Math 662: Homework #1
Solution

#1. (#1.5.1 Text). Since there are 4 each of Aces, Kings, Queens, and Jacks plus $52-4\times 4=36$ other cards

outcome(c)	$X(c)\equiv x$	Prob $p(x)$
Ace	4	$4/52 = 1/13$
King	3	$4/52 = 1/13$
Queen	2	$4/52 = 1/13$
Jack	1	$4/52 = 1/13$
other	0	$36/52 = 9/13$
	Total =	1

The last two columns describe the distribution of X.

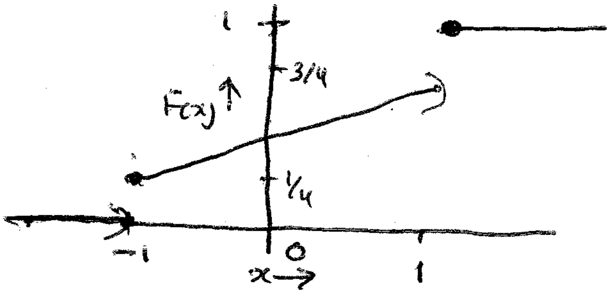
$$\mu \equiv EX = \sum_{x=0}^{4} x\,p(x) = \frac{19}{13}$$

$$EX^2 = \sum x^2 p(x) = \frac{30}{13}$$

$$\Rightarrow \sigma^2 \equiv \operatorname{var} X = EX^2-\mu^2 = \frac{30}{13}-\left(\frac{19}{13}\right)^2 = \frac{29}{169} \approx 0.172$$

#2 (#1.5.8 Text)

$$F(x) = \begin{cases} 0 & \text{if } x<-1 \\ \dfrac{x+2}{4} & \text{if } -1\le x<1 \\ 1 & \text{if } 1\le x \end{cases}$$



Graph of cdf F(x)

RV X with cdf F has discontinuity at $x=\pm 1$

$$P(X=-1) = \text{jump size at } -1 = F(-1)-F(-1-) = \frac{-1+2}{4} - 0 = \frac{1}{4}$$

$$P(X=1) = \text{jump size at } 1 = F(1)-F(1-) = 1-\frac{1+2}{4} = \frac{1}{4}$$

a) $P\left(-\frac{1}{2} < X \le \frac{1}{2}\right) = P\left(X\le \frac{1}{2}\right) - P\left(X<-\frac{1}{2}\right)$

$= F(1/2) - F(-1/2)$, $\left\{\begin{array}{l}\text{since } F \text{ is cont. at } x=-\frac{1}{2} \\ P(X<-\frac{1}{2}) = P(X\le -\frac{1}{2}) \\ \quad = F(-1/2)\end{array}\right.$

$= \left(\frac{1}{8}+\frac{1}{2}\right) - \left(-\frac{1}{8}+\frac{1}{2}\right)$

$= 2/8 = 1/4.$

viz on $[-1,1)$, $F(x) = \dfrac{x+2}{4} = \dfrac{x}{4}+\dfrac{1}{2}$.

b) $P(X=0) = F(0) - F(0-) = 0$, since $F(x)$ is continuous at zero

c) $P(X=1) = F(1) - F(1-) = \frac{1}{4}$, as argued previously

d) $P(2 < X \leq 3) = 0$ since X cannot exceed 1;

or, more formally

$$P(2 < X \leq 3) = P(X \leq 3) - P(X \leq 2) = F(3) - F(2) = 1 - 1 = 0.$$

[Note the RV X in this problem is of "Mixed Type", in the sense that X has a discrete as well as a continuous part. The range space of X is

$$\mathscr{R}(X) = \underbrace{\{-1, 1\}}_{\text{discrete part}} \cup \underbrace{(-1, 1)}_{\text{continuous part}}$$

→ The pmf for the discrete part is $p(-1) = p(1) = \frac{1}{4}$

with total mass in the discrete part $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

→ The pdf for the continuous part on $(-1, 1)$ is:

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}\left(\frac{x+2}{4}\right) = \frac{1}{4}, \quad -1 < x < 1$$

with total mass in the continuous part $= \int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{1}{4} dx = \frac{1}{2}$

Note Total mass of discrete & continuous part put together $= \frac{1}{2} + \frac{1}{2} = 1$.]

2 contd..

(#1.5.9, Text) Total # of slips $= \sum_{i=1}^{100} i = \frac{(100)\cdot(101)}{2} = 5050$

viz, $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

(a) Since all slips are equally likely to be chosen, and there are i-slips with # i on them, X has pmf

$$P(X=x) \equiv p(x) = \begin{cases} \frac{x}{5050} & x = 1, 2, \ldots, 100 \\ 0 & \text{otherwise} \end{cases}$$

(c) Since X can assume only positive integer values; for any real $x \geq 1$ ~~we have~~, and $x \leq 100$; we have

$$F(x) = P(X \leq x) = P(X \leq [x])$$

$$= \sum_{i=1}^{[x]} \frac{i}{5050} = \frac{[x]([x]+1)}{2 \times 5050}$$

$$\Rightarrow \quad F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \dfrac{[x]([x]+1)}{10,100} & \text{if } 1 \leq x < 100 \\ 1 & \text{if } 100 \leq x \end{cases}$$

Clearly F is continuous at $x = 100$. Thus the claim

$$F(x) = \frac{[x]([x]+1)}{10,100}, \quad \text{for } 1 \leq x \leq 100$$

is proved.

#4. $\boxed{\#\ 1.7.6\ (a)\ \text{Text}}$ pdf $f(x) = \begin{cases} \frac{x^2}{18} & \text{if } -3 < x < 3 \\ 0 & \text{else} \end{cases}$

$$\Rightarrow \quad P(|X| < 1) = \int_{-1}^{1} \frac{x^2}{18}\, dx =$$

$P(X^2 < 9) = P(|X| < 3) = 1$, since $f(x)$ vanishes outside $(-3, 3)$

$$P(X^2 < 5) = P(|X| < \sqrt{5}) = \int_{-\sqrt{5}}^{\sqrt{5}} \frac{x^2}{18}\, dx = 2\int_{0}^{\sqrt{5}} \frac{x^2}{18}\, dx =$$

#5. $\boxed{\#\ 1.7.9\ (a) - (c)\ \text{Text}}$

Any value x of a RV X is a median of X, if it satisfies both the inequalities

$$F(x-) = P(X < x) \leq \frac{1}{2} \quad \text{and} \quad F(x) = P(X \leq x) \geq \frac{1}{2}$$

i.e. its left hand limit at x is at most $\frac{1}{2}$, while its right hand limit at x is at least $\frac{1}{2}$.

$\Rightarrow$ if X is discrete, its median may not be unique

if X is a continuous RV, its unique median is the solution of $F(x) = \frac{1}{2}$.

(a) $p(x) = \binom{4}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x} \quad x = 0,1,2,3,4$

This is to so called Binomial Distribution, which we will study later

Numerically computing we have

$$\begin{cases} p(0) = \left(\frac{3}{4}\right)^4 = \frac{81}{256} \\ p(1) = 4\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3 = \frac{108}{256} \end{cases} \qquad \begin{cases} p(2) = 6\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{54}{256} \\ p(3) = 4\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) = \frac{12}{256} \\ p(4) = \left(\frac{1}{4}\right)^4 = \frac{1}{256} \end{cases}$$

leading to the following

x	$p(x)$	$F(x)$
0	$81/256$	$81/256 \approx 0.32$
1	$108/256$	$189/256 \approx 0.74$
2	$54/256$	$243/256$
3	$12/256$	$255/256$
4	$1/256$	$256/256 = 1$

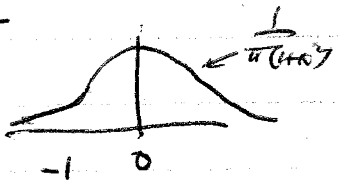
Since $F(x)$ being non-decreasing is flat between consecutive integers in $\{0,1,2,3,4\}$, it is easily seen at $x=1$ is the unique Median for this discrete distribution, since

$$F(1-) \approx 0.32 < \tfrac{1}{2} \quad \text{and} \quad F(1) \approx .74 > \tfrac{1}{2}$$

(b) $\left.\begin{array}{ll} f(x) = 3x^2, & 0<x<1 \\ 0, & \text{else} \end{array}\right\} \Rightarrow$ check cdf $F(x) = \begin{cases} 0, & \text{if } x<0 \\ x^3, & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$

Median x is the solut of $\quad \frac{1}{2} = F(x) = x^3$

$$\Rightarrow \quad x = 2^{-(1/3)} \equiv \frac{1}{\sqrt[3]{2}}$$



(c) $f(x) = \dfrac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{du}{1+u^2} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$$

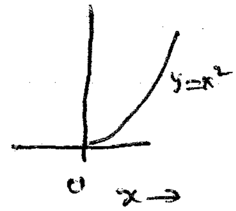
$$\frac{1}{2} = F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x \Rightarrow \tan^{-1} x = 0 \Rightarrow \text{Median} = 0$$

#6

$\boxed{\# 1.7.21 \mid \text{Text}}$ X has pdf $f_X(x) = 2xe^{-x^2}$, $x>0$

On the support (= Range Space) of X, which is $\mathscr{C}(X) = (0,\infty)$, $y = x^2$ is monotone $\uparrow$ with inverse function $x = +\sqrt{y}$

$\Rightarrow Y = X^2$ has pdf



$$f_Y(y) = f_X(\sqrt{y}) \left|\frac{dx}{dy}\right|$$

$$= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} = \begin{cases} 2\sqrt{y}\, e^{-y} \frac{1}{2\sqrt{y}} = e^{-y}, & \text{if } y>0 \\ 0, & \text{otherwise} \end{cases}$$

#7

$\boxed{\# 1.7.22 \mid \text{Text}}$ $y = \tan x$ is $\uparrow$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $x = \tan^{-1} y \in (-\infty, \infty)$

$f_X(x) = \frac{1}{\pi}$, if $-\pi/2 < x < \pi/2$

$$\Rightarrow f_Y(y) = f_X(\tan^{-1} y) \left|\frac{dx}{dy}\right| = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$$

This is the pdf of the standard Cauchy Distribution

#8

$\boxed{\# 1.7.23 \mid \text{Text}}$ (Direct computation of New pdf)

$$\left.\begin{matrix} y = -\ln x^4 \\ 0<x<1 \end{matrix}\right\} \Longleftrightarrow x = (e^{-y})^{1/4} = e^{-y/4}, \quad 0<y<\infty$$

X has pdf $f_X(x) = 4x^3$, $0<x<1$

$$\Rightarrow f_Y(y) = f_X(e^{-y/4}) \left|\frac{dx}{dy}\right| = 4e^{-3y/4} \frac{1}{4} e^{-y/4} = e^{-y}, \quad y>0$$

(Computing via cdf)

$$F_Y(y) = P(-\ln X^4 \le y) = P(X^4 \ge e^{-y}) = P(X \ge e^{-y/4}), \quad y>0$$

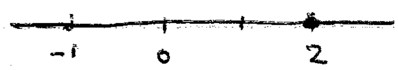
$$= \int_t^1 4x^3\,dx \Big|_{t=e^{-y/4}} = 1 - t^4 \Big|_{t=e^{-y/4}} = 1 - e^{-y}$$

$\Rightarrow Y$ has pdf

$$f_Y(y) = \frac{d}{dy} F_Y(y) = e^{-y}, \quad y>0$$

9(i) $\boxed{\# 1.7.24}$ $\boxed{\text{Text}}$

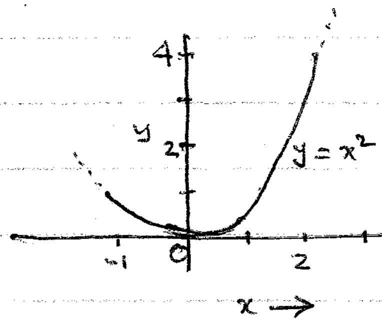
$X \sim$ pdf $f(x) = \begin{cases} \frac{1}{3}, & \text{if } -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$



RV $Y := X^2$ has range space

$\mathcal{E}(Y) = [0, 4)$; see graph of

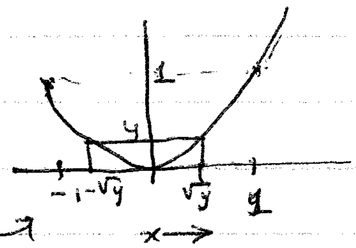
$y = x^2$ at the right side



$$F_Y(y) = \text{cdf of } Y \text{ at } y := P(X^2 \le y)$$
$$= P(|X| \le \sqrt{y})$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

<u>Case 1</u>

For $0 \le y < 1$:
$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\, dx \quad \text{(see graph)}$$
$$= \frac{2}{3}\sqrt{y}$$

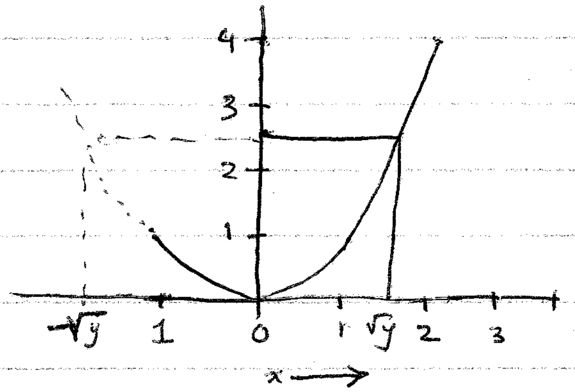


<u>Case 2</u>

For $1 \le y < 4$:
$$F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(-1 \le X \le \sqrt{y})$$

$\left\{ \begin{array}{l} \text{see graph} \rightarrow \\ \text{note} \\ 1 \le \sqrt{y} < 2 \end{array} \right.$

$$= \int_{-1}^{\sqrt{y}} \frac{1}{3}\, dx$$
$$= \frac{1+\sqrt{y}}{3}$$



Thus
$$F_Y(y) := P(X^2 \le y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{2}{3}\sqrt{y}, & \text{if } 0 \le y < 1 \\ \frac{1+\sqrt{y}}{3}, & \text{if } 1 \le y < 4 \\ 1, & \text{if } 4 \le y \end{cases}$$

<u>Note</u>. Both <u>Case 1</u>: $0 \le y < 1$ and <u>Case 2</u> $1 \le y < 4$ can be written together as

$$P(X^2 \le y) = P(|X| \le \sqrt{y}) = \int_{(-\sqrt{y}, y) \cap (-1, 2)} \frac{1}{3}\, dx = \int_{\max(-1, \sqrt{y})}^{\min(\sqrt{y}, 2)} \frac{1}{3}\, dx = \begin{cases} \frac{2}{3}\sqrt{y} & \text{if } 0 \le y < 1 \\ \frac{1+\sqrt{y}}{3} & \text{if } 1 \le y < 4 \end{cases}$$

9(ii) X is continuous RV with cdf $F(x)$, $\Rightarrow$ F is $\uparrow$ strictly

If X has pdf $f_X(x)$, $x \in \mathcal{C}(X)$; since

$$u = F(x) \Rightarrow \frac{du}{dx} = f_X(x)$$

$$\Rightarrow \frac{dx}{du} = \frac{1}{f_X(x)} = \frac{1}{f_X(F^{-1}(u))} > 0, \quad 0<u<1$$

$\Rightarrow$ $U = F(X)$ has pdf

$$f_U(u) = f_X(F^{-1}(u)) \left| \frac{dx}{du} \right| = 1 \quad \text{if } 0<u<1$$

Since $\int_0^1 f_U(u)\,du = \int_0^1 du = 1$, the pdf of U on the whole real line is

$$f_U(u) = \begin{cases} 1 & \text{if } 0<u<1 \\ 0 & \text{elsewhere} \end{cases}$$

Alt: Evaluating $f_U(u)$ via cdf of U

For any $u \in (0,1)$

$$F_U(u) := P(U \le u) = P(F(X) \le u)$$
$$= P(X \le F^{-1}(u)) \quad \text{since } F \text{ is strictly } \uparrow$$
$$= F(F^{-1}(u))$$
$$= u.$$

Then U has cdf
$$F_U(u) = \begin{cases} 0 & \text{if } u<0 \\ u & \text{if } 0 \le u < 1 \\ 1 & \text{if } 1 \le u \end{cases}$$

Note $F_U(0)$ is defined by right-continuity of F, which $\Rightarrow$ $F_U(0) = \lim_{u \to 0+} F_U(u) = \lim_{u \to 0+} u = 0$

Hence U has pdf

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} 1 & \text{if } 0<u<1 \\ 0 & \text{elsewhere} \end{cases}$$