9 Dimensional Analysis and Similitude

Introduction

In the process of constructing a mathematical model, we have seen that the variables influencing the behavior must be identified and classified. We must then determine appropriate relationships among those variables retained for consideration. In the case of a single dependent variable this procedure gives rise to some unknown function:

$$y = f(x_1, x_2, \ldots, x_n)$$

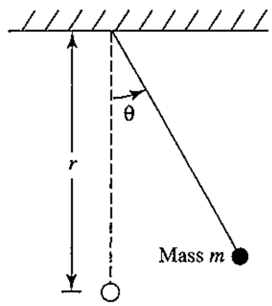
where the x_i measure the various factors influencing the phenomenon under investigation. In some situations the discovery of the nature of the function f for the chosen factors comes about by making some reasonable assumption based on a law of nature or previous experience and construction of a mathematical model. We were able to use this methodology in constructing our model on vehicular stopping distance (see Section 2.2). On the other hand, especially for those models designed to predict some physical phenomenon, we may find it difficult or impossible to construct a solvable or tractable explicative model because of the inherent complexity of the problem. In certain instances we might conduct a series of experiments to determine how the dependent variable y is related to various values of the independent variable(s). In such cases we usually prepare a figure or table and apply an appropriate curve-fitting or interpolation method that can be used to predict the value of y for suitable ranges of the independent variable(s). We employed this technique in modeling the elapsed time of a tape recorder in Sections 4.2 and 4.3.

Dimensional analysis is a method for helping determine how the selected variables are related and for reducing significantly the amount of experimental data that must be collected. It is based on the premise that physical quantities have dimensions and that physical laws are not altered by changing the units measuring dimensions. Thus, the phenomenon under investigation can be described by a dimensionally correct equation among the variables. A dimensional analysis provides qualitative information about the model. It is especially important when it is necessary to conduct experiments in the modeling process because the method is helpful in testing the validity of including or neglecting a particular factor, in reducing the number of experiments to be conducted to make predictions, and in improving the usefulness of the results by providing alternatives for the parameters employed to present them. Dimensional analysis has proved useful in physics and engineering for many years and even now plays a role in the study of the life sciences, economics, and operations research. Let's consider an example illustrating how dimensional analysis can be used in the modeling process to increase the efficiency of an experimental design.

Introductory Example: A Simple Pendulum

Consider the situation of a simple pendulum as suggested in Figure 9.1. Let r denote the length of the pendulum, m its mass, and θ the initial angle of displacement from the vertical. One characteristic that is vital in understanding the behavior of the pendulum is the **period**, which is the time required for the pendulum bob to swing through one complete cycle and return to its original position (as at the beginning of the cycle). We represent the period of the pendulum by the dependent variable t.

Figure 9.1
A simple pendulum



Problem Identification *For a given pendulum system, determine its period.*

Assumptions First, we list the factors that influence the period. Some of these factors are the length r, the mass m, the initial angle of displacement θ, the acceleration due to gravity g, and frictional forces such as the friction at the hinge and the drag on the pendulum. Assume initially that the hinge is frictionless, that the mass of the pendulum is concentrated at one end of the pendulum, and that the drag force is negligible. Other assumptions about the frictional forces will be examined in Section 9.3. Thus, the problem is to determine or approximate the function

$$t = f(r, m, \theta, g)$$

and test its worthiness as a predictor.

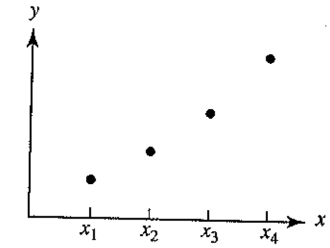
Experimental Determination of the Model Because gravity is essentially constant under the assumptions, the period t is a function of the three variables length r, mass m, and initial angle of displacement θ. At this point we could systematically conduct experiments to determine how t varies with these three variables. We would want to choose enough values of the independent variables to feel confident in predicting the period t over that range. How many experiments will be necessary?

For the sake of illustration, consider a function of one independent variable $y = f(x)$ and assume that four points have been deemed necessary to predict y over a suitable domain for x. The situation is depicted in Figure 9.2. An appropriate curve-fitting or interpolation method could be used to predict y within the domain for x.

Next consider what happens when a second independent variable affects the situation under investigation. We then have a function

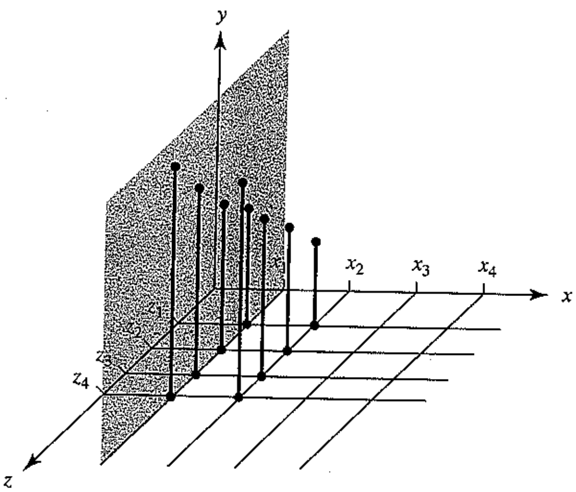
$$y = f(x, z)$$

Figure 9.2
Four points have been deemed necessary to predict y for this function of one variable x.



For each data value of x in Figure 9.2, experiments must be conducted to obtain y for four values of z. Thus, 16 (that is, 4^2) experiments are required. These observations are illustrated in Figure 9.3. Likewise, a function of three variables requires 64 (that is, 4^3) experiments. In general, 4^n experiments are required to predict y when n is the number of arguments of the function, assuming four points for the domain of each argument. Thus, a procedure that reduces the number of arguments of the function f will dramatically reduce the total number of required experiments. Dimensional analysis is one such procedure.

Figure 9.3
Sixteen points are necessary to predict y for this function of the two variables x and z.



The power of dimensional analysis is also apparent when we examine the interpolation curves that would be determined after collecting the data represented in Figures 9.2 and 9.3. Let's assume it is decided to pass a cubic polynomial through the four points shown in Figure 9.2. That is, the four points are used to determine the four constants C_1–C_4 in the interpolating curve:

$$y = C_1x^3 + C_2x^2 + C_3x + C_4$$

Now consider interpolating from Figure 9.3. If for a fixed value of x, say $x = x_1$, we decide to connect our points using a cubic polynomial in z, the equation of the interpolating surface is

$$\begin{aligned} y = {} & D_1x^3 + D_2x^2 + D_3x + D_4 + (D_5x^3 + D_6x^2 + D_7x + D_8)z \\ & + (D_9x^3 + D_{10}x^2 + D_{11}x + D_{12})z^2 + (D_{13}x^3 + D_{14}x^2 + D_{15}x + D_{16})z^3 \end{aligned}$$

Note from the equation that there are 16 constants—$D_1, D_2, \ldots, D_{16}$—to determine, rather than 4 as in the two-dimensional case. This procedure again illustrates the dramatic reduction in effort required when we reduce the number of arguments of the function we will finally investigate.

At this point we make the important observation that the experimental effort required depends more heavily on the number of arguments of the function to be investigated than on the true number of independent variables the modeler originally selected. For example, consider a function of two arguments, say $y = f(x, z)$. The discussion concerning the number of experiments necessary would not be altered if x were some particular combination of several variables. That is, x could be uv/w, where u, v, and w are the variables originally selected in the model.

Consider now the following preview of dimensional analysis, which describes how it reduces our experimental effort. Beginning with a function of n variables (hence, n arguments), the number of arguments is reduced (ordinarily by three) by combining the original variables into products. These resulting $(n - 3)$ products are called **dimensionless products** of the original variables. After applying dimensional analysis, we still need to conduct experiments to make our predictions, but the amount of experimental effort that is required will have been reduced exponentially.

In Chapter 2 we discussed the trade-offs of considering additional variables for increased precision versus neglecting variables for simplification. In constructing models based on experimental data, the preceding discussion suggests that the cost of each additional variable is an exponential increase in the number of experimental trials that must be conducted. In the next two sections, we present the main ideas underlying the dimensional analysis process. You may find that some of these ideas are slightly more difficult than some that we have already investigated, but the methodology is powerful when modeling physical behavior.

9.1 Dimensions as Products

The study of physics is based on abstract concepts such as mass, length, time, velocity, acceleration, force, energy, work, and pressure. To each such concept a unit of measurement is assigned. A physical law such as $F = ma$ is true, provided that the units of measurement are consistent. Thus, if mass is measured in kilograms and acceleration in meters per second squared, then the force must be measured in newtons. These units of measurement belong to the MKS (meter–kilogram–second) mass system. It would be inconsistent with the equation $F = ma$ to measure mass in slugs, acceleration in feet per second squared, and force in newtons. In this illustration, force must be measured in pounds, giving the American Engineering System of measurement. There are other systems of measurement, but all are prescribed by international standards so as to be consistent with the laws of physics.

The three primary physical quantities we consider in this chapter are mass, length, and time. We associate with these quantities the dimensions M, L, and T, respectively. The dimensions are symbols that reveal how the numerical value of a quantity changes when the units of measurement change in certain ways. The dimensions of other quantities follow from definitions or from physical laws and are expressed in terms of M, L, and

T. For example, velocity v is defined as the ratio of distance s (dimension L) traveled to time t (dimension T) of travel—that is, $v = st^{-1}$, so the dimension of velocity is LT^{-1}. Similarly, because area is fundamentally a product of two lengths, its dimension is L^2. These dimension expressions hold true regardless of the particular system of measurement, and they show, for example, that velocity may be expressed in meters per second, feet per second, miles per hour, and so forth. Likewise, area can be measured in terms of square meters, square feet, square miles, and so on.

There are still other entities in physics that are more complex in the sense that they are not usually defined directly in terms of mass, length, and time alone; instead, their definitions include other quantities, such as velocity. We associate dimensions with these more complex quantities in accordance with the algebraic operations involved in the definitions. For example, because momentum is the product of mass with velocity, its dimension is $M(LT^{-1})$ or simply MLT^{-1}.

The basic definition of a quantity may also involve dimensionless constants; these are ignored in finding dimensions. Thus, the dimension of kinetic energy, which is one-half (a dimensionless constant) the product of mass with velocity squared, is $M(LT^{-1})^2$ or simply ML^2T^{-2}. As you will see in Example 2, some constants (dimensional constants), such as gravity g, do have an associated dimension, and these must be considered in a dimensional analysis.

These examples illustrate the following important concepts regarding dimensions of physical quantities.

1. We have based the concept of dimension on three physical quantities: mass m, length s, and time t. These quantities are measured in some appropriate system of units whose choice does not affect the assignment of dimensions. (This underlying system must be linear. A dimensional analysis will not work if the scale is logarithmic, for example.)
2. There are other physical quantities, such as area and velocity, that are defined as simple products involving only mass, length, or time. Here we use the term **product** to include any quotient because we may indicate division by negative exponents.
3. There are still other, more complex physical entities, such as momentum and kinetic energy, whose definitions involve quantities other than mass, length, and time. Because the simpler quantities from (1) and (2) are products, these more complex quantities can also be expressed as products involving mass, length, and time by algebraic simplification. We use the term *product* to refer to any physical quantity from item (1), (2), or (3); a product from (1) is trivial because it has only one factor.
4. To each product, there is assigned a **dimension**—that is, an expression of the form

$$M^n L^p T^q \tag{9.1}$$

where n, p, and q are real numbers that may be positive, negative, or zero.

When a basic dimension is missing from a product, the corresponding exponent is understood to be zero. Thus, the dimension $M^2L^0T^{-1}$ may also appear as M^2T^{-1}. When n, p, and q are all zero in an expression of the form (9.1), so that the dimension reduces to

$$M^0 L^0 T^0 \tag{9.2}$$

the quantity, or product, is said to be *dimensionless*.

Special care must be taken in forming sums of products because just as we cannot add apples and oranges, in an equation we cannot add products that have unlike dimensions. For example, if F denotes force, m mass, and v velocity, we know immediately that the equation

$$F = mv + v^2$$

cannot be correct because mv has dimension MLT^{-1}, whereas v^2 has dimension L^2T^{-2}. These dimensions are unlike; hence, the products mv and v^2 cannot be added. An equation such as this—that is, one that contains among its terms two products having unlike dimensions—is said to be *dimensionally incompatible*. Equations that involve only sums of products having the same dimension are *dimensionally compatible*.

The concept of dimensional compatibility is related to another important concept called **dimensional homogeneity**. In general, an equation that is true regardless of the system of units in which the variables are measured is said to be dimensionally homogeneous. For example, $t = \sqrt{2s/g}$ giving the time a body falls a distance s under gravity (neglecting air resistance) is dimensionally homogeneous (true in all systems), whereas the equation $t = \sqrt{s/16.1}$ is not dimensionally homogeneous (because it depends on a particular system). In particular, if an equation involves only sums of dimensionless products (i.e., it is a dimensionless equation), then the equation is dimensionally homogeneous. Because the products are dimensionless, the factors used for conversion from one system of units to another would simply cancel.

The application of dimensional analysis to a real-world problem is based on the assumption that the solution to the problem is given by a dimensionally homogeneous equation in terms of the appropriate variables. Thus, the task is to determine the form of the desired equation by finding an appropriate dimensionless equation and then solving for the dependent variable. To accomplish this task, we must decide which variables enter into the physical problem under investigation and determine all the dimensionless products among them. In general, there may be infinitely many such products, so they will have to be described rather than actually written out. Certain subsets of these dimensionless products are then used to construct dimensionally homogeneous equations. In Section 9.2 we investigate how the dimensionless products are used to find all dimensionally homogeneous equations. The following example illustrates how the dimensionless products may be found.

EXAMPLE 1 *A Simple Pendulum Revisited*

Consider again the simple pendulum discussed in the introduction. Analyzing the dimensions of the variables for the pendulum problem, we have

Variable	m	g	t	r	θ
Dimension	M	LT^{-2}	T	L	$M^0L^0T^0$

Next we find all the dimensionless products among the variables. Any product of these variables must be of the form

$$m^a g^b t^c r^d \theta^e \tag{9.3}$$

and hence must have dimension

$$(M)^a(LT^{-2})^b(T)^c(L)^d(M^0L^0T^0)^e$$

Therefore, a product of the form (9.3) is dimensionless if and only if

$$M^aL^{b+d}T^{c-2b} = M^0L^0T^0 \tag{9.4}$$

Equating the exponents on both sides of this last equation leads to the system of linear equations

$$\left.\begin{array}{rrrrl} a & & & +0e & =0 \\ & b & +d & +0e & =0 \\ & -2b\;+c & & +0e & =0 \end{array}\right\} \tag{9.5}$$

Solution of the system (9.5) gives $a = 0$, $c = 2b$, $d = -b$, where b is arbitrary. Thus, there are infinitely many solutions. Here are some general rules for selecting arbitrary variables: (1) Choose the dependent variable so it will appear only once, (2) select any variable that expedites the solution of the other equations (i.e., a variable that appears in all equations), and (3) choose a variable that always has a zero coefficient, if possible. Notice that the exponent e does not really appear in (9.4) (because it has a zero coefficient in each equation) so it is also arbitrary. One dimensionless product is obtained by setting $b = 0$ and $e = 1$, yielding $a = c = d = 0$. A second, independent dimensionless product is obtained when $b = 1$ and $e = 0$, yielding $a = 0$, $c = 2$, and $d = -1$. These solutions give the dimensionless products

$$\prod_1 = m^0g^0t^0r^0\theta^1 = \theta$$

$$\prod_2 = m^0g^1t^2r^{-1}\theta^0 = \frac{gt^2}{r}$$

In Section 9.2, we will learn a methodology for relating these products to carry the modeling process to completion. For now, we will develop a relationship in an intuitive manner.

Assuming $t = f(r, m, g, \theta)$, to determine more about the function f, we observe that if the units in which we measure mass are made smaller by some factor (e.g., 10), then the measure of the period t will not change because it is measured in units (T) of time. Because m is the only factor whose dimension contains M, it cannot appear in the model. Similarly, if the scale of the units (L) for measuring length is altered, it cannot change the measure of the period. For this to happen, the factors r and g must appear in the model as r/g, g/r, or, more generally, $(g/r)^k$. This ensures that any linear change in the way length is measured will be canceled. Finally, if we make the units (T) that measure time smaller by a factor of 10, for example, the measure of the period will directly increase by this same factor 10. Thus, to have the dimension of T on the right side of the equation $t = f(r, m, g, \theta)$, g and r must appear as $\sqrt{r/g}$ because T appears to the power -2 in the dimension of g. Note that none of the preceding conditions places any restrictions on the angle θ. Thus, the equation of the period should be of the form

$$t = \sqrt{\frac{r}{g}}\,h(\theta)$$

where the function h must be determined or approximated by experimentation.

We note two things in this analysis that are characteristic of a dimensional analysis. First, in the MLT system, three conditions are placed on the model, so we should generally expect to reduce the number of arguments of the function relating the variables by three. In the pendulum problem we reduced the number of arguments from four to one. Second, all arguments of the function present at the end of a dimensional analysis (in this case, θ) are dimensionless products.

In the problem of the undamped pendulum we assumed that friction and drag were negligible. Before proceeding with experiments (which might be costly), we would like to know whether that assumption is reasonable. Consider the model obtained so far:

$$t = \sqrt{\frac{r}{g}}\, h(\theta)$$

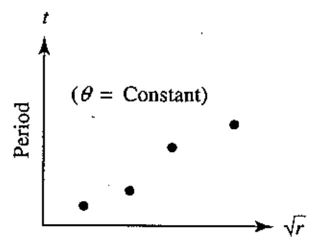
Keeping θ constant while allowing r to vary, form the ratio

$$\frac{t_1}{t_2} = \frac{\sqrt{r_1/g}\, h(\theta_0)}{\sqrt{r_2/g}\, h(\theta_0)} = \sqrt{\frac{r_1}{r_2}}$$

Hence the model predicts that t will vary as $\sqrt{r}$ for constant θ. Thus, if we plot t versus r with fixed θ for some observations, we will expect to get a straight line (Figure 9.4). If we do not obtain a reasonably straight line, then we need to reexamine the assumptions. Note that our judgment here is qualitative. The final measure of the adequacy of any model is always how well it predicts or explains the phenomenon under investigation. Nevertheless, this initial test is useful for eliminating obviously bad assumptions and for choosing among competing sets of assumptions.

Figure 9.4

Testing the assumptions of the simple pendulum model by plotting the period t versus the square root of the length r for constant displacement θ



Dimensional analysis has helped construct a model $t = f(r, m, g, \theta)$ for the undamped pendulum as $t = \sqrt{r/g}\, h(\theta)$. If we are interested in predicting the behavior of the pendulum, we could isolate the effect of h by holding r constant and varying θ. This provides the ratio

$$\frac{t_1}{t_2} = \frac{\sqrt{r_0/g}\, h(\theta_1)}{\sqrt{r_0/g}\, h(\theta_2)} = \frac{h(\theta_1)}{h(\theta_2)}$$

Hence a plot of t versus θ for several observations would reveal the nature of h. This plot is illustrated in Figure 9.5. We may never discover the true function h relating the variables. In such cases, an empirical model might be constructed from the experimental data, as discussed in Chapter 4. When we are interested in using our model to predict t, based on

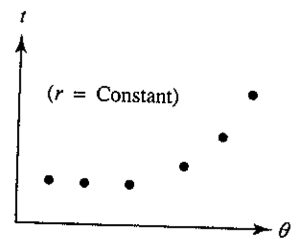


Figure 9.5
Determining the unknown function h

experimental results, it is convenient to use the equation $t\sqrt{g/r} = h(\theta)$ and to plot $t\sqrt{g/r}$ versus θ, as in Figure 9.6. Then, for a given value of θ, we would determine $t\sqrt{g/r}$, multiply it by $\sqrt{r/g}$ for a specific r, and finally determine t.

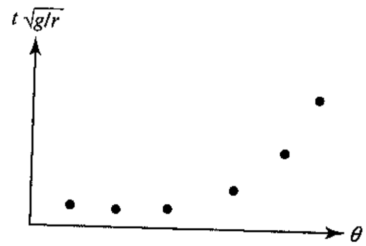


Figure 9.6
Presenting the results for the simple pendulum

EXAMPLE 2 *Wind Force on a Van*

Suppose you are driving a van down a highway with gusty winds. How does the speed of your vehicle affect the wind force you are experiencing?

The force F of the wind on the van is certainly affected by the speed v of the van and the surface area A of the van directly exposed to the wind's direction. Thus, we might hypothesize that the force is proportional to some power of the speed times some power of the surface area; that is,

$$F = kv^a A^b \tag{9.6}$$

for some (dimensionless) constant k. Analyzing the dimensions of the variables gives

Variable	F	k	v	A
Dimension	MLT^{-2}	$M^0L^0T^0$	LT^{-1}	L^2

Hence, dimensionally, Equation (9.6) becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b$$

This last equation cannot be correct because the dimension M for mass does not enter into the right-hand side with nonzero exponent.

So consider again Equation (9.6). What is missing in our assumption concerning the wind force? Wouldn't the strength of the wind be affected by its density? After some reflection we would probably agree that density does have an effect. If we include the density ρ as a factor, then our refined model becomes

$$F = kv^a A^b \rho^c \tag{9.7}$$

Because density is mass per unit volume, the dimension of density is ML^{-3}. Therefore, dimensionally, Equation (9.7) becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b(ML^{-3})^c$$

Equating the exponents on both sides of this last equation leads to the system of linear equations:

$$\left.\begin{aligned} c &= 1 \\ a + 2b - 3c &= 1 \\ -a \qquad &= -2 \end{aligned}\right\} \tag{9.8}$$

Solution of the system (9.8) gives $a = 2$, $b = 1$, and $c = 1$. When substituted into Equation (9.7), these values give the model

$$F = kv^2 A\rho$$

At this point we make an important observation. When it was assumed that $F = kv^a A^b$, the constant was assumed to be dimensionless. Subsequently, our analysis revealed that for a particular medium (so ρ is constant)

$$F \propto Av^2$$

giving $F = k_1 Av^2$. However, k_1 does have a dimension associated with it and is called a **dimensional constant**. In particular, the dimension of k_1 is

$$\frac{MLT^{-2}}{L^2(L^2T^{-2})} = ML^{-3}$$

Dimensional constants contain important information and must be considered when performing a dimensional analysis. We consider dimensional constants again in Section 9.3 when we investigate a damped pendulum.

If we assume the density ρ is constant, our model shows that the force of the wind is proportional to the square of the speed of the van times its surface area directly exposed to the wind. We can test the model by collecting data and plotting the wind force F versus v^2A to determine whether the graph approximates a straight line through the origin. This example illustrates one of the ways in which dimensional analysis can be used to test our assumptions and check whether we have a faulty list of variables identifying the problem. Table 9.1 gives a summary of the dimensions of some common physical entities.

Table 9.1 Dimensions of physical entities in the *MLT* system

Mass	M	Momentum	MLT^{-1}
Length	L	Work	ML^2T^{-2}
Time	T	Density	ML^{-3}
Velocity	LT^{-1}	Viscosity	$ML^{-1}T^{-1}$
Acceleration	LT^{-2}	Pressure	$ML^{-1}T^{-2}$
Specific weight	$ML^{-2}T^{-2}$	Surface tension	MT^{-2}
Force	MLT^{-2}	Power	ML^2T^{-3}
Frequency	T^{-1}	Rotational inertia	ML^2
Angular velocity	T^{-1}	Torque	ML^2T^{-2}
Angular acceleration	T^{-2}	Entropy	ML^2T^{-2}
Angular momentum	ML^2T^{-1}	Heat	ML^2T^{-2}
Energy	ML^2T^{-2}		

9.1 PROBLEMS

1. Determine whether the equation

$$s = s_0 + v_0 t - 0.5gt^2$$

is dimensionally compatible, if s is the position (measured vertically from a fixed reference point) of a body at time t, s_0 is the position at $t = 0$, v_0 is the initial velocity, and g is the acceleration caused by gravity.

2. Find a dimensionless product relating the torque τ (ML^2T^{-2}) produced by an automobile engine, the engine's rotation rate ψ (T^{-1}), the volume V of air displaced by the engine, and the air density ρ.

3. The various constants of physics often have physical dimensions (dimensional constants) because their values depend on the system in which they are expressed. For example, Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the square of the distance between them, or, symbolically,

$$F = \frac{Gm_1m_2}{r^2}$$

where G is the gravitational constant. Find the dimension of G so that Newton's law is dimensionally compatible.

4. Certain stars, whose light and radial velocities undergo periodic vibrations, are thought to be pulsating. It is hypothesized that the period t of pulsation depends on the star's radius r, its mass m, and the gravitational constant G. (See Problem 3 for the dimension of G.) Express t as a product of m, r, and G, so the equation

$$t = m^a r^b G^c$$

is dimensionally compatible.

5. In checking the dimensions of an equation, you should note that derivatives also possess dimensions. For example, the dimension of ds/dt is LT^{-1} and the dimension of d^2s/dt^2 is LT^{-2}, where s denotes distance and t denotes time. Determine whether the equation

$$\frac{dE}{dt} = \left[mr^2\left(\frac{d^2\theta}{dt^2}\right) mgr \sin\theta\right]\frac{d\theta}{dt}$$

for the time rate of change of total energy E in a pendulum system with damping force is dimensionally compatible.

6. For a body moving along a straight-line path, if the mass of the body is changing over time, then an equation governing its motion is given by

$$m\frac{dv}{dt} = F + u\frac{dm}{dt}$$

where m is the mass of the body, v is the velocity of the body, F is the total force acting on the body, dm is the mass joining or leaving the body in the time interval dt, and u is the velocity of dm at the moment it joins or leaves the body (relative to an observer stationed on the body). Show that the preceding equation is dimensionally compatible.

7. In humans, the hydrostatic pressure of blood contributes to the total blood pressure. The hydrostatic pressure P is a product of blood density ρ, height h of the blood column between the heart and some lower point in the body, and gravity g. Determine

$$P = k\rho^a h^b g^c$$

where k is a dimensionless constant.

8. Assume the force F opposing the fall of a raindrop through air is a product of viscosity μ, velocity v, and the diameter r of the drop. Assume that density is neglected. Find

$$F = k\mu^a v^b r^c$$

where k is a dimensionless constant.

9.1 PROJECT

1. Complete the requirements of "Keeping Dimension Straight," by George E. Strecker, UMAP 564. This module is a very basic introduction to the distinction between dimensions and units. It also provides the student with some practice in using dimensional arguments to properly set up solutions to elementary problems and to recognize errors.

9.2 The Process of Dimensional Analysis

In the preceding section we learned how to determine all dimensionless products among the variables selected in the problem under investigation. Now we investigate how to use the dimensionless products to find all possible dimensionally homogeneous equations among the variables. The key result is Buckingham's theorem, which summarizes the entire theory of dimensional analysis.

Example 1 in the preceding section shows that in general, many dimensionless products may be formed from the variables of a given system. In that example we determined every dimensionless product to be of the form

$$g^b t^{2b} r^{-b} \theta^e \tag{9.9}$$

where b and e are arbitrary real numbers. Each one of these products corresponds to a solution of the homogeneous system of linear algebraic equations given by Equation (9.5). The two products

$$\prod_1 = \theta \quad \text{and} \quad \prod_2 = \frac{gt^2}{r}$$

obtained when $b = 0$, $e = 1$, and $b = 1$, $e = 0$, respectively, are special in the sense that any of the dimensionless products (9.9) can be given as a product of some power of $\prod_1$ times some power of $\prod_2$. Thus, for instance,

$$g^3 t^6 r^{-3} \theta^{1/2} = \prod_1^{1/2} \prod_2^3$$

This observation follows from the fact that $b = 0$, $e = 1$ and $b = 1$, $e = 0$ represent, in some sense, independent solutions of the system (9.5). Let's explore these ideas further.

Consider the following system of m linear algebraic equations in the n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{9.10}$$

The symbols a_{ij} and b_i denote real numbers for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The numbers a_{ij} are called the **coefficients** of the system, and the b_i are referred to as the **constants**. The subscript i in the symbol a_{ij} refers to the ith equation of the system (9.10) and the subscript j refers to the jth unknown x_j to which a_{ij} belongs. Thus, the subscripts serve to locate a_{ij}. It is customary to read a_{13} as "a, one, three" and a_{42} as "a, four, two," for example, rather than "a, thirteen" and "a, forty-two."

A **solution** to the system (9.10) is a sequence of numbers $s_1, s_2, \ldots, s_n$ for which $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ solves each equation in the system. If $b_1 = b_2 = \cdots = b_m = 0$, the system (9.10) is said to be **homogeneous**. The solution $s_1 = s_2 = \cdots = s_n = 0$ always solves the homogeneous system and is called the **trivial solution**. For a homogeneous system there are two solution possibilities: Either the trivial solution is the only solution or there are infinitely many solutions.

Whenever $s_1, s_2, \ldots, s_n$ and $s_1', s_2', \ldots, s_n'$ are solutions to the homogeneous system, the sequences $s_1 + s_1', s_2 + s_2', \ldots, s_n + s_n'$, and $cs_1, cs_2, \ldots, cs_n$ are also solutions for any constant c. These solutions are called the **sum** and **scalar multiple** of the original solutions, respectively. If S and S' refer to the original solutions, then we use the notations $S + S'$ to refer to their sum and cS to refer to a scalar multiple of the first solution. If $S_1, S_2, \ldots, S_k$ is a collection of k solutions to the homogeneous system, then the solution

$$c_1S_1 + c_2S_2 + \cdots + c_kS_k$$

is called a **linear combination** of the k solutions, where $c_1, c_2, \ldots, c_k$ are arbitrary real numbers. It is an easy exercise to show that any linear combination of solutions to the homogeneous system is still another solution to the system.

A set of solutions to a homogeneous system is said to be **independent** if no solution in the set is a linear combination of the remaining solutions in the set. A set of solutions is **complete** if it is independent and every solution is expressible as a linear combination of solutions in the set. For a specific homogeneous system, we seek some complete set of solutions because all other solutions are produced from them using linear combinations. For example, the two solutions corresponding to the two choices $b = 0$, $e = 1$ and $b = 1$, $e = 0$ form a complete set of solutions to the homogeneous system (9.5).

It is not our intent to present the theory of linear algebraic equations. Such a study is appropriate for a course in linear algebra. We do point out that there is an elementary algorithm known as Gaussian elimination for producing a complete set of solutions to a given system of linear equations. Moreover, Gaussian elimination is readily implemented on computers and handheld programmable calculators. The systems of equations we will encounter in this book are simple enough to be solved by the elimination methods learned in intermediate algebra.

How does our discussion relate to dimensional analysis? Our basic goal thus far has been to find all possible dimensionless products among the variables that influence the physical phenomenon under investigation. We developed a homogeneous system of linear algebraic equations to help us determine these dimensionless products. This system of equations usually has infinitely many solutions. Each solution gives the values of the exponents that result in a dimensionless product among the variables. If we sum two solutions, we produce another solution that yields the same dimensionless product as does multiplication of the dimensionless products corresponding to the original two solutions. For example, the sum of the solutions corresponding to $b = 0$, $e = 1$ and $b = 1$, $e = 0$ for Equation (9.5) yields the solution corresponding to $b = 1$, $e = 1$ with the corresponding dimensionless product from Equation (9.9) given by

$$gt^2r^{-1}\theta = \prod_1 \prod_2$$

The reason for this result is that the unknowns in the system of equations are the exponents in the dimensionless products, and addition of exponents algebraically corresponds to multiplication of numbers having the same base: $x^{m+n} = x^m x^n$. Moreover, multiplication of a solution by a constant produces a solution that yields the same dimensionless product as does raising the product corresponding to the original solution to the power of the constant. For example, -1 times the solution corresponding to $b = 1$, $e = 0$ yields the solution corresponding to $b = -1$, $e = 0$ with the corresponding dimensionless product

$$g^{-1}t^{-2}r = \prod_2^{-1}$$

The reason for this last result is that algebraic multiplication of an exponent by a constant corresponds to raising a power to a power, $x^{mn} = (x^m)^n$.

In summary, addition of solutions to the homogeneous system of equations results in multiplication of their corresponding dimensionless products, and multiplication of a solution by a constant results in raising the corresponding product to the power given by that constant. Thus, if S_1 and S_2 are two solutions corresponding to the dimensionless products $\prod_1$ and $\prod_2$, respectively, then the linear combination $aS_1 + bS_2$ corresponds to the dimensionless product

$$\prod_1^a \prod_2^b$$

It follows from our preceding discussion that a complete set of solutions to the homogeneous system of equations produces all possible solutions through linear combination. The dimensionless products corresponding to a complete set of solutions are therefore called a *complete set of dimensionless products.* All dimensionless products can be obtained by forming powers and products of the members of a complete set.

Next, let's investigate how these dimensionless products can be used to produce all possible dimensionally homogeneous equations among the variables. In Section 9.1 we defined an equation to be dimensionally homogeneous if it remains true regardless of the system of units in which the variables are measured. The fundamental result in dimensional analysis that provides for the construction of all dimensionally homogeneous equations from complete sets of dimensionless products is the following theorem.

Theorem 1

Buckingham's Theorem An equation is dimensionally homogeneous if and only if it can be put into the form

$$f\left(\prod_1, \prod_2, \ldots, \prod_n\right) = 0 \tag{9.11}$$

where f is some function of n arguments and $\{\prod_1, \prod_2, \ldots, \prod_n\}$ is a complete set of dimensionless products.

Let's apply Buckingham's theorem to the simple pendulum discussed in the preceding sections. The two dimensionless products

$$\prod_1 = \theta \quad \text{and} \quad \prod_2 = \frac{gt^2}{r}$$

form a complete set for the pendulum problem. Thus, according to Buckingham's theorem, there is a function f such that

$$f\left(\theta, \frac{gt^2}{r}\right) = 0$$

Assuming we can solve this last equation for gt^2/r as a function of θ, it follows that

$$t = \sqrt{\frac{r}{g}}h(\theta) \tag{9.12}$$

where h is some function of the single variable θ. Notice that this last result agrees with our intuitive formulation for the simple pendulum presented in Section 9.1. Observe that Equation (9.12) represents only a general form for the relationship among the variables m, g, t, r, and θ. However, it can be concluded from this expression that t does not depend on the mass m and is related to $r^{1/2}$ and $g^{-1/2}$ by some function of the initial angle of displacement θ. Knowing this much, we can determine the nature of the function h experimentally or approximate it, as discussed in Section 9.1.

Consider Equation (9.11) in Buckingham's theorem. For the case in which a complete set consists of a single dimensionless product, for example, $\prod_1$, the equation reduces to the form

$$f\left(\prod_1\right) = 0$$

In this case we assume that the function f has one real root at k (to assume otherwise has little physical meaning). Hence, the solution $\prod_1 = k$ is obtained.

Using Buckingham's theorem, let's reconsider the example from Section 9.1 of the wind force on a van driving down a highway. Because the four variables F, v, A, and ρ were selected and all three equations in (9.8) are independent, a complete set of dimensionless products consists of a single product:

$$\prod_1 = \frac{F}{v^2 A\rho}$$

Application of Buckingham's theorem gives

$$f\left(\prod_1\right) = 0$$

which implies from the preceding discussion that $\prod_1 = k$, or

$$F = kv^2 A\rho$$

where k is a dimensionless constant as before. Thus, when a complete set consists of a *single dimensionless product*, as is generally the case when we begin with four variables, the application of Buckingham's theorem yields the desired relationship *up to a constant of proportionality*. Of course, the predicted proportionality must be tested to determine the adequacy of our list of variables. If the list does prove to be adequate, then the constant of proportionality can be determined by experimentation, thereby completely defining the relationship.

For the case $n = 2$, Equation (9.11) in Buckingham's theorem takes the form

$$f\left(\prod_1, \prod_2\right) = 0 \tag{9.13}$$

If we choose the products in the complete set $\{\prod_1, \prod_2\}$ so that the dependent variable appears in only one of them, for example, $\prod_2$, we can proceed under the assumption that Equation (9.13) can be solved for that chosen product $\prod_2$ in terms of the remaining product $\prod_1$. Such a solution takes the form

$$\prod_2 = H\left(\prod_1\right)$$

and then this latter equation can be solved for the dependent variable. Note that when a complete set consists of more than one dimensionless product, the application of Buckingham's theorem determines the desired relationship *up to an arbitrary function*. After verifying the adequacy of the list of variables, we may be lucky enough to recognize the underlying functional relationship. However, in general we can expect to construct an empirical model, although the task has been eased considerably.

For the general case of n dimensionless products in the complete set for Buckingham's theorem, we again choose the products in the complete set $\{\prod_1, \prod_2, \ldots, \prod_n\}$ so that the dependent variable appears in only one of them, say $\prod_n$ for definiteness. Assuming we can solve Equation (9.11) for that product $\prod_n$ in terms of the remaining ones, we have the form

$$\prod_n = H\left(\prod_1, \prod_2, \ldots, \prod_{n-1}\right)$$

We then solve this last equation for the dependent variable.

Summary of Dimensional Analysis Methodology

STEP 1 Decide which variables enter the problem under investigation.

STEP 2 Determine a complete set of dimensionless products $\{\prod_1, \prod_2, \ldots, \prod_n\}$ among the variables. Make sure the dependent variable of the problem appears in only one of the dimensionless products.

STEP 3 Check to ensure that the products found in the previous step are dimensionless and independent. Otherwise you have an algebra error.

STEP 4 Apply Buckingham's theorem to produce all possible dimensionally homogeneous equations among the variables. This procedure yields an equation of the form (9.11).

STEP 5 Solve the equation in Step 4 for the dependent variable.

STEP 6 Test to ensure that the assumptions made in Step 1 are reasonable. Otherwise the list of variables is faulty.

STEP 7 Conduct the necessary experiments and present the results in a useful format.

Let's illustrate the first five steps of this procedure.

EXAMPLE 1 *Terminal Velocity of a Raindrop*

Consider the problem of determining the terminal velocity v of a raindrop falling from a motionless cloud. We examined this problem from a very simplistic point of view in Chapter 2, but let's take another look using dimensional analysis.

What are the variables influencing the behavior of the raindrop? Certainly the terminal velocity will depend on the size of the raindrop given by, say, its radius r. The density ρ of the air and the viscosity μ of the air will also affect the behavior. (Viscosity measures resistance to motion—a sort of internal molecular friction. In gases this resistance is caused by collisions between fast-moving molecules.) The acceleration due to gravity g is another variable to consider. Although the surface tension of the raindrop is a factor that does influence the behavior of the fall, we will ignore this factor. If necessary, surface tension can be taken into account in a later, refined model. These considerations give the following table relating the selected variables to their dimensions:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

Next we find all the dimensionless products among the variables. Any such product must be of the form

$$v^a r^b g^c \rho^d \mu^e \tag{9.14}$$

and hence must have dimension

$$(LT^{-1})^a (L)^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

Therefore, a product of the form (9.14) is dimensionless if and only if the following system of equations in the exponents is satisfied:

$$\left.\begin{aligned} d + e &= 0 \\ a + b + c - 3d - e &= 0 \\ -a - 2c - e &= 0 \end{aligned}\right\} \tag{9.15}$$

Solution of the system (9.15) gives $b = (3/2)d - (1/2)a$, $c = (1/2)d - (1/2)a$, and $e = -d$, where a and d are arbitrary. One dimensionless product $\prod_1$ is obtained by setting $a = 1$, $d = 0$; another, independent dimensionless product $\prod_2$ is obtained when $a = 0$, $d = 1$. These solutions give

$$\prod_1 = vr^{-1/2}g^{-1/2} \quad \text{and} \quad \prod_2 = r^{3/2}g^{1/2}\rho\mu^{-1}$$

Next, we check the results to ensure that the products are indeed dimensionless:

$$\frac{LT^{-1}}{L^{1/2}(LT^{-2})^{1/2}} = M^0L^0T^0$$

and

$$\frac{L^{3/2}(LT^{-2})^{1/2}(ML^{-3})}{ML^{-1}T^{-1}} = M^0L^0T^0$$

Thus, according to Buckingham's theorem, there is a function f such that

$$f\left(vr^{-1/2}g^{-1/2}, \frac{r^{3/2}g^{1/2}\rho}{\mu}\right) = 0$$

Assuming we can solve this last equation for $vr^{-1/2}g^{-1/2}$ as a function of the second product $\prod_2$, it follows that

$$v = \sqrt{rg}\, h\left(\frac{r^{3/2}g^{1/2}\rho}{\mu}\right)$$

where h is some function of the single product $\prod_2$.

The preceding example illustrates a characteristic feature of dimensional analysis. Normally the modeler studying a given physical system has an intuitive idea of the variables involved and has a working knowledge of general principles and laws (such as Newton's second law) but lacks the precise laws governing the interaction of the variables. Of course, the modeler can always experiment with each independent variable separately, holding the others constant and measuring the effect on the system. Often, however, the efficiency of the experimental work can be improved through an application of dimensional analysis. Although we did not illustrate Steps 6 and 7 of the dimensional analysis process for the preceding example, these steps will be illustrated in Section 9.3.

We now make some observations concerning the dimensional analysis process. Suppose n variables have been identified in the physical problem under investigation. When determining a complete set of dimensionless products, we form a system of three linear algebraic equations by equating the exponents for M, L, and T to zero. That is, we obtain a system of three equations in n unknowns (the exponents). If the three equations are independent, we can solve the system for three of the unknowns in terms of the remaining $n-3$ unknowns (declared to be arbitrary). In this case, we find $n-3$ independent dimensionless products that make up the complete set we seek. For instance, in the preceding example there are five unknowns, a, b, c, d, e, and we determined three of them (b, c, and e) in terms of the remaining $(5-3)$ two arbitrary ones (a and d). Thus, we obtained a complete set of two dimensionless products. When choosing the $n-3$ dimensionless products, we must be sure that the dependent variable appears in only one of them. We can then solve Equation (9.11) guaranteed by Buckingham's theorem for the dependent variable, at least under suitable assumptions on the function f in that equation. (The full story telling when such a solution is possible is the content of an important result in advanced calculus known as the implicit function theorem.)

We acknowledge that we have been rather sketchy in our presentation for solving the system of linear algebraic equations that results in the process of determining all dimensionless products. Recall how to solve simple linear systems by the method of elimination of variables. We conclude this section with another example.

EXAMPLE 2 *Automobile Gas Mileage Revisited*

Consider again the automobile gasoline mileage problem presented in Chapter 2. One of our submodels in that problem was for the force of propulsion F_p. The variables we identified that affect the propulsion force are C_r, the amount of fuel burned per unit time, the amount K of energy contained in each gallon of gasoline, and the speed v. Let's perform a dimensional analysis. The following table relates the various variables to their dimensions:

Variable	F_p	C_r	K	v
Dimension	MLT^{-2}	L^3T^{-1}	$ML^{-1}T^{-2}$	LT^{-1}

Thus, the product

$$F_p^a C_r^b K^c v^d \tag{9.16}$$

must have the dimension

$$(MLT^{-2})^a(L^3T^{-1})^b(ML^{-1}T^{-2})^c(LT^{-1})^d$$

The requirement for a dimensionless product leads to the system

$$\left.\begin{aligned} a \quad\quad + c \quad\quad &= 0 \\ a + 3b - c + d &= 0 \\ -2a - b - 2c - d &= 0 \end{aligned}\right\} \tag{9.17}$$

Solution of the system (9.17) gives $b = -a$, $c = -a$, and $d = a$, where a is arbitrary. Choosing $a = 1$, we obtain the dimensionless product

$$\prod_1 = F_p C_r^{-1} K^{-1} v$$

From Buckingham's theorem there is a function f with $f(\prod_1) = 0$, so $\prod_1$ equals a constant. Therefore,

$$F_p \propto \frac{C_r K}{v}$$

in agreement with the conclusion reached in Chapter 2.

9.2 PROBLEMS

1. Predict the time of revolution for two bodies of mass m_1 and m_2 in empty space revolving about each other under their mutual gravitational attraction.
2. A projectile is fired with initial velocity v at an angle θ with the horizon. Predict the range R.
3. Consider an object that is falling under the influence of gravity. Assume that air resistance is negligible. Using dimensional analysis, find the speed v of the object after it has fallen a distance s. Let $v = f(m, g, s)$, where m is the mass of the object and g is the acceleration due to gravity. Does you answer agree with your knowledge of the physical situation?
4. Using dimensional analysis, find a proportionality relationship for the centrifugal force F of a particle in terms of its mass m, velocity v, and radius r of the curvature of its path.
5. One would like to know the nature of the drag forces experienced by a sphere as it passes through a fluid. It is assumed that the sphere has a low speed. Therefore, the drag force is highly dependent on the viscosity of the fluid. The fluid density is to be neglected. Use the dimensional analysis process to develop a model for drag force F as a function of the radius r and velocity m of the sphere and the viscosity μ of the fluid.
6. The volume flow rate q for laminar flow in a pipe depends on the pipe radius r, the viscosity μ of the fluid, and the pressure drop per unit length dp/dz. Develop a model for the flow rate q as a function of r, μ, and dp/dz.

7. In fluid mechanics, the Reynolds number is a dimensionless number involving the fluid velocity v, density ρ, viscosity μ, and a characteristic length r. Use dimensional analysis to find the Reynolds number.

8. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q in cubic feet per second. Use dimensional analysis to determine an equation for power.

9. Find the volume flow rate dV/dt of blood flowing in an artery as a function of the pressure drop per unit length of artery, the radius r of the artery, the blood density ρ, and the blood viscosity μ.

10. The speed of sound in a gas depends on the pressure and the density. Use dimensional analysis to find the speed of sound in terms of pressure and density.

11. The lift force F on a missile depends on its length r, velocity v, diameter δ, and initial angle θ with the horizon; it also depends on the density ρ, viscosity μ, gravity g, and speed of sound s of the air. Show that

$$F = \rho v^2 r^2 h\left(\frac{\delta}{r}, \theta, \frac{\mu}{\rho v r}, \frac{s}{v}, \frac{rg}{v^2}\right)$$

12. The height h that a fluid will rise in a capillary tube decreases as the diameter D of the tube increases. Use dimensional analysis to determine how h varies with D and the specific weight w and surface tension σ of the liquid.

9.3 A Damped Pendulum

In Section 9.1 we investigated the pendulum problem under the assumptions that the hinge is frictionless, the mass is concentrated at one end of the pendulum, and the drag force is negligible. Suppose we are not satisfied with the results predicted by the constructed model. Then we can refine the model by incorporating drag forces. If F represents the total drag force, the problem now is to determine the function

$$t = f(r, m, g, \theta, F)$$

Let's consider a submodel for the drag force. As we have seen in previous examples, the modeler is usually faced with a trade-off between simplicity and accuracy. For the pendulum it might seem reasonable to expect the drag force to be proportional to some positive power of the velocity. To keep our model simple, we assume that F is proportional to either v or v^2, as depicted in Figure 9.7.

Now we can experiment to determine directly the nature of the drag force. However, we will first perform a dimensional analysis because we expect it to reduce our experimental effort. Assume F is proportional to v so that $F = kv$. For convenience we choose to work with the dimensional constant $k = F/v$, which has dimension MLT^{-2}/LT^{-1}, or simply MT^{-1}. Notice that the dimensional constant captures the assumption about the drag force.

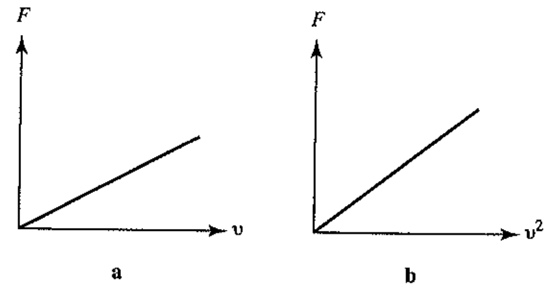


Figure 9.7
Possible submodels for the drag force

Thus, we apply dimensional analysis to the model

$$t = f(r, m, g, \theta, k)$$

An analysis of the dimensions of the variables gives

Variable	t	r	m	g	θ	k
Dimension	T	L	M	LT^{-2}	$M^0L^0T^0$	MT^{-1}

Any product of the variables must be of the form

$$t^a r^b m^c g^d \theta^e k^f \tag{9.18}$$

and hence must have dimension

$$(T)^a (L)^b (M)^c (LT^{-2})^d (M^0L^0T^0)^e (MT^{-1})^f$$

Therefore, a product of the form (9.18) is dimensionless if and only if

$$\left.\begin{aligned} c \qquad\quad + f &= 0 \\ b + d \qquad\quad &= 0 \\ a \quad - 2d - f &= 0 \end{aligned}\right\} \tag{9.19}$$

The equations in the system (9.19) are independent, so we know we can solve for three of the variables in terms of the remaining $(6 - 3)$ three variables. We would like to choose the solutions in such a way that t appears in only one of the dimensionless products. Thus, we choose a, e, and f as the arbitrary variables with

$$c = -f, \; b = -d = \frac{-a}{2} + \frac{f}{2}, \; d = \frac{a}{2} - \frac{f}{2}$$

Setting $a = 1$, $e = 0$, and $f = 0$, we obtain $c = 0$, $b = -1/2$, and $d = 1/2$ with the corresponding dimensionless product $t\sqrt{g/r}$. Similarly, choosing $a = 0$, $e = 1$, and $f = 0$, we get $c = 0$, $b = 0$, and $d = 0$, corresponding to the dimensionless product θ. Finally, choosing $a = 0$, $e = 0$, and $f = 1$, we obtain $c = -1$, $b = 1/2$, and $d = -1/2$, corresponding to the dimensionless product $k\sqrt{r}/m\sqrt{g}$. Notice that t appears in only the