Hamiltonian Systems in the plane.

These notes are not necessarily in the same order that I presented them in class.

Given a continuously differentiable function $H(p,q)$, we define equations of motion by

$$\frac{dp}{dt} = \frac{\partial H}{\partial q}$$
$$\frac{dq}{dt} = -\frac{\partial H}{\partial p} \qquad (1)$$

with this definition, using the chain rule, we find that along a solution trajectory

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\frac{dp}{dt} + \frac{\partial H}{\partial q}\frac{dq}{dt}$$ substitute using (1).

$$= \frac{\partial H}{\partial p}\cdot\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\left(-\frac{\partial H}{\partial p}\right)$$

$$= 0$$

Thus H is conserved along trajectories, or we say H is constant along trajectories. $H(p,q)$ is called the <u>Hamiltonian</u> of the system.

In some cases, $H(p,q)$ can be associated with total energy. In these situations, we say that ~~the~~ equation (1) <u>conserves energy</u>.

Ex $\theta'' + \sin\theta = 0$

$$\Rightarrow \theta' = \psi$$
$$\psi' = -\sin\theta$$

by dividing these two equations and using the Chain rule, we obtain

$$\frac{d\theta}{d\psi} = \frac{\psi}{-\sin\theta}.$$

This is a separable equation which we can integrate.

$$\int -\sin\theta \, d\theta = \int \psi \, d\psi$$

We will call the constant of integration $H(\theta, \psi)$.

$$\Rightarrow H(\theta, \psi) + \cos\theta = \frac{\psi^2}{2}$$

$$\Rightarrow H(\theta, \psi) = \frac{\psi^2}{2} - \cos\theta.$$

Note

$$\frac{dH}{dt} = \frac{\partial H}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial H}{\partial \psi}\frac{d\psi}{dt}$$

$$= \sin\theta\,(\psi) + \psi(-\sin\theta)$$

$$= 0$$

therefore $H(\theta, \psi)$ is a hamiltonian.

we could also use $H(\theta, \psi) = \frac{\psi^2}{2} - \cos\theta + 1$ so that $H(\theta, \psi) \geq 0$.

Using this definition, we find that $H(0,0) = 0$ and $H(\pm\pi, 0) = 1$.

These correspond to the ~~Ex~~ ~~energ~~ energy at the fixed points $(0,0)$ and $(\pm\pi, 0)$.

Phase portraits

Level curves of the function $H(p,q)$ when plotted in the (p,q) plane provide a phase portrait of how solution trajectories behave. Consider the previous example:

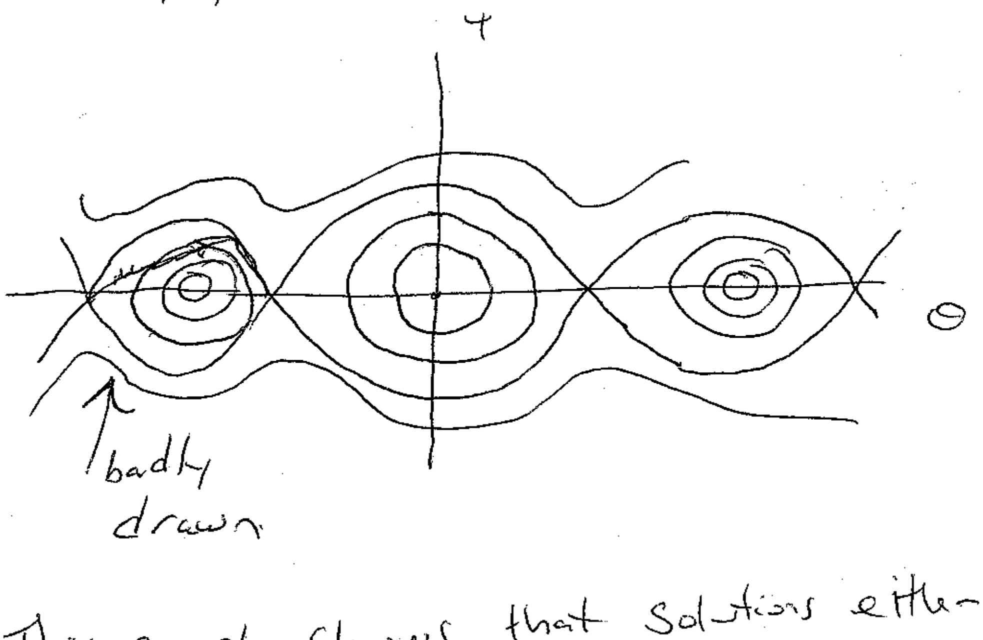
Ex $\theta' = \psi$
$\psi' = -\sin\theta$

$$H(\theta,\psi) = \frac{\psi^2}{2} - \cos\theta + 1.$$

Solve for ψ

$$\Rightarrow \psi = \pm\sqrt{\cos\theta + 2H(\theta,\psi) - 1}$$

Now graph this for different constant values of H, as a function of θ.



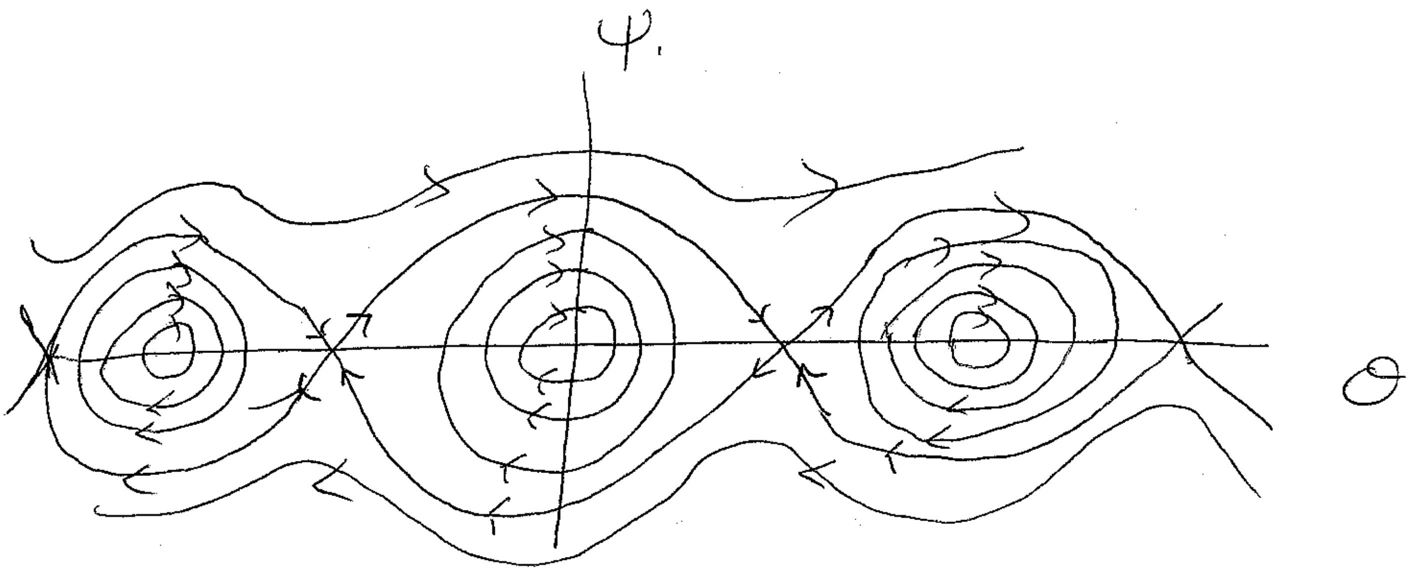
This graph shows that solutions either lie at critical or fixed points, along closed orbits, or along curves that connect distinct fixed points. We need to add the direction or vector field to complete the picture.

The vector field simply provides the instantaneous direction of motion at any point in the plane. For example at the point $(\theta, \psi) = (\pi/2, 0)$ we find

$$\left.\frac{d\theta}{dt}\right|_{(\pi/2,0)} = \left.\psi\right|_{(\pi/2,0)} = 0$$

$$\left.\frac{d\psi}{dt}\right|_{(\pi/2,0)} = \left.-\sin\theta\right|_{(\pi/2,0)} = -1$$

Thus the vector dictating the direction of motion at the point $(\pi/2, 0)$ is $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. This means that there is no instantaneous change in θ at this point and that ψ is decreasing since $-1 < 0$. By checking the vector field at various key places we find the following phase portrait.



General rules about phase portraits:

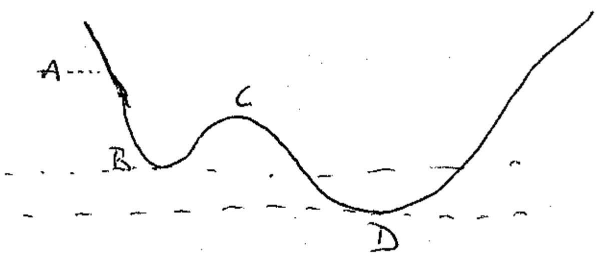
1. Different solution trajectories cannot cross.
2. Solution trajectories may converge to the same fixed point, but this convergence takes an infinite amount of time.
3. Periodic solutions must enclose a fixed point.
4. Nearby trajectories must have the same or similar direction of motion unless, perhaps, a fixed point lies in a neighborhood of them.

Hamiltonian systems ~~to~~ in the plane have stereotypical kinds of solutions which are of the following type.

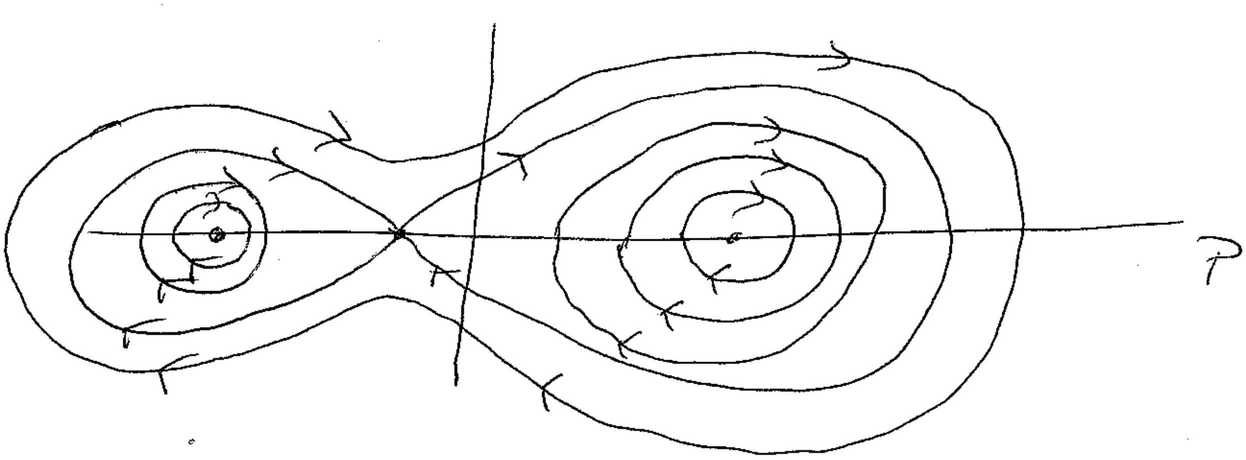
1. Fixed points; called either centers or saddles.
2. Center points are always surrounded by periodic orbits.
3. Saddle points have trajectories that converge as $t \to \pm\infty$ to the same (homoclinic orbit) or different (heteroclinic orbit) saddle points.

Energy Landscape

Imagine a ball rolling without friction along a curve called the "energy landscape". You can think of a particle trapped in a ~~potential~~ potential well.



Depending on the starting location and velocity, the ball will exhibit various dynamics. But energy will be conserved since there is no friction. Thus a ball released at A with 0 velocity will roll down the hill, pass B, up to C, over that hill, pass D and make it to the same height as A, but now on the right side of the landscape. It will then turn around and repeat. If p measures location and $q = p'$ measures velocity, the phase portrait is



This phase portrait has 2 centers corresponding to the local minima at B and D, a saddle point corresponding to the local maxima at C. There are periodic orbits surrounding the centers and homoclinic orbits connecting the saddle to itself. The origin of the (P, θ) plane is arbitrarily chosen. Thus given an energy landscape, we can easily find the phase portrait and vice versa.

In general a local minimum of the energy landscape corresponds to a center of the phase portrait. A local maxima corresponds to a saddle point.