3

Modeling Change One Step at a Time

Mathematics required:
derivatives as limits to differences
Mathematics developed:
solution methods for linear first-order ordinary differential equations; solution to difference equations through iteration

3.1 Introduction

Let us do something simple but practical for a change: compound interest and mortgage payments. Without a strategy, one can get overwhelmed by the complexity very quickly even for this simple problem. We will introduce the *difference equation* as a modeling tool. It allows us to consider changes in time from one step to another, *one step at a time.*

3.2 Compound Interest and Mortgage Payments

Consider three problems of increasing complexity, all involving compound interest.

Your Bank Account

Let $P(t)$ be your account balance at time t. $P(0) = P_0$ is your initial deposit. From then on, it is earning interest at a fixed *interest rate* r. The quantity r has the dimension of $(\text{time})^{-1}$. For example, $r = 6\%$ per year. Without compounding, the interest you earn in one year is simply $(rP_0 \cdot 1\text{yr})$ and so

$$P(1\text{yr}) = P_0(1 + r \cdot 1\text{yr}).$$

Often bank savings and certificates of deposit carry compound interest. Let Δt be the *compounding interval.* If your balance is compounded monthly, then $\Delta t = 1\text{month} = \frac{1}{12}\text{yr}$. One month after your initial deposit, you earn interest of

$$P_0 \cdot r \cdot \Delta t.$$

That is added to your account, which becomes

$$P_0 \cdot (1 + r\Delta t).$$

This larger amount then earns interest for the next month at the rate r. At the end of two months, your balance becomes

$$P(2\Delta t) = P(\Delta t) \cdot (1 + r\Delta t),$$

and so on.

Since there is no interest compounding within Δt, the equation governing the change of your balance in one Δt step is actually quite simple:

$$\boxed{P(t + \Delta t) = P(t) \cdot (1 + r\Delta t).} \tag{3.1}$$

This is a *difference equation*. It describes how $P(t)$ evolves from one time step to another. We have accomplished our modeling process for this problem of compound interest once we have written down Eq. (3.1). Solving Eq. (3.1) is simply a mathematical exercise.

Solution

Equation (3.1) can be solved either by assuming $P(t) = c\lambda^t$ as before, or by iteration, starting with $t = 0$:

$$\begin{aligned} P(\Delta t) &= P_0 \cdot (1 + r\Delta t), \\ P(2\Delta t) &= P(\Delta t) \cdot (1 + r\Delta t) = P_0 \cdot (1 + r\Delta t)^2, \\ &\vdots \\ P(m\Delta t) &= P_0 \cdot (1 + r\Delta t)^m. \end{aligned}$$

This last equation can be rewritten as

$$\boxed{P(t) = P(0) \cdot (1 + r\Delta t)^{t/\Delta t}.} \tag{3.2}$$

This yields the balance at time t when the interest rate is r and the compounding period is Δt.

Your Mortgage Payments, Monthly Interest Compounding

Suppose you borrow P_0 from the bank to buy a house and agree to pay interest at the fixed rate r compounded monthly. You also agree to pay the bank a fixed amount M monthly. Suppose the loan is for 30 years. After 360 monthly payments you pay off both the original principal and interest completely. What is M, and how much total interest have you paid when it is all over?

The problem is simpler if we consider the change over a single Δt. In one Δt, the principal increases because of the simple interest, but reduces by the monthly payment of M. Thus

$$P(t+\Delta t) = P(t) \cdot (1 + r\Delta t) - M. \tag{3.3}$$

After we solve Eq. (3.3), we set $P(0) = P_0$ and $P(30\,\text{years}) = 0$, and we will get M.

Solution

Let $R = (1 + r\Delta t)$ be the growth factor due to interest accrued during Δt, just before a payment is made. Equation (3.3) is, in terms of R,

$$P(t+\Delta t) = P(t) \cdot R - M.$$

It is solved by iteration:

$$\begin{aligned}
P(\Delta t) &= P_0 \cdot R - M,\\
P(2\Delta t) &= P(\Delta t) \cdot R - M = [P_0 \cdot R - M] \cdot R - M,\\
P(3\Delta t) &= P(2\Delta t) \cdot R - M = P_0 \cdot R^3 - M \cdot [1 + R + R^2],\\
&\vdots\\
P(m\Delta t) &= P_0 \cdot R^m - M \cdot [1 + R + R^2 + \cdots + R^{m-1}].
\end{aligned}$$

The sum inside the square brackets is a geometric series, which can be summed exactly using the following trick. Let

$$\begin{aligned}
S &\equiv 1 + R + R^2 + \cdots + R^{m-1},\\
RS &= R + R^2 + \cdots + R^{m-1} + R^m,\\
S - RS &= 1 - R^m.
\end{aligned}$$

Solving for S, we get

$$S = (1 - R^m)/(1 - R).$$

Therefore,

$$\boxed{P(m\Delta t) = P_0 \cdot R^m - M\frac{1 - R^m}{1 - R}.} \tag{3.4}$$

In a 30-year mortgage, we want

$$P(360\,\text{months}) = 0,$$

$$0 = P_0 \cdot R^{360} - M\frac{1 - R^{360}}{1 - R},$$

and we can solve for the correct monthly payment that will allow you to pay off your loan in 30 years:

$$\boxed{M = P_0 \cdot R^{360} \cdot \frac{(R - 1)}{(R^{360} - 1)}.} \tag{3.5}$$

Your Mortgage Payments, Daily Interest Compounding

Most mortgages carry a daily compounding interest. The repayment is still on a monthly basis. This introduces a complication because the interval over which interest compounds is different from the payment interval. We choose Δt to be the interval between mortgage payments. We will have, one Δt later,

$$P(t + \Delta t) = P(t) \cdot R - M.$$

However, the growth factor R is now due to interest compounded daily over a period of one month. For now we take one year to be 365 days and one month to be 365 days/12. Thus, from Eq. (3.2), for compounding interest without repayment, we have

$$R = \left(1 + r \cdot \frac{1}{365}\text{yr}\right)^{365/12}. \tag{3.6}$$

The solution is still Eq. (3.4) and Eq. (3.5), but with R replaced by Eq. (3.6).

3.3 Some Examples

Example 1

Suppose you borrowed \$100,000 to buy a condominium at 10% annual interest, compounded monthly. What should your monthly payment be if you want to pay off the loan in 30 years?

Solution

We want to find M such that $P(360\,\text{months}) = 0$. This yields from Eq. (3.5):

$$\begin{aligned} M &= P_0 \cdot R^{360} \cdot \frac{(R-1)}{(R^{360}-1)} \\ &= \$100{,}000 \cdot \left(1+\frac{0.1}{12}\right)^{360} \cdot \left(\frac{0.1}{12}\right) \Big/ \left[\left(1+\frac{0.1}{12}\right)^{360} - 1\right] \\ &= \$877.57. \end{aligned}$$

This is the required monthly payment if the \$100,000 original loan is to be paid off in 360 equal monthly installments.

Example 2

What is your monthly mortgage payment if you borrow \$100,000 for 30 years at 10% annual interest, compounded daily?

Solution

We use the same formula as in Example 1, except with

$$R = \left(1+\frac{0.1}{365}\right)^{365/12} = 1.008367.$$

So

$$\begin{aligned} M &= \$100{,}000 \cdot (1.00836)^{360} \cdot (0.00836)/[(1.00836)^{360} - 1] \\ &= \$880.55. \end{aligned}$$

The required monthly mortgage payment is \$881. You pay about \$[illegible] more per month with daily compounding of the money you owe compared with monthly compounding.

3.4 Compounding Continuously

We would like to calculate the account balance for various compounding frequencies n within a year. We have found, from (3.2), that the balance after one year is

$$P(1\text{yr}) = P_0 \cdot \left(1 + r \cdot \left(\frac{1}{n}\text{yr}\right)\right)^n,$$

where n is the number of times within a year that interest is deposited into the account for the purpose of subsequent compounding. Table 3.1 lists the year-end balance for an initial \$1,000 deposit under various compounding frequencies.

Continuous Compounding

If the interest is compounded continuously, $n \to \infty$. We have

$$P(1\text{yr}) = P_0 \cdot \lim_{n\to\infty} \left(1 + r \cdot \left(\frac{1}{n}\text{yr}\right)\right)^n.$$

The symbol e, in honor of the Swiss mathematician Leonhard Euler (1707–1783), is assigned to the limit:

$$\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m = e = 2.71828\ldots.$$

By letting $\frac{1}{m} = r \cdot (\frac{1}{n}\text{yr})$, we can show that

$$\lim_{n\to\infty} \left(1 + r \cdot \left(\frac{1}{n}\text{yr}\right)\right)^n = \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m(r\cdot 1\text{yr})} = \left[\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m\right]^{(r\cdot 1\text{yr})}$$

$$= e^{(r\cdot 1\text{yr})}.$$

So the formula for the balance after one year of continuous compounding is

$$P(1\text{yr}) = P_0 e^{r\cdot(1\text{yr})}.$$

It can be shown easily that at any time t,

$$\boxed{P(t) = P(0)e^{rt}.} \qquad (3.7)$$

It is even simpler than the "discrete compounding" formula we wrote down earlier, as it only depends on r (no n-dependence) (see Table 3.1).

TABLE 3.1
Compounding interest at $r = 6\%$ on an original principal $P_0 = \$1{,}000$

Compounding Frequency	*n*	*Year-End Balance*	*Annual Yield*
Annually	1	\$1,060.00	6%
Semiannually	2	\$1,060.90	6.090%
Quarterly	4	\$1,061.36	6.136%
Monthly	12	\$1,061.68	6.168%
Daily	365	\$1,061.83	6.183%
Instantly	∞	\$1,061.84	6.184%

The "annual yield" is calculated from $(P - P_0)/P_0$ and is listed in the right column.

There is actually very little difference (0.016%) between the interest earned from continuous compounding and that from daily compounding. It amounts to about 1¢ on a principal of \$1,000.

Double My Money: "Rule of 72," or Is It "Rule of 69"?

In financial circles there is a mythical "rule of 72." It says that if you divide the APR, the annual percentage rate (i.e., 100 times r times 1 year), into 72, you will get the number of years it takes to double your money. As is often the case with these "rules" created by nonscientists, there is no specification of the conditions under which the formula is valid. The accuracy of the rule actually depends on the magnitude of r and how it is compounded, as we will see. In any case, the rule is supposed to work this way: if you are earning 6% interest annually, APR = 6. Divide 6 into 72, and you get 12. So it takes 12 years to double your money at a 6% interest rate. Let us see if it is right.

First consider the case where the annual interest rate r is compounded daily (this is the case in most bank savings accounts). Since we have shown that there is very little difference in the yield between daily compounding and continuous compounding, we will use the latter to approximate the former. Starting with a principal of $P(0)$, after t years we have in our savings account

$$P(t) = P(0)e^{rt}.$$

We want to find the $t = \tau$ for which $P(\tau)/P(0) = 2$. Thus

$$2 = e^{r\tau}.$$

Take the natural log on both sides, and since $\ln 2 = 0.693$, we have

$$r\tau = \ln 2 = 0.693,$$

yielding

$$(\tau/\text{yr}) = 69/\text{APR}.$$

It seems that the rule should have been called the "rule of 69," and that it takes 11.5 years to double your money at an APR of 6.

Well, perhaps the financial people who came up with this rule of 72 were not thinking of daily or continuous compounding. Perhaps there was no compounding within each year. To resolve this ambiguity, let's use, instead of APR, the AY, the annual yield (in percent). So, regardless of how often we compound within a year, we always have

$$P(t) = P(0)\left(1 + \frac{AY}{100}\right)^{(t/\text{yr})}.$$

Again we seek the $t = \tau$ for which $P(\tau)/P(0) = 2$:

$$\ln 2 = (\tau/\text{yr})\ln\left(1 + \frac{AY}{100}\right),$$

so

$$(\tau/\text{yr}) = 0.693/\ln\left(1 + \frac{AY}{100}\right).$$

This formula is not so easy to use. We should not forget that most people on Wall Street may not know what ln is. Since an approximate formula will suffice, and we know that $\ln(1 + x) \cong x$ for $|x| \ll 1$, we approximate the above formula by

$$\tau \cong 69/AY \text{ years, provided that } |AY/100| \ll 1.$$

Thus we again obtained the rule of 69, for small annual yields.

Table 3.2 compares various approximations with the exact value of τ, the number of years it takes to double the original investment. Neither approximation is perfect, but the rule of 72 appears to give a better approximation to the exact value for AY between 6 and 8, a more commonly encountered range for interest rates, while the rule of 69 gives a better approximation for AY of 2 or less.

We thus see that the rule of 72 is an ad hoc formula because it is not derived mathematically. The rule of 69, on the other hand, is a mathematically correct (asymptotic) approximation for small AY; the approximation is better the smaller the AY. The rule of 69 is also almost exact for any interest rate as long as the compounding frequency is daily or more often.

TABLE 3.2
Years to double an original investment for various annual yields

AY	*Exact*	*72/AY*	*69/AY*
2	35.0	36	34.5
4	17.7	18	17.3
6	11.9	12	11.5
8	9.0	9.0	8.6
10	7.3	7.2	6.9
12	6.1	6.0	5.8
14	5.3	5.1	4.9
16	4.7	4.5	4.3
18	4.2	4.0	3.8
20	3.8	3.6	3.5
25	3.1	2.9	2.8
30	2.6	2.4	2.3
35	2.3	2.1	2.0
40	2.1	1.8	1.7
45	1.9	1.6	1.5
50	1.7	1.4	1.4
60	1.5	1.2	1.2
70	1.3	1.0	1.0
80	1.2	0.9	0.9
90	1.1	0.8	0.8
100	1.0	0.7	0.7

3.5 Rate of Change

The difference equation (3.1) for compound interest can also be written in the form of a rate of change:

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t). \tag{3.8}$$

Equation (3.8) says that the time rate of change of $P(t)$ is proportional to $P(t)$ itself, and the proportionality constant is r. If the rate of change of $P(t)$ is in addition reduced by withdrawal at the *rate* of $w(t) \equiv W(t)/\Delta t$, we say

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = r\,P(t) - w(t). \tag{3.9}$$

Equation (3.9) is the same as Eq. (3.3); it is just rewritten in a form that emphasizes that it is the rate of change that is being modeled.

Continuous Change

It is now quite easy to go to the continuous limit. We simply take the limit as $\Delta t \to 0$ and recognize that

$$\lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = \frac{d}{dt} P(t).$$

Equation (3.8) reduces to the *differential* equation:

$$\frac{d}{dt} P(t) = r P(t), \tag{3.10}$$

whose solution is (see Appendix A)

$$P(t) = P_0 e^{rt},$$

the same as Eq. (3.7), obtained earlier.

Alternatively, one could view the solution to Eq. (3.10) as the continuous limit of Eq. (3.2) as $\Delta t \to 0$. The limiting process done to arrive at (3.7) then tells us that $P(t) = P_0 e^{rt}$ is the solution to Eq. (3.10) because that equation is the limit of the discrete equation (3.8) whose solution is (3.2). Equation (3.9) becomes, in the continuous limit:

$$\frac{d}{dt} P(t) = r P(t) - w(t).$$

Similar equations are used to model a variety of phenomena. One example that we will discuss later is the harvesting of fish (where $w(t)$ is the harvesting rate and $P(t)$ is the fish population), although the way the fish population "compounds" needs some modification.

3.6 Chaotic Bank Balances

Although we will wait until chapter 7 to discuss the phenomenon of *deterministic chaos*, we cannot resist the temptation to give one example here. Normally we expect our bank account balances to be precisely predictable (to the cent). Here is an example where a precise formula for calculating balances can yield chaotic values.

Most banks do not pay interest on "dormant accounts." In the state of Washington, your bank account is considered "dormant" if there are no deposits or withdrawals for 5 years. One possible reason for this practice is to avoid the astronomical sums that can build up through interest compounding over a long period of time, which can bankrupt the bank if later claimed.

An alternative way to avoid this is to specify a maximum amount, K, that a bank is willing to pay out to any account, and have the interest rate reduced gradually as the account balance approaches K. For continuously compounded interest, a suitable formula for the account balance $P(t)$ at an initial interest rate r is

$$\frac{d}{dt}P(t) = r \cdot P(t) \cdot \left(1 - \frac{P(t)}{K}\right).$$

The depositor and the bank can agree on this formula beforehand. The balance can then be calculated automatically without human intervention.

Now, since most banks do not pay interest continuously, a discrete version is more appropriate. We consider the following, rather reasonable (it seems), discrete formula:

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = r \cdot P(t) \cdot \left(1 - \frac{P(t)}{K}\right),$$

where Δt is the compounding period, taken to be one year for our present problem. It will also be convenient to normalize all $P(t)$ by K. For example, if $K = \$1$ million, then all dollar amounts will be measured in millions of dollars. So, for $p(t) \equiv P(t)/K$, the formula can be rewritten as

$$p(t+\Delta t) = p(t) + r\,\Delta t \cdot p(t) \cdot (1 - p(t)).$$

The bank pays a huge interest of 300% per year compounded yearly (i.e., $r\,\Delta t = 3$). Sensing an opportunity, you sell your house and put the proceeds, \$58,000 ($p(0) = 0.058$), in the bank.

a. Calculate your account balance after 45 years. Since most banking institutions want to be accurate to the penny, keep nine decimal places in your calculator for $p(t)$.
b. You want to be more accurate and keep an extra digit. Do the calculation in (a) again, but this time keep 10 decimal places in your calculator.
c. Suppose you want to withdraw your money after 43 years. If you had a choice, would you prefer to have the more "accurate" way of computing interest (i.e., keep 10 decimal places instead of 9)?

Solution

Table 3.3 lists the values of $P_n \equiv p(n\Delta t)$ for Calculator 1, which keeps 9 digits after the decimal point for $P(t)$ (and hence is accurate to the cent) and for Calculator 2, which keeps 10 digits (and hence is accurate to 0.1 cent). $n = 1$ is one year later, $n = 2$ is two years later, etc.

TABLE 3.3
Balance P_n after

n	P_n (Calculator
0	\$58,000
1	\$221,908
2	\$739,902
3	\$1,317,242
4	\$63,585
5	\$242,211
6	\$792,846.
7	\$1,285,569.
8	\$184,212.
9	\$635,047.
10	\$1,330,333.
11	\$11,970.
12	\$47,452.
13	\$183,053.
14	\$631,688.
15	\$1,329,662.
16	\$14,641.
17	\$57,922.
18	\$221,626.
19	\$739,150.
20	\$1,317,571.
21	\$62,301.
22	\$237,560.

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3.7 Exercises

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TABLE 3.3
Balance P_n after n years

n	P_n (Calculator 1)	P_n (Calculator 2)	n	P_n (Calculator 1)	P_n (Calculator 2)
0	\$58, 000.000	\$58, 000.0000	23	\$780, 938.022	\$780, 094.1514
1	\$221, 908.000	\$221, 908.0000	24	\$1, 294, 159.505	\$1, 294, 735.9504
2	\$739, 902.518	\$739, 902.5186	25	\$152, 091.546	\$149, 920.2578
3	\$1, 317, 242.863	\$1, 317, 242.8633	26	\$538, 970.668	\$532, 252.7801
4	\$63, 585.171	\$63, 585.1704	27	\$1, 284, 414.529	\$1, 279, 132.0546
5	\$242, 211.462	\$242, 211.4599	28	\$188, 496.069	\$207, 991.7790
6	\$792, 846.671	\$792, 846.6656	29	\$647, 391.971	\$702, 185.3756
7	\$1, 285, 569.152	\$1, 285, 569.1569	30	\$1, 332, 218.791	\$1, 329, 548.5972
8	\$184, 212.474	\$184, 212.4560	31	\$4, 454.442	\$15, 095.9718
9	\$635, 047.189	\$635, 047.1371	32	\$17, 758.241	\$59, 700.2221
10	\$1, 330, 333.959	\$1, 330, 333.9493	33	\$70, 086.898	\$228, 108.5388
11	\$11, 970.508	\$11, 970.5472	34	\$265, 611.072	\$756, 333.6387
12	\$47, 452.152	\$47, 452.3067	35	\$850, 796.563	\$1, 309, 212.8357
13	\$183, 053.487	\$183, 054.0625	36	\$1, 231, 621.877	\$94, 736.5953
14	\$631, 688.210	\$631, 689.8806	37	\$375, 810.164	\$352, 021.3137
15	\$1, 329, 662.856	\$1, 329, 663.2066	38	\$1, 079, 540.817	\$1, 036, 328.2389
16	\$14, 641.492	\$14, 640.0974	39	\$821, 938.141	\$923, 384.2993
17	\$57, 922.848	\$57, 917.3922	40	\$1, 261, 005.641	\$1, 135, 621.5046
18	\$221, 626.223	\$221, 606.2958	41	\$273, 616.884	\$673, 577.4132
19	\$739, 150.343	\$739, 097.1321	42	\$869, 868.938	\$1, 333, 190.0580
20	\$1, 317, 571.683	\$1, 317, 594.8163	43	\$1, 209, 459.844	\$573.0397
21	\$62, 301.312	\$62, 210.9653	44	\$449, 460.003	\$2, 291.1736
22	\$237, 560.887	\$237, 233.2485	45	\$1, 191, 797.168	\$9, 148.9459

Calculator 1 is accurate to the cent, while Calculator 2 is accurate to 0.1 cent.

a. An initial deposit of \$58,000 grows to \$1.19 million after 45 years.
b. It is, however, only \$9,149 after 45 years if we keep 10 digits after the decimal point.
c. After 43 years, I would prefer to keep the \$1.209 million using the bank's method. My "more accurate" method gives me only \$573, a tenth of my original deposit!

3.7 Exercises

1. *Mortgage*

a. You borrowed \$200,000 on a 15-year mortgage at a 4.75% annual interest rate compounded daily. What is your monthly mortgage payment?

b. Same as in (a), except that it is a 30-year mortgage at 5.25%. What is your monthly payment? (Typically, the interest rate for a 30-year mortgage is higher than for a 15-year mortgage by about 0.50%.)

c. How much more total interest will you pay on a 30-year mortgage compared with a 15-year mortgage ((b) vs. (a))?

2. ***Biweekly mortgage***

Let M be the monthly payment for a 30-year mortgage at r annual interest rate compounded daily. Your employer pays you on a biweekly schedule instead of on a monthly schedule. You want to make a mortgage payment every two weeks in the amount of $M/2$. How many years sooner can you finish paying off your mortgage? Use $r = 10\%$ per year, $P_0 = \$100{,}000$. Your answer should be independent of P_0.

3. Redo problem 1, but use continuously compounding interest as an approximation to daily compounding. What errors would you have in your answers for (a), (b), and (c)?

4. ***Lottery winner***

You are the winner of the $10 million lottery jackpot. The first decision you need to make is whether to take your winnings in 25 annual payments of $400,000 each, or to elect the $10 million lump sum up front. Discuss how you arrive at your decision. Your decision should be dependent on the prevailing interest rate for safe investments.

5. ***Power of compounding***

A professor's daughter is now at a private college that costs $30,000 per year. When she was born her grandparents put $10,000 in a college fund for her, investing it in a mutual fund that has had an average annual return of 18% for the past 18 years. Ignore the year-to-year fluctuations of the return. Does she have enough money in her account for four years of college expenses if she entered college at 18 years of age?

6. ***Power of compounding***

a. You borrowed $1,000 from a loan shark at 5% *monthly* interest. How much do you owe four years later?

b. You are trying to build a nest egg for your retirement. You have estimated that you will need income of $5,000 a month in order to live comfortably after retirement. How large a nest egg (principal) must you have at retirement? Assume that at your retirement you p that money, P_0, in an annuity with a guaranteed annual return of 8%. And suppose you think you will live forever.

c. You are now 25 and plan to retire at age 65. You want to start saving so that you can build a nest egg of $1 million. How much should you save each month? Assume that your savings will be earning 10% interest each year, compounded monthly.

7. ***Present value of money***

You have a contract that entitles you to receive $1 million 20 years from now. But you can't wait and want your money now. You want to sell your contract. What is a fair price for it? Assume the risk-free, inflation-adjusted interest rate is 3% per year, compounded continuously.