first of these products. From Buckingham's theorem, there is a function h with

$$h\left(t\sqrt{g/r}, \theta, \frac{k\sqrt{r}}{m\sqrt{g}}\right) = 0$$

Assuming we can solve this last equation for $t\sqrt{g/r}$, we obtain

$$t = \sqrt{r/g}\, H\left(\theta, \frac{k\sqrt{r}}{m\sqrt{g}}\right)$$

for some function H of two arguments.

Testing the Model (Step 6)

Given $t = \sqrt{r/g}\, H(\theta, k\sqrt{r}/m\sqrt{g})$, our model predicts that $t_1/t_2 = \sqrt{r_1/r_2}$ if the parameters of the function H (namely, θ and $k\sqrt{r}/m\sqrt{g}$) could be held constant. Now there is no difficulty with keeping θ and k constant. However, varying r while simultaneously keeping $k\sqrt{r}/m\sqrt{g}$ constant is more complicated. Because g is constant, we could try to vary r and m in such a manner that $\sqrt{r}/m$ remains constant. This might be done using a pendulum with a hollow mass to vary m without altering the drag characteristics. Under these conditions we would expect the plot in Figure 9.8.

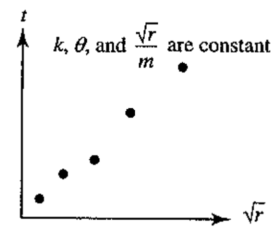


Figure 9.8
A plot of t versus $\sqrt{r}$ keeping the variables k, θ, and $\sqrt{r}/m$ constant

Presenting the Results (Step 7)

As was suggested in predicting the period of the undamped pendulum, we can plot $t\sqrt{g/r} = H(\theta, k\sqrt{r}/m\sqrt{g})$. However, because H is here a function of two arguments, this would yield a three-dimensional figure that is not easy to use. An alternative technique is to plot $t\sqrt{g/r}$ versus $k\sqrt{r}/m\sqrt{g}$ for various values of θ. This is illustrated in Figure 9.9. To be safe

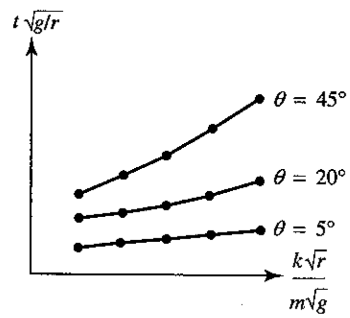


Figure 9.9
Presenting the results

in predicting t over the range of interest for representative values of θ, it would be necessary to conduct sufficient experiments at various values of $k\sqrt{r}/m\sqrt{g}$. Note that once data are collected, various empirical models could be constructed using an appropriate interpolating scheme for each value of θ.

Choosing Among Competing Models

Because dimensional analysis involves only algebra, it is tempting to develop several models under different assumptions before proceeding with costly experimentation. In the case of the pendulum, under different assumptions, we can develop the following three models (see Problem 1 in the Section 9.3 problem set):

A: $t = \sqrt{r/g}\, h(\theta)$ — No drag forces

B: $t = \sqrt{r/g}\, h\left(\theta, \dfrac{k\sqrt{r}}{m\sqrt{g}}\right)$ — Drag forces proportional to v: $F = kv$

C: $t = \sqrt{r/g}\, h\left(\theta, \dfrac{k_1 r}{m}\right)$ — Drag forces proportional to v^2: $F = k_1 v^2$

Because all the preceding models are approximations, it is reasonable to ask which, if any, is suitable in a particular situation. We now describe the experimentation necessary to distinguish among these models, and we present some experimental results.

Model A predicts that when the angle of displacement θ is held constant, the period t is proportional to $\sqrt{r}$. Model B predicts that when θ and $\sqrt{r}/m$ are both held constant, while maintaining the same drag characteristics k, t is proportional to $\sqrt{r}$. Finally, Model C predicts that if θ, r/m, and k_1 are held constant, then t is proportional to $\sqrt{r}$.

The following discussion describes our experimental results for the pendulum.[1] Various types of balls were suspended from a string in such a manner as to minimize the friction at the hinge. The kinds of balls included tennis balls and various types and sizes of plastic balls. A hole was made in each ball to permit variations in the mass without altering appreciably the aerodynamic characteristics of the ball or the location of the center of mass. The models were then compared with one another. In the case of the tennis ball, Model A proved to be superior. The period was independent of the mass, and a plot of t versus $\sqrt{r}$ for constant θ is shown in Figure 9.10.

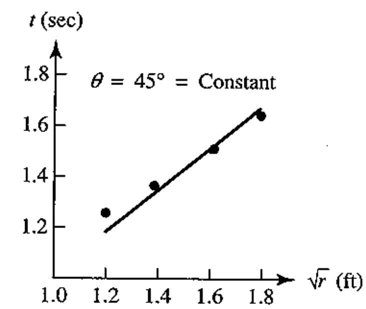
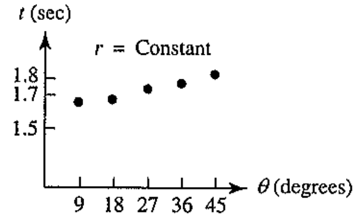


Figure 9.10
Model A for a tennis ball

[1] Data collected by Michael Jaye.

Having decided that $t = \sqrt{r/g}h(\theta)$ is the best of the models for the tennis ball, we isolated the effect of θ by holding r constant to gain insight into the nature of the function h. A plot of t versus θ for constant r is shown in Figure 9.11.

Figure 9.11
Isolating the effect of θ



Note from Figure 9.11 that for small angles of initial displacement θ, the period is virtually independent of θ. However, the displacement effect becomes more noticeable as θ is increased. Thus, for small angles we might hypothesize that $t = c\sqrt{r/g}$ for some constant c. If one plots t versus $\sqrt{r}$ for small angles, the slope of the resulting straight line should be constant.

For larger angles, the experiment demonstrates that the effect of θ needs to be considered. In such cases, one may desire to estimate the period for various angles. For example, if $\theta = 45°$ and we know a particular value of $\sqrt{r}$, we can estimate t from Figure 9.10. Although not shown, plots for several different angles can be graphed in the same figure.

Dimensional Analysis in the Model-Building Process

Let's summarize how dimensional analysis assists in the model-building process. In the determination of a model we must first decide which factors to neglect and which to include. A dimensional analysis provides additional information on how the included factors are related. Moreover, in large problems, we often determine one or more submodels before dealing with the larger problem. For example, in the pendulum problem we had to develop a submodel for drag forces. A dimensional analysis helps us choose among the various submodels.

A dimensional analysis is also useful for obtaining an initial test of the assumptions in the model. For example, suppose we hypothesize that the dependent variable y is some function of five variables, $y = f(x_1, x_2, x_3, x_4, x_5)$. A dimensional analysis in the MLT system in general yields $\prod_1 = h(\prod_2, \prod_3)$, where each $\prod_i$ is a dimensionless product. The model predicts that $\prod_1$ will remain constant if $\prod_2$ and $\prod_3$ are held constant, even though the components of $\prod_2$ and $\prod_3$ may vary. Because there are, in general, an infinite number of ways of choosing $\prod_i$, we should choose those that can be controlled in laboratory experiments. Having determined that $\prod_1 = h(\prod_2, \prod_3)$, we can isolate the effect of $\prod_2$ by holding $\prod_3$ constant and vice versa. This can help explain the functional relationship among the variables. For instance, we say in our example that the period of the pendulum did not depend on the initial displacement for small displacements.

Perhaps the greatest contribution of dimensional analysis is that it reduces the number of experiments required to predict the behavior. If we wanted to conduct experiments to predict values of y for the assumed relationship $y = f(x_1, x_2, x_3, x_4, x_5)$ and it was decided that 5 data points would be necessary over the range of each variable, 5^5, or 3125, experiments would be necessary. Because a two-dimensional chart is required to interpolate conveniently,

y might be plotted against x_1 for five values of x_1, holding x_2, x_3, x_4, x_5 constant. Because x_2, x_3, x_4, and x_5 must vary as well, 5^4, or 625, charts would be necessary. However, after a dimensional analysis yields $\prod_1 = h(\prod_2, \prod_3)$, only 25 data points are required. Moreover, $\prod_1$ can be plotted versus $\prod_2$, for various values of $\prod_3$, on a single chart. Ultimately, the task is far easier after dimensional analysis.

Finally, dimensional analysis helps in presenting the results. It is usually best to present experimental results using those $\prod_i$ that are classical representations within the field of study. For instance, in the field of fluid mechanics there are eight factors that might be significant in a particular situation: velocity v, length r, mass density ρ, viscosity μ, acceleration of gravity g, speed of sound c, surface tension σ, and pressure p. Thus, a dimensional analysis could require as many as five independent dimensionless products. The five generally used are the Reynolds number, Froude number, Mach number, Weber number, and pressure coefficient. These numbers, which are discussed in Section 9.5, are defined as follows:

Reynolds number	$\dfrac{vr\rho}{\mu}$
Froude number	$\dfrac{v^2}{rg}$
Mach number	$\dfrac{v}{c}$
Weber number	$\dfrac{\rho v^2 r}{\sigma}$
Pressure coefficient	$\dfrac{p}{\rho v^2}$

Thus, the application of dimensional analysis becomes quite easy. Depending on which of the eight variables are considered in a particular problem, the following steps are performed.

1. Choose an appropriate subset from the preceding five dimensionless products.
2. Apply Buckingham's theorem.
3. Test the reasonableness of the choice of variables.
4. Conduct the necessary experiments and present the results in a useful format.

We illustrate an application of these steps to a fluid mechanics problem in Section 9.5.

9.3 PROBLEMS

1. For the damped pendulum,
   a. Assume that F is proportional to v^2 and use dimensional analysis to show that $t = \sqrt{r/g}\,h(\theta, rk_1/m)$.
   b. Assume that F is proportional to v^2 and describe an experiment to test the model $t = \sqrt{r/g}\,h(\theta, rk_1/m)$.

2. Under appropriate conditions, all three models for the pendulum imply that t is proportional to $\sqrt{r}$. Explain how the conditions distinguish among the three models by considering how m must vary in each case.

3. Use a model employing a differential equation to predict the period of a simple frictionless pendulum for small initial angles of displacement. (*Hint*: Let $\sin\theta = \theta$.) Under these conditions, what should be the constant of proportionality? Compare your results with those predicted by Model A in the text.[2]

9.4 Examples Illustrating Dimensional Analysis

EXAMPLE 1 *Explosion Analysis*[3]

In excavation and mining operations, it is important to be able to predict the size of a crater resulting from a given explosive such as TNT in some particular soil medium. Direct experimentation is often impossible or too costly. Thus, it is desirable to use small laboratory or field tests and then scale these up in some manner to predict the results for explosions far greater in magnitude.

We may wonder how the modeler determines which variables to include in the initial list. Experience is necessary to intelligently determine which variables can be neglected. Even with experience, however, the task is usually difficult in practice, as this example will illustrate. It also illustrates that the modeler must often change the list of variables to get usable results.

Problem Identification *Predict the crater volume V produced by a spherical explosive located at some depth d in a particular soil medium.*

Assumptions and Model Formulation Initially, let's assume that the craters are geometrically similar (see Chapter 2), where the crater size depends on three variables: the radius r of the crater, the density ρ of the soil, and the mass W of the explosive. These variables are composed of only two primary dimensions, length L and mass M, and a dimensional analysis results in only one dimensionless product (see Problem 1a in the Section 9.4 problem set):

$$\prod_r = r\left(\frac{\rho}{W}\right)^{1/3}$$

According to Buckingham's theorem, $\prod_r$ must equal a constant. Thus, the crater dimensions of radius or depth vary with the cube root of the mass of the explosive. Because the crater

[2] For students who have studied differential equations.

[3] This example is adapted with permission from R. M. Schmidt, "A Centrifuge Cratering Experiment: Development of a Gravity-Scaled Yield Parameter." In *Impact and Explosion Cratering*, edited by D. J. Roddy, R. O. Pepin, and R. B. Merrill (New York: Pergamon, 1977), pp. 1261–1276.

volume is proportional to r^3, it follows that the volume of the crater is proportional to the mass of the explosive for constant soil density. Symbolically, we have

$$V \propto \frac{W}{\rho} \tag{9.20}$$

Experiments have shown that the proportionality (9.20) is satisfactory for small explosions (less than 300 lb of TNT) at zero depth in soils, such as moist alluvium, that have good cohesion. For larger explosions, however, the rule proves unsatisfactory. Other experiments suggest that gravity plays a key role in the explosion process, and because we want to consider extraterrestial craters as well, we need to incorporate gravity as a variable.

If gravity is taken into account, then we assume crater size to be dependent on four variables: crater radius r, density of the soil ρ, gravity g, and charge energy E. Here, the charge energy is the mass W of the explosive times its specific energy. Applying a dimensional analysis to these four variables again leads to a single dimensionless product (see Problem 1b in the 9.4 problem set):

$$\prod_{rg} = r\left(\frac{\rho g}{E}\right)^{1/4}$$

Thus, $\prod_{rg}$ equals a constant and the linear crater dimensions (radius or depth of the crater) vary with the one-fourth root of the energy (or mass) of the explosive for a constant soil density. This leads to the following proportionality known as the quarter-root scaling and is a special case of *gravity scaling*:

$$V \propto \left(\frac{E}{\rho g}\right)^{3/4} \tag{9.21}$$

Experimental evidence indicates that gravity scaling holds for large explosions (more than 100 tons of TNT) where the stresses in the cratering process are much larger than the material strengths of the soil. The proportionality (9.21) predicts that crater volume decreases with increased gravity. The effect of gravity on crater formation is relevant in the study of extraterrestial craters. Gravitational effects can be tested experimentally using a centrifuge to increase gravitational accelerations.[4]

A question of interest to explosion analysts is whether the material properties of the soil do become less important with increased charge size and increased gravity. Let's consider the case in which the soil medium is characterized only by its density ρ. Thus, the crater volume V depends on the explosive, soil density ρ, gravity g, and the depth of burial d of the charge. In addition, the explicit role of material strength or cohesion has been tested and the strength–gravity transition is shown to be a function of charge size and soil strength.

We now describe our explosive in more detail than in previous models. To characterize an explosive, three independent variables are needed: size, energy field, and explosive density δ. The size can be given as charge mass W, as charge energy E, or as the radius α of the spherical explosive. The energy yield can be given as a measure of the specific

[4]See the papers by R. M. Schmidt (1977, 1980) and by Schmidt and Holsapple (1980), cited in Further Reading, which discuss the effects when a centrifuge is used to perform explosive cratering tests under the influence of gravitational acceleration up to 480 G, where 1 G is the terrestrial gravity field strength of 981 cm/sec^2.

energy Q_e or the energy density per unit volume Q_V. The following equations relate the variables:

$$W = \frac{E}{Q_e}$$
$$Q_V = \delta Q_e$$
$$\alpha^3 = \left(\frac{3}{4\pi}\right)\left(\frac{W}{\delta}\right)$$

One choice of these variables leads to the model formulation

$$V = f(W, Q_e, \delta, \rho, g, d)$$

Because there are seven variables under consideration and the MLT system is being used, a dimensional analysis generally will result in four (7 − 3) dimensionless products. The dimensions of the variables are shown in the following table.

Variable	V	W	Q_e	δ	ρ	g	d
Dimension	L^3	M	L^2T^{-2}	ML^{-3}	ML^{-3}	LT^{-2}	L

Any product of the variables must be of the form

$$V^a W^b Q_e^c \delta^e \rho^f g^k d^m \tag{9.22}$$

and hence have dimensions

$$(L^3)^a (M^b)(L^2T^{-2})^c (ML^{-3})^{e+f} (LT^{-2})^k (L)^m$$

Therefore, a product of the form (9.22) is dimensionless if and only if the exponents satisfy the following homogeneous system of equations:

$$\begin{aligned} M:&\quad b + e + f = 0 \\ L:&\quad 3a + 2c - 3e - 3f + k + m = 0 \\ T:&\quad -2c - 2k = 0 \end{aligned}$$

Solution to this system produces

$$b = \frac{k-m}{3} - a, \quad c = -k, \quad e = a - f + \frac{k-m}{3}$$

where a, f, k, and m are arbitrary. By setting one of these arbitrary exponents equal to 1 and the other three equal to 0, in succession, we obtain the following set of dimensionless products:

$$\frac{V\delta}{W}, \quad \left(\frac{g}{Q_e}\right)\left(\frac{W}{\delta}\right)^{1/3}, \quad d\left(\frac{\delta}{W}\right)^{1/3}, \quad \frac{\rho}{\delta}$$

(Convince yourself that these are dimensionless.) Because the dimensions of ρ and δ are equal, we can rewrite these dimensionless products as follows:

$$\prod_1 = \frac{V\rho}{W}$$

$$\prod_2 = \left(\frac{g}{Q_e}\right)\left(\frac{W}{\delta}\right)^{1/3}$$

$$\prod_3 = d\left(\frac{\rho}{W}\right)^{1/3}$$

$$\prod_4 = \frac{\rho}{\delta}$$

so $\prod_1$ is consistent with the dimensionless product implied by Equation (9.20). Then, applying Buckingham's theorem, we obtain the model

$$h\left(\prod_1, \prod_2, \prod_3, \prod_4\right) = 0 \tag{9.23}$$

or

$$V = \frac{W}{\rho} H\left(\frac{gW^{1/3}}{Q_e\delta^{1/3}}, \frac{d\delta^{1/3}}{W^{1/3}}, \frac{\rho}{\delta}\right)$$

Presenting the Results For oil-base clay, the value of ρ is approximately 1.53 g/cm^2; for wet sand, 1.65; and for desert alluvium, 1.60. For TNT, δ has the value 2.23 g/cm^3. Thus, $0.69 < \prod_4 < 0.74$, so for simplicity we can assume that for these soils and TNT, $\prod_4$ is constant. Then, Equation (9.23) becomes

$$h\left(\prod_1, \prod_2, \prod_3\right) = 0 \tag{9.24}$$

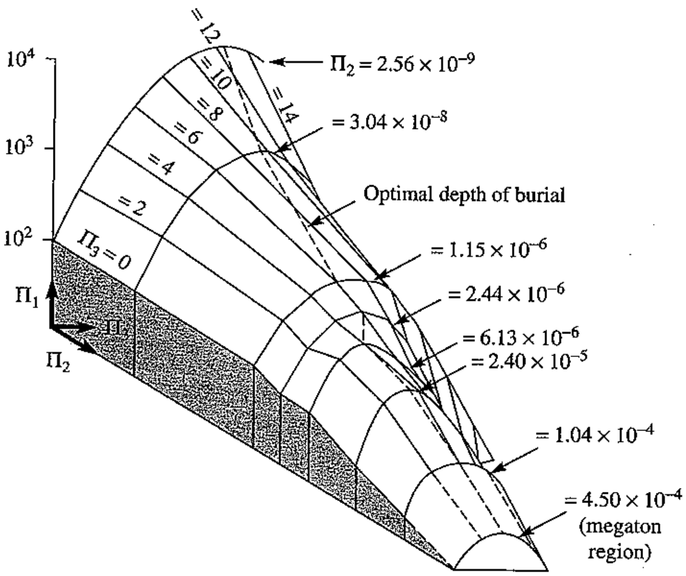


Figure 9.12
A plot of the surface $h(\prod_1, \prod_2, \prod_3) = 0$, showing the crater volume parameter $\prod_1$ as a function of gravity-scaled yield $\prod_2$ and depth of burial parameter $\prod_3$ (reprinted by permission of R. M. Schmidt)

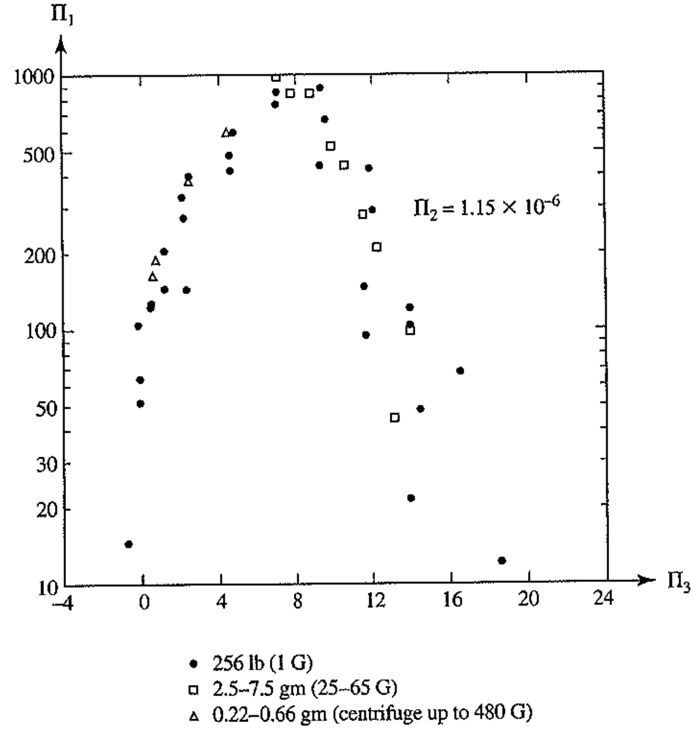


Figure 9.13
Data values for a cross section of the surface depicted in Figure 9.12 (data reprinted by permission from R. M. Schmidt)

R. M. Schmidt gathered experimental data to plot the surface described by Equation (9.24). A plot of the surface is depicted in Figure 9.12, showing the crater and volume parameter $\prod_1$ as a function of the scaled energy charge $\prod_2$ and the depth of the burial parameter $\prod_3$. Cross-sectional data for the surface parallel to the $\prod_1 \prod_3$ plane when $\prod_2 = 1.15 \times 10^{-6}$ are depicted in Figure 9.13.

Experiments have shown that the physical effect of increasing gravity is to reduce crater volume for a given charge yield. This result suggests that increased gravity can be compensated for by increasing the size of the charge to maintain the same cratering efficiency. Note also that Figures 9.12 and 9.13 can be used for prediction once an empirical interpolating model is constructed from the data. Holsapple and Schmidt (1982) extend these methods to impact cratering, and Housen, Schmidt, and Holsapple (1983) extend them to crater ejecta scaling.

EXAMPLE 2 *How Long Should You Roast a Turkey?*

One general rule for roasting a turkey is the following: Set the oven to 400 °F and allow 20 min per pound for cooking. How good is this rule?

Assumptions Let t denote the cooking time for the turkey. Now, on what variables does t depend? Certainly the size of the turkey is a factor that must be considered. Let's assume that the turkeys are geometrically similar and use l to denote some characteristic dimension of the uncooked meat; specifically, we assume that l represents the length of the turkey. Another factor is the difference between the temperature of the raw meat and the oven ΔT_m. (We know from experience that it takes longer to cook a bird that is nearly frozen than it does to cook one that is initially at room temperature.) Because the turkey will have to reach a certain interior temperature before it is considered fully cooked, the difference ΔT_c between the temperature of the cooked meat and the oven is a variable determining the cooking time. Finally, we know that different foods require different cooking times independent of size; it takes only 10 min or so to bake a pan of cookies, whereas a roast beef or turkey requires several hours. A measure of the factor representing the differences between foods is the *coefficient of heat conduction* for a particular uncooked food. Let k denote the coefficient of heat conduction for a turkey. Thus, we have the following model formulation for the cooking time:

$$t = f(\Delta T_m, \Delta T_c, k, l)$$

Dimensional Analysis Consider the dimensions of the independent variables. The temperature variables ΔT_m and ΔT_c measure the energy per volume and therefore have the dimension ML^2T^{-2}/L^3, or simply $ML^{-1}T^{-2}$. Now, what about the heat conduction variable k? **Thermal conductivity** k is defined as the amount of energy crossing one unit cross-sectional area per second divided by the gradient perpendicular to the area. That is,

$$k = \frac{\text{energy}/(\text{area} \times \text{time})}{\text{temperature}/\text{length}}$$

Accordingly, the dimension of k is $(ML^2T^{-2})(L^{-2}T^{-1})/(ML^{-1}T^{-2})(L^{-1})$, or simply L^2T^{-1}. Our analysis gives the following table:

Variable	ΔT_m	ΔT_c	k	l	t
Dimension	$ML^{-1}T^{-2}$	$ML^{-1}T^{-2}$	L^2T^{-1}	L	T

Any product of the variables must be of the form

$$\Delta T_m^a \Delta T_c^b k^c l^d t^e \tag{9.25}$$

and hence have dimension

$$(ML^{-1}T^{-2})^a(ML^{-1}T^{-2})^b(L^2T^{-1})^c(L)^d(T)^e$$

Therefore, a product of the form (9.25) is dimensionless if and only if the exponents satisfy

$$\begin{aligned} M:&\quad a + b &&= 0\\ L:&\quad -a - b + 2c + d &&= 0\\ T:&\quad -2a - 2b - c + e &&= 0 \end{aligned}$$

Solution of this system of equations gives

$$a = -b, \quad c = e, \quad d = -2e$$

where b and e are arbitrary constants. If we set $b = 1$, $e = 0$, we obtain $a = -1$, $c = 0$, and $d = 0$; likewise, $b = 0$, $e = 1$ produces $a = 0$, $c = 1$, and $d = -2$. These independent solutions yield the complete set of dimensionless products:

$$\prod_1 = \Delta T_m^{-1} \Delta T_c \quad \text{and} \quad \prod_2 = kl^{-2}t$$

From Buckingham's theorem, we obtain

$$h\left(\prod_1, \prod_2\right) = 0$$

or

$$t = \left(\frac{l^2}{k}\right) H\left(\frac{\Delta T_c}{\Delta T_m}\right) \tag{9.26}$$

The rule stated in our opening remarks gives the roasting time for the turkey in terms of its weight w. Let's assume the turkeys are geometrically similar, or $V \propto l^3$. If we assume the turkey is of constant density (which is not quite correct because the bones and flesh differ in density), then, because weight is density times volume and volume is proportional to l^3, we get $w \propto l^3$. Moreover, if we set the oven to a constant baking temperature and specify that the turkey must initially be near room temperature (65 °F), then $\Delta T_c / \Delta T_m$ is a dimensionless constant. Combining these results with Equation (9.26), we get the proportionality

$$t \propto w^{2/3} \tag{9.27}$$

because k is constant for turkeys. Thus, the required cooking time is proportional to weight raised to the two-thirds power. Therefore, if t_1 hours are required to cook a turkey weighing w_1 pounds and t_2 is the time for a weight of w_2 pounds,

$$\frac{t_1}{t_2} = \left(\frac{w_1}{w_2}\right)^{2/3}$$

it follows that a doubling of the weight of a turkey increases the cooking time by the factor $2^{2/3} \approx 1.59$.

How does our result (9.27) compare to the rule stated previously? Assume that ΔT_m, ΔT_c, and k are independent of the length or weight of the turkey, and consider cooking a 23-lb turkey versus an 8-lb bird. According to our rule, the ratio of cooking times is given by

$$\frac{t_1}{t_2} = \left(\frac{20 \cdot 23}{20 \cdot 8}\right) = 2.875$$

On the other hand, from dimensionless analysis and Equation (9.27),

$$\frac{t_1}{t_2} = \left(\frac{23}{8}\right)^{2/3} \approx 2.02$$

Thus, the rule predicts it will take nearly three times as long to cook a 23-lb bird as it will to cook an 8-lb turkey. Dimensional analysis predicts it will take only twice as long. Which rule is correct? Why have so many cooks overcooked a turkey?

Testing the Results Suppose that turkeys of various sizes are cooked in an oven preheated to 325 °F. The initial temperature of the turkeys is 65 °F. All the turkeys are removed from the oven when their internal temperature, measured by a meat thermometer, reaches 195 °F. The (hypothetical) cooking times for the various turkeys are recorded as shown in the following table.

w (lb)	5	10	15	20
t (hr)	2	3.4	4.5	5.4

A plot of t versus $w^{2/3}$ is shown in Figure 9.14. Because the graph approximates a straight line through the origin, we conclude that $t \propto w^{2/3}$, as predicted by our model.

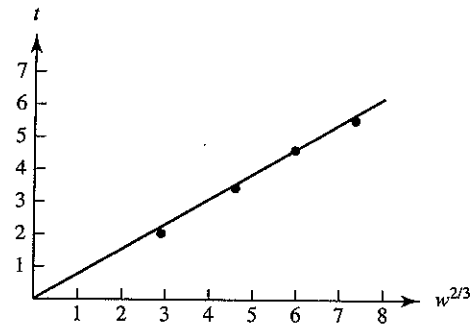


Figure 9.14
Plot of cooking times versus weight to the two-thirds power reveals the predicted proportionality.

9.4 PROBLEMS

1. **a.** Use dimensional analysis to establish the cube-root law

$$r\left(\frac{\rho}{W}\right)^{1/3} = \text{constant}$$

for scaling of explosions, where r is the radius or depth of the crater, ρ is the density of the soil medium, and W the mass of the explosive.

b. Use dimensional analysis to establish the one-fourth-root law

$$r\left(\frac{\rho g}{E}\right)^{1/4} = \text{constant}$$

for scaling of explosions, where r is the radius or depth of the crater, ρ is the density of the soil medium, g is gravity, and E is the charge energy of the explosive.

2. **a.** Show that the products $\prod_1, \prod_2, \prod_3, \prod_4$ for the refined explosion model presented in the text are dimensionless products.

b. Assume ρ is essentially constant for the soil being used and restrict the explosive to a specific type, say TNT. Under these conditions, ρ/δ is essentially constant, yielding $\prod_1 = f(\prod_2, \prod_3)$. You have collected the following data with $\prod_2 = 1.5 \times 10^{-6}$.

$\prod_3$	0	2	4	6	8	10	12	14
$\prod_1$	15	150	425	750	825	425	250	90

i. Construct a scatterplot of $\prod_1$ versus $\prod_3$. Does a trend exist?

ii. How accurate do you think the data are? Find an empirical model that captures the *trend* of the data with accuracy commensurate with your appraisal of the accuracy of the data.

iii. Use your empirical model to predict the volume of a crater using TNT in desert alluvium with (CGS system) $W = 1500$ g, $\rho = 1.53$ g/cm^3, and $\prod_3 = 12.5$.

3. Consider a zero-depth burst, spherical explosive in a soil medium. Assume the value of the crater volume V depends on the explosive size, energy yield, and explosive energy, as well as on the strength Y of the soil (considered a resistance to pressure with dimensions $ML^{-1}T^{-2}$), soil density ρ, and gravity g. In this problem, assume

$$V = f(W, Q_e, \delta, Y, \rho, g)$$

and use dimensional analysis to produce the following *mass set* of dimensionless products.

$$\prod_1 = \frac{V\rho}{W} \qquad \prod_2 = \left(\frac{g}{Q_e}\right)\left(\frac{W}{\delta}\right)^{1/3}$$

$$\prod_3 = \frac{Y}{\delta Q_e} \qquad \prod_4 = \frac{\rho}{\delta}$$

4. For the explosion process and material characteristics discussed in Problem 3, consider

$$V = f(E, Q_v, \delta, Y, \rho, g)$$

and use dimensional analysis to produce the following *energy set* of dimensionless products.

$$\overline{\prod}_1 = \frac{VQ_v}{E} \qquad \overline{\prod}_2 = \frac{\rho g E^{1/3}}{Q_v^{4/3}}$$

$$\overline{\prod}_3 = \frac{Y}{Q_v} \qquad \overline{\prod}_4 = \frac{\rho}{\delta}$$

5. Repeat Problem 4 for

$$V = f(E, Q_e, \delta, Y, \rho, g)$$

and use dimensional analysis to produce the following *gravity set* of dimensionless products.

$$\overline{\prod}_1 = V\left(\frac{\rho g}{E}\right)^{3/4} \qquad \overline{\prod}_2 = \left(\frac{1}{Q_e}\right)\left(\frac{g^3 E}{\delta}\right)^{1/4}$$

$$\overline{\prod}_3 = \frac{Y}{\delta Q_e} \qquad \overline{\prod}_4 = \frac{\rho}{\delta}$$

6. An experiment consists of dropping spheres into a tank of heavy oil and measuring the times of descent. It is desired that a relationship for the time of descent be determined and verified by experimentation. Assume the time of descent is a function of mass m, gravity g, radius r, viscosity μ, and distance traveled d. Neglect fluid density. That is,

$$t = f(m, g, r, \mu, d)$$

   a. Use dimensional analysis to find a relationship for the time of descent.

   b. How will the spheres be chosen to verify that the time of descent relationship is independent of fluid density? Assuming you have verified the assumptions on fluid density, describe how you would determine the nature of your function experimentally.

   c. Using differential equations techniques, find the velocity of the sphere as a function of time, radius, mass, viscosity, gravity, and fluid density. Using this result and that found in part (a), predict under what conditions fluid density may be neglected. (*Hints:* Use the results of Problem 5 in Section 9.2 as a submodel for drag force. Consider the buoyant force.)[5]

7. A windmill is rotated by air flow to produce power to pump water. It is desired to find the power output P of the windmill. Assume that P is a function of the density of the air ρ, viscosity of the air μ, diameter of the windmill d, wind speed v, and the rotational speed of the windmill ω (measured in radians per second). Thus,

$$P = f(\rho, \mu, d, v, \omega)$$

   a. Using dimensional analysis, find a relationship for P. Be sure to check your products to make sure that they are dimensionless.

   b. Do your results make common sense? Explain.

   c. Discuss how you would design an experiment to determine the nature of your function.

8. For a sphere traveling through a liquid, assume that the drag force F_D is a function of the fluid density ρ, fluid viscosity μ, radius of the sphere r, and speed of the sphere v. Use dimensional analysis to find a relationship for the drag force

$$F_D = f(\rho, \mu, r, v)$$

   Make sure you provide some justification that the given independent variables influence the drag force.

9.4 PROJECT

1. Complete the requirements for the module, "Listening to the Earth: Controlled Source Seismology," by Richard G. Montgomery, UMAP 292-293. This module develops the elementary theory of wave reflection and refraction and applies it to a model of the earth's subsurface. The model shows how information on layer depth and sound velocity may

[5]For students who have studied differential equations.

be obtained to provide data on width, density, and composition of the subsurface. This module is a good introduction to controlled seismic methods and requires no previous knowledge of either physics or geology.

9.4 Further Reading

Holsapple, K. A., & R. M. Schmidt. "A Material-Strength Model for Apparent Crater Volume." *Proc. Lunar Planet Sci. Conf.* 10 (1979): 2757–2777.

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Schmidt, R. M. "A Centrifuge Cratering Experiment: Development of a Gravity-Scaled Yield Parameter." In *Impact and Explosion Cratering*, edited by D. J. Roddy et al., pp. 1261–1278. New York: Pergamon, 1977.

Schmidt, R. M. "Meteor Crater: Energy of Formation—Implications of Centrifuge Scaling." *Proc. Lunar Planet Sci. Conf.* 11 (1980): 2099–2128.

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9.5 Similitude

Suppose we are interested in the effects of wave action on a large ship at sea, heat loss of a submarine and the drag force it experiences in its underwater environment, or the wind effects on an aircraft wing. Quite often, because it is physically impossible to duplicate the actual phenomenon in the laboratory, we study a scaled-down model in a simulated environment to predict accurately the performance of the physical system. The actual physical system for which the predictions are to be made is called the **prototype**. How do we scale experiments in the laboratory to ensure that the effects observed for the model will be the same effects experienced by the prototype?

Although extreme care must be exercised in using simulations, the dimensional products resulting from dimensional analysis of the problem can provide insight into how the scaling for a model should be done. The idea comes from Buckingham's theorem. If the physical system can be described by a dimensionally homogeneous equation in the variables, then it can be put into the form

$$f\left(\prod_1, \prod_2, \ldots, \prod_n\right) = 0$$

for a complete set of dimensionless products. Assume that the independent variable of the problem appears only in the product $\prod_n$ and that

$$\prod_n = H\left(\prod_1, \prod_2, \ldots, \prod_{n-1}\right)$$

For the solution to the model and the prototype to be the same, it is sufficient that the value of all independent dimensionless products $\prod_1, \prod_2, \ldots, \prod_{n-1}$ be the same for the model and the prototype.

For example, suppose the Reynolds number $vr\rho/\mu$ appears as one of the dimensionless products in a fluid mechanics problem, where v represents fluid velocity, r a characteristic dimension (such as the diameter of a sphere or the length of a ship), ρ the fluid density, and μ the fluid viscosity. These values refer to the prototype. Next, let v_m, r_m, ρ_m, and μ_m denote the corresponding values for the scaled-down model. For the effects on the model and the prototype to be the same, we want the two Reynolds numbers to agree so that

$$\frac{v_m r_m \rho_m}{\mu_m} = \frac{vr\rho}{\mu}$$

The last equation is referred to as a *design condition* to be satisfied by the model. If the length of the prototype is too large for the laboratory experiments so that we have to scale down the length of the mode, say $r_m = r/10$, then the same Reynolds number for the model and the prototype can be achieved by using the same fluid ($\rho_m = \rho$ and $\mu_m = \mu$) and varying the velocity, $v_m = 10v$. If it is impractical to scale the velocity by the factor of 10, we can instead scale it by a lesser amount $0 < k < 10$ and use a different fluid so that the equation

$$\frac{k\rho_m}{10\mu_m} = \frac{\rho}{\mu}$$

is satisfied. We do need to be careful in generalizing the results from the scaled-down model to the prototype. Certain factors (such as surface tension) that may be negligible for the prototype may become significant for the model. Such factors would have to be taken into account before making any predictions for the prototype.

EXAMPLE 1 *Drag Force on a Submarine*

We are interested in the drag forces experienced by a submarine to be used for deep-sea oceanographic explorations. We assume that the variables affecting the drag D are fluid velocity v, characteristic dimension r (here, the length of the submarine), fluid density ρ, the fluid viscosity μ, and the velocity of sound in the fluid c. We wish to predict the drag force by studying a model of the prototype. How will we scale the experiments for the model?

A major stumbling block in our problem is in describing shape factors related to the physical object being modeled—in this case, the submarine. Let's consider submarines that are ellipsoidal in shape. In two dimensions, if a is the length of the major axis and b is the length of the minor axis of an ellipse, we can define $r_1 = a/b$ and assign a characteristic dimension such as r, the length of the submarine. In three dimensions, define also $r_2 = a/b'$, where a is the original major axis and b' is the second minor axis. Then r, r_1, and r_2 describe the shape of the submarine. In a more irregularly shaped object, additional shape factors would be required. The basic idea is that the object can be described using a characteristic dimension and an appropriate collection of shape factors. In the case of our three-dimensional ellipsoidal submarine, the shape factors r_1 and r_2 are needed. These shape factors are dimensionless constants.

Returning to our list of six fluid mechanics variables D, v, r, ρ, μ, and c, notice that we are neglecting surface tension (because it is small) and that gravity is not being considered. Thus, it is expected that a dimensionless analysis will produce three $(6 - 3)$ independent dimensionless products. We can choose the following three products for convenience:

Reynolds number	$R = \dfrac{vr\rho}{\mu}$
Mach number	$M = \dfrac{v}{c}$
Pressure coefficient	$P = \dfrac{p}{\rho v^2}$

The added shape factors are dimensionless so that Buckingham's theorem gives the equation

$$h(P, M, R, r_1, r_2) = 0$$

Assuming that we can solve for P yields

$$P = H(M, R, r_1, r_2)$$

Substituting $P = p/\rho v^2$ and solving for p gives

$$p = \rho v^2 H(R, M, r_1, r_2)$$

Remembering that the total drag force is the pressure (force per unit area) times the area (which is proportional to r^2 for geometrically similar objects) and gives the proportionality $D \propto pr^2$, or

$$D = kpv^2r^2H(R, M, r_1, r_2) \tag{9.28}$$

Now a similar equation must hold to give the proportionality for the model

$$D_m = kp_m v_m^2 r_m^2 H(R_m, M_m, r_{1m}, r_{2m}) \tag{9.29}$$

Because the prototype and model equations refer to the same physical system, both equations are identical in form. Therefore, the design conditions for the model require that

Condition (a)	$R_m = R$
Condition (b)	$M_m = M$
Condition (c)	$r_{1m} = r_1$
Condition (d)	$r_{2m} = r_2$

Note that if conditions (a)–(d) are satisfied, then Equations (9.28) and (9.29) give

$$\frac{D_m}{D} = \frac{\rho_m v_m^2 r_m^2}{\rho v^2 r^2} \tag{9.30}$$

Thus, D can be computed once D_m is measured. Note that the design conditions (c) and (d) imply geometric similarity between the model and the prototype submarine

$$\frac{a_m}{b_m} = \frac{a}{b} \quad \text{and} \quad \frac{a_m}{b'_m} = \frac{a}{b'}$$

If the velocities are small compared to the speed of sound in a fluid, then v/c can be considered constant in accordance with condition (b). If the same fluid is used for both the model and prototype, then condition (a) is satisfied if

$$v_m r_m = vr$$

or

$$\frac{v_n}{v} = \frac{r}{r_m}$$

which states that the velocity of the model must increase inversely as the scaling factor r_m/r. Under these conditions, Equation (9.30) yields

$$\frac{D_m}{D} = \frac{\rho_m v_m^2 r_m^2}{\rho v^2 r^2} = 1$$

If increasing the velocity of the scaled model proves unsatisfactory in the laboratory, then a different fluid may be considered for the scaled model ($\rho_m \neq \rho$ and $\mu_m \neq \mu$). If the ratio v/c is small enough to neglect, then both v_m and r_m can be varied to ensure that

$$\frac{v_m r_m \rho_m}{\mu_m} = \frac{vr\rho}{\mu}$$

in accordance with condition (a). Having chosen values that satisfy design condition (a), and knowing the drag on the scaled model, we can use Equation (9.30) to compute the drag on the prototype. Consider the additional difficulties if the velocities are sufficiently great that we must satisfy condition (b) as well.

A few comments are in order. One distinction between the Reynolds number and the other four numbers in fluid mechanics is that the Reynolds number contains the viscosity of the fluid. Dimensionally, the Reynolds number is proportional to the ratio of the inertia forces of an element of fluid to the viscous force acting on the fluid. In certain problems the numerical value of the Reynolds number may be significant. For example, the flow of a fluid in a pipe is virtually always parallel to the edges of the pipe (giving *laminar flow*) if the Reynolds number is less than 2000. Reynolds numbers in excess of 3000 almost always indicate turbulent flow. Normally, there is a critical Reynolds number between 2000 and 3000 at which the flow becomes turbulent.

The design condition (a) mentioned earlier requires the Reynolds number of the model and the prototype to be the same. This requirement precludes the possibility of laminar flow in the prototype being represented by turbulent flow in the model, and vice versa. The equality of the Reynolds number for a model and prototype is important in all problems in which viscosity plays a significant role.

The Mach number is the ratio of fluid velocity to the speed of sound in the fluid. It is generally important for problems involving objects moving with high speed in fluids, such as projectiles, high-speed aircraft, rockets, and submarines. Physically, if the Mach number is the same in model and prototype, the effect of the compressibility force in the fluid relative to the inertia force will be the same for model and prototype. This is the situation that is required by condition (b) in our example on the submarine.

9.5 PROBLEMS

1. A model of an airplane wing is tested in a wind tunnel. The model wing has an 18-in. chord, and the prototype has a 4-ft chord moving at 250 mph. Assuming the air in the wind tunnel is at atmospheric pressure, at what velocity should wind tunnel tests be conducted so that the Reynolds number of the model is the same as that of the prototype?

2. Two smooth balls of equal weight but different diameters are dropped from an airplane. The ratio of their diameters is 5. Neglecting compressibility (assume constant Mach number), what is the ratio of the terminal velocities of the balls? Are the flows completely similar?

3. Consider predicting the pressure drop Δp between two points along a smooth horizontal pipe under the condition of steady laminar flow. Assume

$$\Delta p = f(s, d, \rho, \mu, v)$$

where s is the control distance between two points in the pipe, d is the diameter of the pipe, ρ is the fluid density, μ is the fluid viscosity, and v is the velocity of the fluid.

   a. Determine the design conditions for a scaled model of the prototype.

   b. Must the model be geometrically similar to the prototype?

   c. May the same fluid be used for model and prototype?

   d. Show that if the same fluid is used for both model and prototype, then the equation is

$$\Delta p = \frac{\Delta p_m}{n^2}$$

   where $n = d/d_m$.

4. It is desired to study the velocity v of a fluid flowing in a smooth open channel. Assume that

$$v = f(r, \rho, \mu, \sigma, g)$$

where r is the characteristic length of the channel cross-sectional area divided by the wetted perimeter, ρ is the fluid density, μ is the fluid viscosity, σ is the surface tension, and g is the acceleration of gravity.

   a. Describe the appropriate pair of shape factors r_1 and r_2.

   b. Show that

$$\frac{v^2}{gr} = H\left(\frac{\rho v r}{\mu}, \frac{\rho v^2 r}{\sigma}, r_1, r_2\right)$$

   Discuss the design conditions required of the model.