THE EFFECT OF RESIDUAL STRESSES ON WAVE SPEED IN ARTERIES

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(condensed title: Wave Speed in Arteries)

Abstract

The traditional approach to calculating stress distribution in arteries has been to assume (incorrectly) that the unloaded intact artery is stress-free. We consider the unloaded intact artery to have initial (i.e., residual) stresses and study how this affects the calculated wave speed of the arterial pulse. We use a set of equations that describe, in a simplified way, the blood flow in arteries and apply non-linear elasticity theory to derive a formula for wave speed. We compare wave speed calculations under two assumptions (considering unloaded intact arteries as stress-free and considering these arteries to have residual stresses). We find that wave speeds calculated assuming residual stresses are more realistic. Clinical applications of this work are suggested.

Keywords: residual stresses, wave speed, arteries, blood flow

1 Introduction

The accurate evaluation of wave speed in arteries by noninvasive methods is important to discover and explain diseases of the arteries, such as, arteriosclerosis, stenosis and aneurysm. Understanding of pulse waves also can be used to locate sites that need surgical treatment and to infer the condition of the heart. In addition, changes in wave speed can be important in evaluating the effect of therapeutic exercise, medication and paced breathing on peripheral arterial behavior. This is important in rehabilitation medicine, since experimental work has shown that wave speed decreases as an artery becomes more elastic.

The traditional approach to the study of stress distribution in arteries was to assume that unloaded intact arteries have no residual stresses. This leads to unrealistically high stress concentrations at the inner wall of an artery as well as high mean stress. Chuong and Fung [5, 6] found that considering the unloaded intact artery to have residual stresses leads to a reduction in the calculated stress and strain distribution, yielding more realistic results. We demonstrate in this paper that considering the unloaded intact artery to have residual stresses, similarly, will lead to a reduction in the calculated value of wave speed.

Theoretical values for wave speeds obtained in earlier attempts have not been satisfactory. Thomas Young's [20] formula leads to a decrease in wave speed with increasing pressure, contrary to experimental observations [10]. Bergel [4], Opatowski [16] and others have made modifications in this formula with limited success. All these works considered the stress-strain relation of the blood vessel to be linear, whereas the relation is nonlinear. A thorough discussion of theoretical predictions and experimental results on wave propagation in arteries can be found in [3] and [15].

In this paper, we apply well-known equations for blood flow in arteries. From these equations, the theory of PDEs is used to find the formula for wave speed. One unknown function relating the effect of pressure on arterial cross-sectional area is needed to perform the calculation. We derive this function using the data for stretch ratio vs. pressure when unloaded intact arteries are assumed to be stress-free [5] and when they are assumed to have residual stresses [6]. We compare wave speed calculations using these two assumptions and find that calculated wave speeds are more realistic if it is assumed that unloaded intact arteries have residual stresses. Therefore, we conclude that residual stresses must be taken into account to evaluate the wave speed correctly. We also find that the wave speed increases with stiffening of the arteries under either assumption.

2 Calculation of Wave Speed

Blood flow in arteries can be described by the following equations [1, 17, 18]

$$A_t + (Au)_x + \psi = 0 \tag{2.1}$$

$$(\rho u A)_t + (\rho u^2 A + pA)_x - pA_x = F \tag{2.2}$$

The first equation represents conservation of mass, where $A = A(x,t)$ is the cross sectional area of the arterial lumen as a function of axial coordinate, x, and time, t. $u = u(x,t)$ is the velocity which is assumed to be uniform in any cross-section normal to the axial direction. This assumption is reasonable for large arteries. ψ is the rate of outflow (leakage) per unit length of the artery . The second equation represents conservation of momentum, where $p = p(x,t)$ is the pressure, and F represents the effect of forces other than pressure. We also consider blood to be a homogeneous and incompressible fluid, i.e. the density, ρ, is constant. The subscripts denote partial differentiation with respect to that subscript.

Substituting equation (1) into equation (2) yields

$$u_t \; + \; u \; u_x \; + \; \frac{p_x}{\rho} \; = \; \frac{F \; + \; \rho u \psi}{\rho A} \tag{2.3}$$

We assume that pressure, p, is the only applied force acting on the arterial walls at a specific distance, x, from the heart. Longitudinal coupling and the weight of the blood in the artery are ignored, and the artery is assumed to be long compared to its radius. Furthermore, the amplitude of the pressure pulse and the wavelength / radius ratio are assumed to be small. Under these assumptions the inner radius, r and the cross-sectional area $A(x,t)$ of the artery in the loaded state depend on the elasticity, $E(x)$ and the pressure $p(x,t)$. Thus,

$$\frac{A(x,t)}{A_0(x)} \; = \; f(p, E, x) \tag{2.4}$$

where $A_0(x)$ is the cross-sectional area of the artery in the unloaded state at position x. Since $A_0(x)$ is a function of x, the artery may be tapered. With elasticity E a function of position x, we write

$$\frac{A(x,t)}{A_0(x)} \; = \; f(p,x) \qquad \text{and} \qquad \frac{\partial p}{\partial A} \; = \; \frac{1}{A_0(x)(\frac{\partial f}{\partial p})_x} \tag{2.5}$$

where the subscript x denotes that location is fixed. Substituting equation (2.5) into equation (2.3) and rewriting equation (2.1) yields the set of conservation laws

$$\begin{pmatrix} u \\ A \end{pmatrix}_t + \begin{pmatrix} u & \frac{1}{\rho}\frac{\partial p}{\partial A} \\ A & u \end{pmatrix} \begin{pmatrix} u \\ A \end{pmatrix}_x = \begin{pmatrix} \frac{F+\rho u \psi}{\rho A} \\ -\psi \end{pmatrix} \tag{2.6}$$

The speeds at which information propagates along the characteristics of these equations are given by the eigenvalues of the 2×2 matrix in equation (2.6) (see e.g. [7]). Using equations (2.5) and (2.6), the wave speeds, c, are given by

$$c(x,t) \; = \; u(x,t) \; \pm \left[\frac{A}{\rho}\frac{\partial p}{\partial A}\right]^{1/2} \; = \; u(x,t) \; \pm \left[\frac{f(p,x)}{\rho(\frac{\partial f}{\partial p})_x}\right]^{1/2} \tag{2.7}$$

The middle expression for wave speed is the same as obtained by Anliker [1], (p. 223).

Table 1: Circumferential Stretch Ratios vs. Pressure

Pressure (mm Hg)	λ_θ **where** $\Theta_0 = 71.4^0$	λ_θ **where** $\Theta_0 = 180^0$
0.0	0.875	1.000
60.0	1.583	1.820
120.0	1.729	1.927
160.0	1.750	1.962

Since the flow velocity in an artery is typically much smaller than the wave speed, we will refer to the wave speed as

$$c(x,t) \;=\; \left[f(p,x)/\rho \left(\frac{\partial f}{\partial p}\right)_x \right]^{1/2} \tag{2.8}$$

from here on. From this expression, it is clear that to compute the wave speed in an artery one needs only the function $f(p)$ at a specific position x.

To find $f(p)$, the work of Chuong and Fung [5, 6] can be used. Using nonlinear elasticity theory, they studied how the artery stretches in the radial, circumferential and axial directions when subjected to different pressures. In particular, the stretch ratio in the circumferential direction, λ_θ, is of use. Chuong and Fung derived

$$\lambda_\theta \;=\; \left(\frac{\pi}{\Theta_0}\right)\frac{r}{R} \tag{2.9}$$

where $2\Theta_0$ is the angle subtended by the sector of the artery in its reference state (Fig. 1). By reference state, we mean the state in which the artery is assumed to have no residual stresses. R is the radial coordinate of a point in the artery in the reference state and r is the radius of the same point in the loaded artery at any time. Two reference states were considered. For the case of the reference state being the unloaded closed artery, (i.e. under the assumption that the unloaded intact artery is stress free) $\Theta_0 \;=\; 180^0$ [5] (Fig. 1, State 1). If the unloaded intact artery is assumed to have residual stresses, this artery must be cut open and allowed to find its natural stress free state (Fig. 1, State 0). The shape the artery attains in experiments is approximately that of a sector of a circle. For this reference state, $\Theta_0 \;=\; 71.4^0$ for the cut open thoracic aorta of the rabbit [6]. The values of λ_θ at the inner wall of the thoracic aorta are taken from graphs in [5] and [6] for the four values of pressure considered by Chuong and Fung (Table 1).

From here on, λ_θ is the circumferential stretch ratio at the inner wall of the thoracic aorta (at a distance x from the heart) and the subscript i denotes the inner wall of the artery. Since $\lambda_\theta = \pi r_i/\Theta_0 R_i$ (equation 2.9); $(r_i/R_i)^2 = A/A_0$, and $A/A_0 = f(p)$ at a given distance x (equation 2.5), we obtain

$$f(p) = (\Theta_0 \lambda_\theta/\pi)^2 \tag{2.10}$$

At low pressure, elastin fibers in the arterial wall are easily stretched. However, when the pressure increases, collagen is encountered, thereby limiting the artery's ability to stretch. Therefore, we model the dependence of λ_θ on pressure by

$$\frac{d\lambda_\theta}{dp} = k(\lambda_{max} - \lambda_\theta) \tag{2.11}$$

where k is a constant and λ_{max} is the maximum value that stretch ratio λ_θ can take. This differential equation models the rate at which the artery stretches with respect to pressure as proportional to how far it is from its maximum ability to stretch. The solution to equation (2.11) is

$$\lambda_\theta = \lambda_{max} + b\, e^{-kp} \tag{2.12}$$

This functional form yields a good fit to the data (Fig. 2) and has an asymptotic limit λ_{max} as $p \to \infty$, i.e. there is a limit to the amount an artery can stretch under high pressure.

The flattening of the circumferential stretch ratio versus pressure curve (Fig. 2) with higher pressure can be interpreted physically as stiffening of the arteries. We expect this approximation (equation 2.12) to hold only in the physiological range of pressure (60 - 160 mm Hg) since at higher pressures longitudinal coupling may become more important.

We mention here that in calculating λ_θ, all the stresses in the arterial wall, i.e. radial, circumferential and longitudinal stresses, have been taken into account (see [5, 6]) without assuming that the wall thickness, h, is small.

Using equations (2.8), (2.10) and (2.12), and taking $\rho = 1.06$ gm/cm^3 as the density of blood, we establish the formula for wave speed as

$$c = \left(\frac{\lambda_{max} + be^{-kp}}{-2.12\ bke^{-kp}} \right)^{1/2} \tag{2.13}$$

3 Results

We use the data in Table 1 to calculate the optimal values for λ_{max}, b and k, using the method of least squares and the functional form of equation (2.12). When the unloaded intact artery is considered to have residual stresses, the optimal values are

Table 2: Wave Speed vs. Pressure

Pressure (mm Hg)	c **(cm/sec) WRS-NC**	c **(cm/sec) NRS-NC**	c **(cm/sec) WRS-WC**	c **(cm/sec) NRS-WC**
40	335	353	337	333
60	455	503	405	416
80	607	705	480	512
100	803	979	563	623
120	1055	1354	658	754
140	1382	1867	764	909
160	1807	2573	884	1092

WRS denotes intact artery having residual stresses, NRS denotes intact artery having no residual stresses, NC denotes computations without correction, WC denotes computations allowing ± 0.075 correction in λ_θ

$\lambda_{max} = 1.764$, $b = -0.889$ and $k = 1.998 \times 10^{-5}$ whereas when the unloaded intact state is assumed to be stress free, $\lambda_{max} = 1.959$, $b = -0.959$ and $k = 2.401 \times 10^{-5}$. Here, to have a consistent set of units, pressure in mm Hg was converted into dynes/cm^2, (1 mm Hg $=$ 1330 dynes/cm^2).

Using the values obtained from the exponential fit (equation 2.12), we calculate wave speed versus pressure in the physiological range using equation (2.13). (See Table 2 and Fig. 3). It is worthwhile to note some of the possible sources of error in λ_θ in the data (Table 1) used in our calculations; the assumption of a circular form (i.e. sector) for the stress free reference state 0 of the cut open thoracic aorta as well as possible errors in the exponential fitting form of the strain energy function used in Chuong and Fung [5, 6] to calculate the material constants from experiments. Allowing for an error up to ± 0.075 in each value of λ_θ in Table 1, new values for λ_{max}, b and k for both residual and non residual stress cases are calculated which yield more realistic wave speeds (Table 2 and Fig. 4). Also appearing in Fig. 4, are experimental results [1] (p. 220) for wave speeds in an anesthetized dog (since this is the only data available for comparison). It is clear that the results considering the unloaded intact artery to have residual stresses are significantly closer to the experimental results than considering such an artery to be stress free, with or without corrections to the values of λ_θ.

Equations (2.12 and 2.13) reveal that our method leads to higher wave speeds as the artery becomes stiffer (i.e. p gets large) as occurs in arteriosclerosis (see also [9], p. 151), qualitatively validating our strategy. In addition, the strategy applied leads to a curve of wave speed versus pressure with positive slope and concavity similar to the experimental curve.

4 Discussion and Conclusions

We have demonstrated that considering the unloaded states to have residual stresses leads to a reduction in calculated wave speeds compared to calculations in which the intact artery is assumed to be stress free. The wave speed in an artery can be found using the function $f(p)$, equation (2.10) in conjunction with equation (2.12), which depends upon the elastic parameters, λ_{max}, b and k. We expect these parameter values to vary slowly with distance from the heart. Thus, the wave speed will vary with distance from the heart. The values of λ_{max}, b and k can be found at a particular location if experimental values of the circumferential stretch ratio, λ_θ, are known at that point.

The goal of this work was to study whether considering unloaded intact arteries to have residual stresses leads to significantly different values of calculated wave speeds compared to assuming these arteries to be stress free and if so, which values of wave speed are closer to experimental observations. Our calculations reveal that considering residual stresses gives values closer to the measured values of wave speed (Fig. 4). Although, we would have preferred to compare our theoretical results for the thoracic aorta of the rabbit with the corresponding experimental results, such data is not available. We, therefore, compared our calculated results with experimental data from an anesthetized dog [1]. We also note that measured wave speeds for humans and dogs lie in almost the same range [15] (p. 104, reproduced in Table 3). Therefore, it is not unreasonable to compare qualitatively wave speeds in rabbits with those of dogs. The deviation of our theoretical results from the measured values may be attributed to a number of sources: e.g., the anesthetizing of the dog in the experiments; experimental errors in measuring λ_θ in [5] and [6]; assuming a circular form (i.e. sector) for the stress free reference state 0 of the cut open thoracic aorta; neglecting the weight of the blood in the model. Neglecting longitudinal coupling may cause errors which increase with pressure. Since the calculated wave speeds are very sensitive to variations in the data for λ_θ vs. p, the effects of the above sources of error are magnified.

Accurate wave speed information can be useful in evaluating the effect of therapeutic exercise, medication and paced breathing on peripheral arterial properties. This is because the wave speed is a measure of the elasticity of arteries. Lower wave speeds indicate more elastic arteries. We suggest that this information may be used to evaluate whether therapeutic exercise in the long term rehabilitation process contributes to increasing the elasticity of arteries.

This work represents the first attempt to include the effect of residual stresses on cal-

Table 3: Wave Velocities in Thoracic Aorta

Species	Wave Velocity (cm/sec)	Source
Human	550-650	[19]
Human	400	[13]
Human	530	[12]
Dog	400-700	[8]
Dog	480-550	[11]
Dog	410-480	[14]
Dog	400-500	[2]

culations of wave speed. Even with our simplified approach, the calculations demonstrate that considering unloaded intact arteries to have residual stresses yields calculated wave speeds that are closer to measured values compared to wave speeds obtained when the arteries are considered to be stress free. We note that our results lie below experimental values for low pressures and above for moderate and high pressures. This is likely due to the exponential variation of λ_θ with pressure and by neglecting longitudinal coupling (which has greater influence at high pressure) in our analysis.

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Figure Captions

Fig. 1 Cross-sectional representation of arteries at the stress free reference state (State 0) under the hypothesis that the unloaded intact artery has residual stresses, for the hypothesis that the unloaded intact artery is stress free (State 1) and for the subsequent loaded state (State 2) under transmural pressure and axial stretch.

Fig. 2 Circumferential stretch ratio, λ_θ, versus pressure at the inner wall of the thoracic aorta of the rabbit considering the unloaded intact artery to be stress free and considering it to have residual stresses.

Fig. 3 Comparison of calculated wave speeds at the thoracic aorta of the rabbit.

Fig. 4 Comparison of calculated wave speeds at the thoracic aorta of the rabbit with measured wave speed in an anesthetized dog under transmural pressure allowing for ± 0.075 correction to data in Table 1. (See also, Table 3). Open circles represent experimental data [1].

State 0

State 1

State 2