \geq 0$. Coupled with the observation that $S_n \uparrow$ in n w.p.1 (by non-negativity of $\{Y_n, n \geq 1\}$), this implies $I(S_n) \leq I(S_{n+1})$ w.p.1. Hence,

$$\Delta H(n) = EI(S_n) \leq EI(S_{n+1}) = \Delta H(n+1),$$

i.e., $\{H(n), n \geq 0\}$ is a convex sequence. By an appeal to (2.15), this implies

$$Eh(U) = Eh(\sum_{i=1}^M Y_i) = EH(M) \leq EH(N) = Eh(\sum_{i=1}^N Y_i),$$

which completes our argument for (2.17). □

As an application of the foregoing theorem, we show that the class of HNBUE life distributions, is closed under formation of randomly stopped sums of independent HNBUE components, which are also independent of the stopping variable. For any $c > 0$, let $(Exp)_c$ denote an exponentially distributed r.v., with mean c, and $(Geom)_c$ a discrete r.v. that is geometrically distributed on the nonnegative integers with mean c.

Corollary 2.5 $\{Y_n, n \geq 1\}$ *independent HNBUE, M discrete HNBUE and independent of* $\{Y_n, n \geq 1\} \Rightarrow U := \sum_{i=1}^M Y_i$, *is HNBUE*

Proof. Take $X_n \stackrel{d}{=} (Exp)_{EY_n}, n \geq 1$ and $M \stackrel{d}{=} N \stackrel{d}{=} (Geom)_{EN}$. Then,

$$\begin{aligned} EU &= P(M=0) + \sum_{n=1}^{\infty} \sum_{i=1}^{n} (EY_i) P(M=n) \\ &= P(N=0) + \sum_{n=1}^{\infty} \sum_{i=1}^{n} (EX_i) P(N=n) = EV. \end{aligned}$$

Hence, theorem 2.4 gives $U <^{*}_{hmrl} V$, where $V = \sum_{i=1}^{N} X_i \stackrel{d}{=} (Exp)_{EV} = (Exp)_{EU}$. Thus V has the HNBUE-property. □

3. Geometric compounding and exponentiality

When N is geometrically distributed on the positive integers and the summands are i.i.d., the set of r.v.s. $V = \Sigma_{i=1}^{N} X_i$ in (1.1) defines the family of 'geometric compounds' with d.f.

$$C(x) = (1-\rho) \sum_{n=1}^{\infty} \rho^{n-1} F^{*n}(x), \quad 0 < \rho < 1, \ x \geq 0 \tag{3.1}$$

where F is the underlying d.f. of the i.i.d. r.v.s. X_n. When the latter satisfies suitably weak nonparametric assumptions, we can characterize exponential distributions within the family of geometric compound d.f.s C in (3.1) via simple moment conditions on F. Let (C_r) be the moment condition

$$(C_r): \quad EX_1^r = \Gamma(r+1)(EX_1)^r, \quad r > -1, \ r \neq 0, \ r \neq 1,$$

that the r-th. moment of F equals the corresponding moment of an exponential distribution with the same mean as that of F. Note that the condition is trivially true for $r=0$ and is free for $r=1$.

Corresponding to F with mean $\mu_F \equiv EX_1$, let TF denote the induced equilibrium distribution

$$TF(x) := \mu_F^{-1} \int_0^x \{1 - F(t)\} dt, \quad x \geq 0.$$

Let $(Exp)_c$, which we have used to denote an exponentially distributed r.v. with mean $c > 0$, also denote the corresponding distributions without ambiguity. Before stating our results relating to exponentiality, it is useful to recall the HNBUE (harmonic new beter than used in expectaion) and class $\mathcal{L}$ properties of d.f.s on the half-line (Klefsjö [9], [10]), viz., X_1 with d.f. F is HNBUE if

$$\int_0^x \frac{dt}{E(X_1 - t|X_t > t)} \geq \frac{x}{EX_1}, \; x > 0 \Leftrightarrow TF <_{st} (Exp)_{\mu_F}, \tag{3.2}$$

and $F \in \mathcal{L}$ if

$$E(e^{-sX_1}) \leq \frac{1}{1 + s(EX_1)}, \; s > 0 \Leftrightarrow (Exp)_{\mu_F} <_L F \tag{3.3}$$

A solidarity theorem on geometric compounds and its i.i.d. summands sharing the class $\mathcal{L}$ property, proved in [1], can be recovered as a special case of our results. Choose $M \stackrel{d}{=} N$ with a geometric distribution and $\{Y_n\}$ i.i.d., such that $Y_n \stackrel{d}{=} (Exp)_{EX_1}$. Our theorem 2.1 **A** (iii) then specializes to the following :

Corollary 3.1. (Bhattacharjee *et al.* [1]) *Suppose* $V = \sum_{i=1}^{N} X_i$, *is a Geometric Compound r.v. Then,*

$$V \in \mathcal{L} \text{ if and only if } X_1 \in \mathcal{L}$$

Corollary 3.2. *Consider the d.f.* C *in* (3.1) *induced by* F.

(i) *Suppose* $F \in$ *HNBUE. If* (C_r) *holds for some* $r > -1, r \neq 0$, *then* C *is exponential.*

(ii) *Suppose* $F \in \mathcal{L}$. *If* (C_r) *holds for some* $r \in (-1,1)\backslash\{0\}$, *then* C *is exponential.*

(iii) *Suppose* X_1 *with d.f.* F, *and coefficient of variation (c.v.)* η, *is such that* $TF \in \mathcal{L}$. *If*

$$EX_1^r = \Gamma(r+1)(EX_1)^r \left\{\frac{1}{2}(1+\eta^2)\right\}^{r-1}, \quad \textit{for some } r \in (0,2)\backslash\{1\}, \tag{3.4}$$

then C *is exponential.*

The above claims follow by using the results of Bhattacharjee and Sethuraman [6], Bhattacharjee [4] characterizing exponentiality within aging classes via moment conditions together with the fact that, in (3.1).

$$C \text{ is exponential } \Leftrightarrow F \text{ is exponential}, \tag{3.5}$$

i.e. the geometrically stopped sum V in (1.1) is exponentially distributed if and only if the underlying common distribution of X_n is exponential, which is well known. A simple argument for (3.5) is as follows :

$$\begin{aligned} C = (Exp)_{EV} \quad &\Leftrightarrow \quad \frac{1}{1+s(EV)} = L_C(s) = \frac{(1-\rho)L(s)}{1-\rho L(s)}, \text{ by} \\ &\Leftrightarrow \quad L(s) = \{\rho + (1-\rho)(1+sEV)\}^{-1} = \frac{1}{1+s(EX_1)} \\ &\Leftrightarrow \quad F = (Exp)_{EX_1}, \end{aligned}$$

To justify hypothesis $TF \in \mathcal{L}$ in Corollary 3.2, which may appear to be contrived and artificial at first right; note that it is indeed a non-empty class with the following interpretation. A lifetime d.f. on $[0, \infty)$ is in this class, if under a policy of repeated renewals on failure, the long run distribution of an unit currently in use has the class-$\mathcal{L}$ aging property. To see that

$$\mathcal{L}_0 := \{F \text{ on } [0, \infty) : TF \in \mathcal{L}\}$$

is not vacuous, note $\mathcal{L}_0 \supset \mathcal{L}_D := \{F : F \in \mathcal{L}, \quad TF \in \mathcal{L}$ which has been shown to be non empty with the property $F \in \mathcal{L}_D$ iff $C \in \mathcal{L}_D$ (Theorem 2.2 in [1]), where C is the compound geometric d.f. (3.1). Another argument to show that $\mathcal{L}_0$ is not empty follows from the observation that if $F \in \overline{\mathcal{L}}$ (dual class of $\mathcal{L}$) with coefficient of variation $\eta = 1$, then $TF \in \mathcal{L}$, viz., for $s > 0$,

$$E_{TF}(e^{-sX}) = \frac{1 - E_F(e^{-sX})}{sE_F X} \leq \frac{1}{sE_F X}\left(1 - \frac{1}{1+sE_F X}\right) = \frac{1}{1+sE_{TF}X},$$

since $\eta = 1$ iff $E_{TF}X = E_F X$.

Proof of Corollary 3.2. To prove claim (iii), first note that the moments of TF and F are related by

$$E_{TF}X^r = \frac{EX^{r+1}}{(r+1)EX}. \tag{3.6}$$

Writing $\mu_r \equiv E_F X^r$ brevity, note that if $TF \in \mathcal{L}$, then (3.4) together with Corollary 2.5 in Bhattacharjee [4] implies that

$$\begin{aligned} TF = (Exp)_{\mu_{TF}} \quad &\Leftrightarrow \quad E_{TF}X^r = \Gamma(r+1)(E_{TF}X)^r, \text{ for some } r \in (-1,1)\backslash\{0\} \\ &\Leftrightarrow \quad \mu_{r+1} = \Gamma(r+2)\mu^{r+1}\left(\frac{\mu_2}{2\mu_1^2}\right)^r, \text{ for some } r \in (-1,1)\backslash\{0\} \\ &\Leftrightarrow \quad \mu_p = \Gamma(p+1)\mu^p\left\{\frac{1}{2}(1+\eta^2)\right\}^{p-1}, \text{ for some } p \in (0,2)\backslash\{1\} \end{aligned}$$

which proves the desired conclusion, since F is exponential if and only if TF is exponential. □

Claim (ii) follows by combining (3.5) with a result of Bhattacharjee (Corollary 2.5, [4]) which proves sufficiency of condition (C_r) for some $r \in (-1,1)\backslash\{0\}$ under a class $\mathcal{L}$ hypothesis, Claim (i) similarly follows from (4.5) together with a known result (Theorem 2.3 in [6]). We may also directly argue claim (i) as follows.

Write $G \equiv (Exp)_{\mu_F}$ for brevity, and note that exponential distributions have finite moments of all orders $r > -1$. If F satisfies condition (C_r) for some $r > -1 (\neq 0, \neq 1)$ then (3.6) implies

$$\begin{aligned} E_{TF}X^p &= \frac{E_F X^{p+1}}{(p+1)E_F X}, \quad p = r+1 \\ &= \Gamma(p)(E_F X)^{p-1} = E_G X^p, \end{aligned}$$

so that,

$$\int_0^\infty x^{p-1}\left\{\overline{G}(x) - \overline{TF}(x)\right\} dx = p^{-1}(E_G X^p - E_{TF} X^p) = 0$$

where $\overline{G}$ and $\overline{TF}$ denote the tail functions of G and TF respectively.

Since the integrand is non-negative when F is HNBUE (*viz.*, (3.2)), it must vanish a.e. Hence $\overline{TF} = \overline{G}$ poinwise, which holds iff F (and hence C in (3.1)) is exponential. □

Remark. (i) Since the moment condition (C_r) is satisfied by the exponential distributions, for all $r > -1$, it follows that the results in Corollary 3.2 *characterize* the exponentials among compound geometric distributions.

(ii) Note that HNBUE class contains all standard aging classes IFR, DMRL, IRFA, NBU, NBUE. Hence Corollary 3.2 remains valid when F has any of the above stronger aging properties. Distributions in class $\mathcal{L}$ need only have a finite variance (Bhattacharjee and Sengupta [5]); while all positive order moments exist for HNBUE class of distributions with a finite mean [9], and hence the same is true for all stronger aging classes which are nested within HNBUE. Note that DMRL, NBUE, HNBUE and $\mathcal{L}$ aging properties require assuming a finite mean; while for NBU (and hence for IFRA and IFR classes, which are nested within NBU); the existence of a finite mean is free (Theorem 3.3 (ii), in [7]).

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