Spring 2012
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Mathematics 768(102)- Probability Theory

Homework #1 (Post Date: 01/29/12)
(**Due Date: 02/08/12**)

1. Let f be a function mapping Ω to another space E with a σ-field $\mathcal{E}$. Let

$$\mathcal{A} = \{A \subset \Omega : \text{ there exists } B \text{ with } A = f^{-1}(B)\}.$$

Show that $\mathcal{A}$ is a σ-field on Ω. How can you characterize this σ-field among all σ-fields $\mathcal{F}$ such that the map $f : (\Omega,\text{F}) \longmapsto (E, \mathcal{E})$ is measurable ?

2. Let $\mathcal{A}$ be a σ-field of subsets of a space Ω, and let $B \in \mathcal{A}$. Show that

$$\mathcal{F} = \{A \bigcap B : A \in \mathcal{A}\}$$

is a σ-field of subsets of B. (We say $\mathcal{A}_B := \mathcal{F}$ is the σ-field $\mathcal{A}$ *relativized* to B. It is also called the *trace* of $\mathcal{A}$ on B.)

If $B \subset \Omega$ but, $B \notin \mathcal{A}$, is $\mathcal{F}$ still a σ-field ?

3. Let $\mathcal{F}$ be a σ-field on $[0, 1]$ such that $[\frac{1}{n+1}, \frac{1}{n}] \in \mathcal{F}$ for all $n = 1, 2, \cdots$. Prove each of the following sets belong to the σ-field $\mathcal{F}$:

a) $\{0\}$
b) $\{\frac{1}{n} : n = 2, 3, \cdots\}$
c) $\{(\frac{1}{n}, 1]\}$, for all $n = 1, 2, \cdots$
d) $\{(0, \frac{1}{n}]\}$, for all $n = 1, 2, \cdots$

4. (i) Show by an example that the *independence assumption* of the events $A_n; n = 1, 2, \cdots$ *in the second Borel Cantelli Lemma* :

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Longrightarrow P(A_n \text{ i.o. }) = 1$$

cannot be entirely dispensed with.

Continued ...

(ii) Think of an infinite sequence of independent coin tosses with a coin which falls heads with a probability p.

a) Show that, so long as p is strictly positive, it is ceratin that heads occour infinitely often in the sequence.

b) Verbally interpret the statement that the complementary event corresponding to a) has probability zero.

c) How often can the tails occur ? (Quantify by a probability statement.)

(iii) Consider the coin-toss sequence as in (ii) above, but which need not be independent, and uses a different coin for each toss. If the coin used for the n-th toss falls heads with probability $p_n := \frac{1}{(n+1)^2}$; $n = 1, 2, \cdots$; what can you say about the frequency of heads (tails) in the sequence of observed results?

5. Consider the sequence of sets $\{A_n \ : \ n \geq 1\}$ on the real line,

$$A_n = \begin{cases} \left(\frac{1}{n} \ , \ \frac{2}{3} - \frac{1}{n}\right), & \text{if } n \text{ odd} \\ \left(\frac{1}{3} - \frac{1}{n} \ , \ 1 + \frac{1}{n}\right), & \text{if } n \text{ even} \end{cases}$$

Find $\limsup_{n \to \infty} A_n$ and $\liminf_{n \to \infty} A_n$.
Do the sets A_n converge to a limit ?