Mathematics 691(101)- Stochastic Processes & Applications

Homework #4 (due 11/24//09)
Note, Tuesday Nov. 24, 2009 follows a **Thursday schedule.**

1. Let Y_n; $n = 0, 1, 2, \cdots$ be i.i.d. Bernoulli variables (assuming values 0,1 with probabilities $(1-p)$ and $p \in (0,1)$ respectively. Let

$$X_n = Y_n + Y_{n+1}\ , \quad n = 0, 1, 2, \cdots$$

Show that the process $\{X_n; n = 0, 1, 2, \cdots\}$ is *not* a Markov Chain (MC).

2. Show that you can *get a Markov Chain* (MC) in the setup of the preceeding example, *by enlarging the state space*, as follows. Let Y_n; $n = 0, 1, 2, \cdots$ be i.i.d. Bernoulli, as in Problem #1. Define the states of the process as $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. Then show that the process $\{(Y_n, Y_{n+1});\ n = 0, 1, 2, \cdots$ is a Markov Chain.

3. A classic example of a Markov (MC) in Physics, is the "*Ehrenfest model*" of diffusion of a gas through a membrane separating two containers (represented by urns A, B say). The gas molecules are represented by balls numbered $1, 2, \cdots, N$ distributed between the two containers.

The state of the system is defined by the number of gas molecules present in one of the urns (say urn A). Thus $S = \{0, 1, 2, \cdots, N\}$. The gas molecules move *randomly* between the two containers (urns) through the membrane. This random movement between the two containers can be modeled as follows. Pick an integer at random between 1 and N; then move the ball (gas molecule) that bears this number from its current urn to the other urn. If the system is currently in state i, then the chance that ball picked is in urn A is (i/N), and moves to urn B, decreasing the number of balls in urn A to $(i-1)$. Thus,

$$p_{i,i-1} = \frac{i}{N} \quad \text{if } 0 < i \le N.$$

By similar reasoning, we also have,

$$p_{i,i+1} = 1 - \frac{i}{N} \quad \text{if } 0 \le i < N.$$

Analyse this model. What is its long run behavior ? Interpret your results. Note that this idealized discrete model of diffusion of gas molecules is an example of a random walk with *reflecting barriers* at 0 and N. Compare your results with the standard model of gambler's ruin with *absorbing boundaries*.

4. (Allen, #22, p.86) A Markov Chain model for the growth and replacement of trees considers 4 stages (*states*) in the life of trees : seedling (1), young (2), mature (3) and old (4). A tree progreeses through these stages, and an old tree at death is replaced by a seedling. Thus the MC has a transition matrix of the form

$$P = \begin{pmatrix} p_{11} & 1-p_{11} & 0 & 0 \\ 0 & p_{22} & 1-p_{22} & 0 \\ 0 & 0 & p_{33} & 1-p_{33} \\ 1-p_{44} & 0 & 0 & p_{44} \end{pmatrix}$$

a) Suppose $p_{ii} \in (0,1) \forall i = 1,2,3,4$. Show that the MC is irriducible and aperiodic. Find the unique stationary distribution.

b) Suppose $p_{44} = 1$, but all other assumptions in a) remain. What do these assumptions imply about the growth and replacement of trees ? Show that

$$p_{ii}^{(n)} = p_{ii}^n \text{ for each state } i.$$

Identify the communicating classes and classify the states.