Stresses and Strains in the Left Ventricular Wall Approximated as a Thick Conical Shell Using Large Deformation Theory

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(condensed title: Stresses and Strains in the Left Ventricle)

Abstract

In this paper, stress and strain equations are developed for the left ventricle mainly to find the influence of the ventricle's shape on wall stresses. Here, the ventricle is assumed to be a thick-walled truncated conical shell and large elastic deformation theory is applied. Our model is compared to corresponding results approximating the left ventricle as a spherical shell. Clinically relevant parameters such as the myocardial stiffness constant, the stretch ratios and the stresses and strains have been computed using available canine data. The conical model leads to more realistic results than the spherical model and enables one to evaluate stresses and strains from base to apex instead of only at the equatorial region as in a cylindrical model.

Keywords: stress, strain, large deformation theory, left ventricle

1 Introduction

Cardiologists, physiologists and experimental researchers require an accurate mathematical model of left ventricular stress and strain distribution. Stress and strain interaction due to the ventricular contraction of the heart tissue results in cardiac compliance changes. These compliance changes vary in the young, the old and the diseased heart. Pathological conditions such as ischemic heart disease, ventricular hypertrophy and congestive heart failure frequently result in a stiffer myocardium. Decreased coronary artery perfusion associated with a stiffer myocardium may contribute to angina and the development of cardiac arrhythmias. Researchers attempting to predict the effects of medical interventions upon cardiac compliance would benefit from having a physiologically accurate mathematical model of the myocardium.

The first attempt was made to estimate stresses in the ventricular wall using the classical elasticity results of Lamé [1]. In these efforts, the ventricular wall was approximated as a spherical shell and the ventricle as a spherical cavity. The solution to the spherical problem does not explicitly contain the material constants. However, if the stress-strain relation is non-linear or the shape of the ventricle is not spherical, the solution depends explicitly on the material constants [2]. Sandler [3], approximated the ventricular wall as an ellipsoidal shell and used the law of Laplace in calculating the stresses and tension in the wall. Wong and Rautaharju [4], noting that the Laplace Law is not appropriate for describing thick-walled structures, approximated the left ventricle as a thick ellipsoidal shell in analyzing the stresses. However, they assumed that the myocardium obeys Hooke's law and that elastic deformations are small.

The stress-strain relation is non-linear in the ventricular wall and the ventricle takes on large deformations on the order of 50-100% of its initial dimensions [6]. Mirsky [6], was the first to compute ventricular stresses based on large deformation theory. Interested in the qualitative aspects of stress distribution, Mirsky assumed the shape of the ventricle to be spherical. He found that the tangential stress at the endocardial surface was ten times as high as that obtained using small deformation (classical) elasticity theory. Research to calculate stresses in the ventricle considering its shape to be other than spherical or ellipsoidal has been performed by Janz [7], Taber [8] and Azhari *et. al.* [9]. Janz [7] developed two formulas for estimating local average circumferential stress in spherical, cylindrical, conical and ellipsoidal shapes. These formulas do not predict the stress variation through the wall and ignore shear stresses. His main objective was simplifying formulas. Taber [8] approximated the ventricle as a thick-walled cylinder consisting of nine incompressible laminae of equal thickness. He developed a non-linear shell theory to determine stresses in the left ventricular wall. Taber noted that his results decrease in accuracy as the shell grows thicker. This is always the case with shell theories. Azhari *et. al.*, noting that a cylindrical model of the left ventricle does not accurately predict the non-uniformity of its twisting motion during loading, developed a conical model to understand better this non-uniformity. However, these authors used linear elasticity theory and assumed that the ventricular wall material is isotropic and homogeneous. Other researchers also have noted that the left ventricle can be described more accurately as conical with a circular cross-section rather than spherical or ellipsoidal [10, 11, 12]. Since the shape of the ventricular wall is an important factor in the resulting stress distribution in the wall [5], we develop stress and strain equations valid for the left ventricular wall in this paper by approximating the wall as a thick conical shell and we use the large elastic deformation theory of Green and Zerna [13].

Many other studies relevant to investigation of stresses in the left ventricular wall have been published [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Some of these authors have employed the finite element method successfully for three-dimensional analysis of stress distribution in the left ventricular wall. The effects of anisotropy due to muscle fiber orientation on the wall stresses have also been examined. Although these models are more extensive than ours, they do not consider analytically the effect of the conical shape of the ventricle. We, therefore, in this paper, examine the effect of the conical shape on the myocardial properties using large deformation theory and find an almost closed form solution. Although it is known that the ventricle is not homogeneous and isotropic, we neglect the anisotropy and nonhomogeneity of the wall material to make the mathematics tractable and focus mainly on the effects due to the geometry of the ventricle. We find that the conical model leads to more realistic calculated results than the spherical model and enables one to evaluate stresses and strains from base to apex instead of only at the equatorial region as in a cylindrical model.

2 Method

We approximate the left ventricle as a circular-conical shell (closed at the apical end) (Fig. 1) and assume that it behaves as an incompressible, homogeneous, isotropic, elastic material. Regarding the thickness of the wall, Thomas [26] observed that the wall is thicker at the base than at the apical end. Peskin [12] concluded that the wall thickness is proportional to the radial distance of the middle surface from the symmetry axis. Thus, at the apical end, the thickness is almost zero.

The radius $r = r(z)$ and distance z at any point on the surface of the inner cone are related by

$$r = \frac{r_i}{h_1}(h_1 - z) \tag{1}$$

and that of the outer cone by

$$r = \frac{r_o}{h_1}(h_1 - z) \tag{2}$$

where h_1 is the height of the circular cone, r_i is the inner radius of the conical shell and r_o is the outer radius of the shell, all in the deformed state (see Fig. 1). We take $h_1 = 4r_i$ so that the volume of the cone with radius r_i is the same as the volume of the sphere with radius r_i. The corresponding radii in the undeformed state are R_i and R_o with height h_1/λ where λ is the extension ratio (assumed to be constant) in the axial direction.

We identify $(\theta_1, \theta_2, \theta_3)$ of standard elastic large deformation theory notation (see Appendix, eq. (33)) with the polar coordinates (r, θ, z) in the deformed state and with the coordinates (R, Θ, Z) in the undeformed state, so that

$$\begin{aligned} y_1 &= r cos\theta, \quad y_2 = r sin\theta, \quad y_3 = z \\ x_1 &= R cos\Theta, \quad x_2 = R sin\Theta, \quad x_3 = z/\lambda \end{aligned} \tag{3}$$

We further consider the unloaded state to be subjected to the following deformation to simulate twisting motion, inflation and extension as observed in the passive filling of the ventricle. (If the twisting motion is neglected and the inflation, under constant extension ratio, is such that the conical shape is maintained, the stresses would not depend on Z.) Thus, $R = Q(r)$, $\theta = \Theta + \psi Z + \omega(R)$, $z = \lambda Z + \mathrm{w}(R)$. In view of this, equation (3) can be written as

$$\begin{aligned} y_1 &= r cos\theta, \qquad y_2 = r sin\theta, \qquad y_3 = z \\ x_1 &= Q(r) cos\left(\theta - \frac{\psi}{\lambda}(z - \mathrm{w}(r)) - \omega(r)\right), \\ x_2 &= Q(r) sin\left(\theta - \frac{\psi}{\lambda}(z - \mathrm{w}(r)) - \omega(r)\right), \\ x_3 &= \frac{z - \mathrm{w}(r)}{\lambda} \end{aligned} \tag{4}$$

From (4) and (34) (see Appendix), we obtain

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad G^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tag{5}$$

$$g_{ij} = \begin{pmatrix} R'^{\,2} + R^2(\psi_1 \mathrm{w}_r - \omega_r)^2 + \frac{\mathrm{w}_r^2}{\lambda^2}; & R^2(\psi_1 \mathrm{w}_r - \omega_r); & -\left[R^2\psi_1(\psi_1 \mathrm{w}_r - \omega_r) + \frac{\mathrm{w}_r}{\lambda^2}\right] \\ R^2(\psi_1 \mathrm{w}_r - \omega_r) & R^2 & -R^2\psi_1 \\ -\left[R^2\psi_1(\psi_1 \mathrm{w}_r - \omega_r) + \frac{\mathrm{w}_r}{\lambda^2}\right] & -R^2\psi_1 & R^2\psi_1^2 + \frac{1}{\lambda^2} \end{pmatrix}; \tag{6}$$

where $R' = \frac{dR}{dr}$, $\mathrm{w}_r = \frac{d\mathrm{w}}{dr}$, $\omega_r = \frac{d\omega}{dr}$ and $\psi_1 = \frac{\psi}{\lambda}$.

We assume that the shell material of the ventricle wall is incompressible since it is approximately 90% fluid. The incompressibility condition is given by

$$\frac{\pi h_1}{3}(R_o^2 - R_i^2) = \frac{\pi h_1}{3}\lambda(r_o^2 - r_i^2) \tag{7}$$

or

$$\lambda r_o^2 - R_o^2 = \lambda r_i^2 - R_i^2 = \lambda r^2 - R^2 \tag{8}$$

And $g = G$ gives

$$R' = \lambda r/R \tag{9}$$

from which (7) can also be obtained by integration.

Using (9) in (6), g_{ij} and g^{ij} become

$$g_{ij} = \begin{pmatrix} \frac{\lambda^2 r^2}{R^2} + R^2(\psi_1 \mathrm{w}_r - \omega_r)^2 + \frac{\mathrm{w}_r^2}{\lambda^2}; & R^2(\psi_1 \mathrm{w}_r - \omega_r); & -\left[R^2\psi_1(\psi_1 \mathrm{w}_r - \omega_r) + \frac{\mathrm{w}_r}{\lambda^2}\right] \\ R^2(\mathrm{w}_r - \omega_r) & R^2 & -R^2\psi_1 \\ -\left[R^2\psi_1(\psi_1 \mathrm{w}_r - \omega_r) + \frac{\mathrm{w}_r}{\lambda^2}\right] & -R^2\psi_1 & R^2\psi_1^2 + \frac{1}{\lambda^2} \end{pmatrix}; \tag{10}$$

and

$$g^{ij} = \begin{pmatrix} r'^{\,2} & r'\omega' & r'\mathrm{w}' \\ r'\omega' & \psi^2 + \omega'^{\,2} + R^{-2} & \psi\lambda + \mathrm{w}'\omega' \\ r'\mathrm{w}' & \psi\lambda + \mathrm{w}'\omega' & \lambda^2 + \mathrm{w}'^{\,2} \end{pmatrix} \tag{11}$$

where $\omega' = \frac{d\omega}{dR}$, $\mathrm{w}' = \frac{d\mathrm{w}}{dR}$, and $r' = \frac{dr}{dR}$.

In our present analysis, the strain energy function is assumed to be a function of the strain invariant I_1 only (see eq. (35) in Appendix and [6] p. 1143). Using (35), (5) and (11), we find that I_1 is given by

$$I_1 = r'^{\,2} + \psi^2 r^2 + r^2\omega'^{\,2} + \frac{r^2}{R^2} + \lambda^2 + \mathrm{w}'^{\,2} \tag{12}$$

In what follows, we shall denote I_1 by I. Using (11) and (37), the non-zero components of stresses are

$$\sigma_{11} = \tau^{11} = 2\frac{\partial W}{\partial I} r'^{\,2} - p(r)$$

$$
\begin{aligned}
\sigma_{22} \;=\; r^2\tau^{22} &\;=\; 2\frac{\partial W}{\partial I}\left[(\psi r)^2 \;+\; (\omega' r)^2 \;+\; (r/R)^2\right] \;-\; p(r) \\
\sigma_{33} \;=\; \tau^{33} &\;=\; 2\frac{\partial W}{\partial I}(\lambda^2 \;+\; \mathrm{w}'^{\,2}) \;-\; p(r) \\
\sigma_{12} \;=\; r\tau^{12} &\;=\; 2\frac{\partial W}{\partial I}(r r' \omega') \\
\sigma_{13} \;=\; \tau^{13} &\;=\; 2\frac{\partial W}{\partial I} r' \mathrm{w}' \\
\sigma_{23} \;=\; r\tau^{23} &\;=\; 2\frac{\partial W}{\partial I}(r\psi\lambda \;+\; r\omega' \mathrm{w}')
\end{aligned}
\tag{13}
$$

where $\frac{\partial W}{\partial I}$ is given by (32), σ_{ij} (i, j take the values $1, 2, 3$ where 1 refers to r, 2 refers to θ and 3 refers to z) are the physical components of the stresses and $p(r)$ is the undetermined hydrostatic pressure. Substituting (13), (41) and (42) into (40) (see Appendix), and setting $F^i \;=\; f^j \;=\; 0$ for equilibrium in the absence of body forces, the equations of equilibrium are obtained as

$$
\begin{aligned}
\frac{d}{dr}\sigma_{11} \;-\; \frac{\sigma_{22} \;-\; \sigma_{11}}{r} &\;=\; 0 \\
\frac{d}{dr}\sigma_{12} \;+\; 2\frac{\sigma_{12}}{r} &\;=\; 0 \\
\frac{d}{dr}\sigma_{13} \;+\; \frac{\sigma_{13}}{r} &\;=\; 0
\end{aligned}
\tag{14}
$$

(See [13] p. 96 for a similar calculation.)

Integrating the second and third of (14), we have

$$
\begin{aligned}
r^2\sigma_{12} &\;=\; A \text{ (a constant)} \\
r\sigma_{13} &\;=\; B \text{ (a constant)}
\end{aligned}
\tag{15}
$$

and integrating the first of (14), we get

$$
p(r) \;=\; 2\frac{\partial W}{\partial I} r'^{\,2} \;-\; \int_{r_i}^{r} H(r)dr \;-\; K \tag{16}
$$

where K is the constant of integration and r_i is the inner radius at any cross-section and

$$
H(r) \;=\; \frac{\sigma_{22} - \sigma_{11}}{r} \tag{17}
$$

Using the boundary conditions at any cross-section, i.e. $\sigma_{11}(r_i) \;=\; -P_i$ and $\sigma_{11}(r_o) \;=\; 0$ where P_i is the uniform pressure at the inner surface and the outer surface $r \;=\; r_o$ is free of external pressure, we find that

$$
P_i \;=\; \int_{r_i}^{r_o} H(r)dr \tag{18}
$$

$$
p(r) \;=\; 2\frac{\partial W}{\partial I} r'^{\,2} \;+\; \int_{r}^{r_o} H(r)dr \tag{19}
$$

Using (18), (19), and (13), the Cauchy stressses can be written in terms of the given pressure P_i as

$$\begin{aligned}
\sigma_{11} &= \int_{r_i}^{r} H(r)dr \; - \; P_i \\
\sigma_{22} &= 2\frac{\partial W}{\partial I}\left[(\psi r)^2 \; + \; (\omega' r)^2 \; + \; (r/R)^2 \; - \; r'^{\,2}\right] \; + \; \sigma_{11} \\
\sigma_{33} &= 2\frac{\partial W}{\partial I}(\lambda^2 \; + \; \mathrm{w}'^{\,2} \; - \; r'^{\,2}) \; + \; \sigma_{11} \\
\sigma_{12} &= 2\frac{\partial W}{\partial I}(rr'\omega') \\
\sigma_{13} &= 2\frac{\partial W}{\partial I}r'\mathrm{w}' \\
\sigma_{23} &= 2\frac{\partial W}{\partial I}(r\psi\lambda \; + \; r\omega'\mathrm{w}')
\end{aligned} \tag{20}$$

In order to find quantitatively the stresses, we need to know $\frac{d\omega}{dR}$ and $\frac{d\mathrm{w}}{dR}$. For the sake of solution, we use the values $\frac{d\mathrm{w}}{dR} \; = \; 0.2$ and $\frac{d\omega}{dR} \; = \; 0.1$ radians/cm, $\psi \; = \; 1.7^o$ per cm. at R_i [27]. Now using, (15) and (13) we find that

$$\begin{aligned}
(2\tfrac{\partial W}{\partial I}\mathrm{w}'r'r)|_{R_i} &= A \\
(2\tfrac{\partial W}{\partial I}\omega' r' r^3)|_{R_i} &= B
\end{aligned} \tag{21}$$

Using (21), with the above parameters, the values of A and B become known. Also, from (21) we find that

$$r\omega' \; = \; \left(\frac{B}{Ar}\right)\mathrm{w}' \tag{22}$$

Using (22), in (12), I becomes

$$I \; = \; r'^2 \; + \; \left(\frac{r}{R}\right)^2 \; + \; r^2\psi^2 \; + \; \lambda^2 \; + \; \left[1 \; + \; \left(\frac{B}{Ar}\right)^2\right]x^2 \tag{23}$$

where $x \; = \; \mathrm{w}'$. Then the first of (21) becomes

$$f(x) \; = \; 2\frac{\partial W}{\partial I}rr'x \; - \; A \; = \; 0 \tag{24}$$

where $\frac{df}{dx} \; = \; 2rr'(\frac{\partial W}{\partial I} \; + \; x\frac{d}{dx}\frac{\partial W}{\partial I})$.

Now using the Newton-Raphson method, we solve the uncoupled nonlinear algebraic equation (24) for x, i.e., w', at any radial location. Then, ω' is determined from (22). With known values of w' and ω', the stresses given by (13) can be evaluated at any location. The strains can be evaluated using (45), (5) and (11).

The externally applied loads to maintain this deformation can be determined, for example, the axial load L and moment M at the apex are given by the equations

$$\begin{aligned}
L &= 2\pi \textstyle\int_{r_i}^{r_o}(\sigma_{33}) \; r \; dr \; - \; P_i\pi r_i^2 \\
M &= 2\pi \textstyle\int_{r_i}^{r_o}(r^2 \; \sigma_{23}) \; dr
\end{aligned} \tag{25}$$

In order to avoid having a singularity at the apical end and to be consistent with the actual shape of the ventricle, we consider the cone to be truncated. The height of the truncated cone is taken to be three times the radius at the base of the cone and the radius at the apical end is taken as one-fourth of the radius at the base. Thus, $z = 3h_1/4$, $r = r_i/4$ on the inner surface and $z = 3h_1/4$, $r = r_o/4$ on the outer surface of the truncated cone (see Fig. 1).

3 Results

To determine the magnitude of the stresses, we need to have pressure-volume data for the left ventricle as well as an expression for $\frac{\partial W}{\partial I}$. Since $\frac{\partial W}{\partial I}$ is not available for a conical shell, we adopt the pressure-volume data obtained by Spotnitz et. al. [28], see Table 1.

We take the inner radius at the base of the conical shell to be equal to the inner radius of the spherical shell, i.e., $R_i = 1.42$ cm. at zero pressure. Now, the volume of the truncated conical cavity in the deformed state is given by

$$\frac{\pi}{3}\left[r_i^2 h_1 - \left(\frac{r_i}{4}\right)^2 \frac{h_1}{4}\right] = \frac{\pi}{3} r_i^2 h_1 \left(\frac{63}{64}\right) = \frac{21}{16}\pi r_i^3 \qquad (26)$$

From Table 1, we use the data for pressure $P_i = 20$ mm Hg to compare our results with those of Mirsky [6]. Using (26) and volume of 52 ml at $P_i = 20$ mm Hg, we have

$$52 = \frac{21}{16}\pi r_i^3. \qquad (27)$$

Thus, $r_i = 2.327$ cm. and $h_1 = 4r_i = 9.308$ cm. at $P_i = 20$ mm Hg. From the second expression of (26), we write the expression for the volume of the conical cavity in the undeformed state

$$\frac{1}{3}\pi R_i^2 \frac{h_1}{\lambda}\left(\frac{63}{64}\right) \qquad (28)$$

where we have replaced r_i by R_i and h_1 by h_1/λ, its height in the undeformed state. This must equal the volume of the spherical cavity with inner radius $R_i = 1.42$ cm. so that

$$\frac{1}{3}\pi R_i^2 \frac{h_1}{\lambda}\left(\frac{63}{64}\right) = \frac{4\pi}{3} R_i^3 \quad \text{or} \quad \lambda = 0.984\left(\frac{h_1}{4R_i}\right) \qquad (29)$$

Using $R_i = 1.42$ and $h_1 = 9.308$ we find from (29) that $\lambda = 1.613$ for $P_i = 20$ mm Hg. The value of λ will change as P changes. From the incompressibility condition (7), we obtain

$$r_o^2 - r_i^2 = \frac{1}{\lambda}(R_o^2 - R_i^2) \qquad (30)$$

Using $r_i = 2.327$, found above, $R_i = 1.42$, $R_o = 2.99$ cm. (see Table 1) and $\lambda = 1.613$ found above, we have from (30), that $r_o = 3.116$ cm. Thus, the thickness T of the conical shell at the base at $P_i = 20$ mm Hg is given by

$$T = r_o - r_i = 0.789\text{cm} \qquad (31)$$

Table 1: Pressure - Volume Data for the Left Ventricle of Canines

(At zero pressure R_i (inner radius) is 1.42 cm and R_o (outer radius) is 2.99 cm.)

Pressure *(mm Hg)*	**Volume** *ml*
0	12.0
5	31.5
10	40.0
15	46.7
20	52.0
25	56.5
30	60.0

Table 2: Pressure - Volume - Radius Data for the Left Ventricle as a Conical Shell

Pressure *(mm Hg)*	**Volume** *ml*	r_i *cm*	r_o *cm*	I_M	$\left(\frac{\partial W}{\partial I}\right)_M$	λ
5	31.5	1.960	2.986	3.724	6.840	1.365
10	40.0	2.132	3.038	4.151	13.009	1.478
15	46.7	2.245	3.081	4.479	19.373	1.556
20	52.0	2.327	3.116	4.736	25.899	1.613
25	56.5	2.392	3.146	4.950	32.550	1.658
30	60.0	2.441	3.170	5.117	39.284	1.692

We have applied the method described above to find r_i, r_o and λ for the other values of diastolic pressure P_i given in Table 1. Then, using (12), we evaluate $I_M = (I_1 + I_2)/2$ for each value of P_i, where I_1 is the value of the strain invariant I at r_i, I_2 is the strain invariant at r_o and I_M is the mean of I_1 and I_2.

In order to evaluate $\frac{\partial W}{\partial I}$, we use equation (18) by assuming $\frac{\partial W}{\partial I}$ to be constant over the range of integration and equal to the value in the middle of the range of integration, as in [6]. This assumption yields values for pressure P_i to within 10% [6]. The values for $\frac{\partial W}{\partial I}$ found using the above procedure are given in Table 2.

Using the data for $\left(\frac{\partial W}{\partial I}\right)_M$ for various values of I_M, the best fit for $\frac{\partial W}{\partial I}$ of the form $\frac{\partial W}{\partial I} = A + Be^{CI}$ is found using least squares.

$$\frac{\partial W}{\partial I} = -5.329 + 0.391e^{0.925I} \tag{32}$$

We are now in a position to evaluate the stresses at different locations through the ventricular wall. For instance, using (32), (20), (8) and (12) we may evaluate all the stresses through the wall.

The values of the stresses in the wall for other pressures similarly can be found. The strains and stretch ratios also can be evaluated by using (45), (47), (5) and (11). In the following section, we compare computed results for such parameters with the corresponding results for a spherical shell, using corresponding equations valid for a spherical shell, for $P_i = 20$ mm Hg.

4 Conclusions

Computations have been performed to determine clinically relevant parameters such as circumferential stress-strain relations, the myocardial stiffness constant, stretch ratios, variation of wall thickness and modulus of chamber stiffness using available canine data [28].

In Fig. 2, we present the non-linear circumferential stress-strain relation through the left ventricular wall at a pressure of 20 mm Hg, for the conical model (at the base) and the spherical model. We also present the result for the conical model without twist, i.e., ϕ, w(r) and $\omega(r)$ are zero. If the twisting motion is neglected, the stresses do not depend on Z. Because the lowest stresses and strains occur at the epicardial surface and the highest stresses and strains occur at the endocardial surface, the curve goes from the epicardial to the endocardial surface from left to right. It is observed that the curve is steeper for the spherical model for strains greater than 0.5 although the stresses are higher for the conical model. The results are consistent with the observation made by Mirsky [5], that for a given pressure and thickness to radius ratio, the stresses are minimal in a sphere, maximal in a cylinder and intermediate in an ellipsoid of revolution. Elastic stiffness, defined as the slope of the stress vs. strain curve can be determined from the curve in Fig. 2 and is presented in Fig. 6.

In Fig. 3, we plot the circumferential stress σ_{22} through the ventricular wall for $P_i = 20$ mm Hg. It is observed that the stress decreases as we go from the base to the apex. The decrease is noticeable from the base to the middle region but the change is much smaller from the middle region to the apex.

In Fig. 4, we plot the circumferential strain e_{22} through the ventricular wall for $P_i = 20$ mm Hg. As in the case of stress, the stress also decreases from base to apex. The decrease is noticeable from the base to the middle region but the change is much smaller from the middle region to the apex.

In Fig. 5, all stresses are plotted through the wall thickness. The thin solid curve represents σ_{11}. The thin dashed curve represents σ_{22}. The thin dotted curve represents σ_{33}. The thick solid curve represents σ_{12}. The thick dashed curve represents σ_{13}. The thick dotted curve represents σ_{23}.

In Fig. 6, we find the myocardial stiffness constant, k_m, at $P_i = 20$ mm Hg. This is defined as the slope of the relation between the elastic stiffness and stress. The myocardial stiffness constant describes the extensibility of a single unit of heart muscle [29]. It is noted that this constant for a spherical shell is higher than that for the conical model. Therefore, the conical model compliance is greater than the spherical model compliance.

In Fig. 7, we present a plot of stretch ratios λ_r and λ_θ in the radial and tangential directions, respectively, through the ventricular wall at a pressure of 20 mm Hg. Stretch ratio is defined as the ratio of stretched length to initial length. Although λ_r is higher for the spherical model than for the conical model, no substantial difference is observed for λ_θ.

The variation of circumferential stress through the wall at 20 mm Hg is presented in Fig. 8, both for the spherical and conical models. The curve for the spherical model is steeper than that for the conical model. The more gradual slope of the conical model stress curve more closely approximates the normal heart [14]. The conical model stresses are more evenly distributed between the endocardial and epicardial surfaces than stresses calculated using the spherical model. High stress concentrations at the endocardial surface are observed for both models. Such high concentrations theoretically can be reduced by considering the residual stresses in the wall [30, 31]. Experimental data valid for a conical model that would make it possible to evaluate the effect of residual stresses is not available. We note, however, that recent advances have been made in measuring ventricular strains in the rat ventricle [32].

Fig. 9 gives the distribution of axial stress for the conical model through the wall at a pressure of 20 mm Hg. The maximum axial stress occurs at the endocardial surface and decreases gradually through the wall. Finally, in Fig. 10, we plot chamber stiffness versus pressure to determine the modulus of chamber stiffness (k_c) which is defined as the slope of the relation between chamber stiffness $\frac{dP}{dV}$ and pressure P [29].

5 Discussion

We have presented a conical model of the left ventricle that can be used to evaluate stresses, strains and other clinically relevant parameters of the myocardium. For this model, the axial stress (in place of meridional stress for the spherical model) and the stretch ratios in the axial direction are the new quantities which must be taken into account. We have assumed the ventricular wall material to be homogeneous, elastic and isotropic. However, the ventricular wall can better be described as non-homogeneous, anisotropic and viscoelastic. The wall is non-homogeneous and anisotropic since it depends upon the varying spatial orientation of muscle fibers through the wall. The location and magnitude of the maximum stresses may be affected by considering the nonhomogeneity of the material, its anisotropy [5] and its residual stresses [30, 31]. Angiography reveals that the left ventricle is irregular in shape. Therefore, idealized geometry such as spherical, cylindrical, ellipsoidal or, for that matter, conical also is not truly representative of the left ventricle. Our analytical approach using large deformation theory, assuming a thick conical ventricular wall, is more likely to be in agreement with the physiological heart parameters than previous analytical spherical or ellipsoidal models. In addition, the analytic solution provides some important insight which cannot be assessed using numerical solutions only. For example, from (15), one can predict that for the type of deformation discussed, the shear stress σ_{12} is inversely proportional the square of the radius, whereas σ_{13} is inversely proportional to the radius. The conical model also enables one to determine stresses and strains from base to apex instead of only in the equatorial region, as in a cylindrical model.

6 Appendix

We adopt the notation and method of Green and Zerna [13], wherein we suppose that a body of perfectly elastic material is deformed such that a point initially at x_i with respect to a fixed rectangular Cartesian coordinate system moves to y_i in the same coordinate system. The curvilinear convected coordinates θ_i are related to x_i and y_i by

$$x_i = x_i(\theta_1, \theta_2, \theta_3); \qquad y_i = y_i(\theta_1, \theta_2, \theta_3) \tag{33}$$

The metric tensors g_{ij} and G_{ij} in the undeformed and deformed states, respectively, are given by

$$g_{ij} = \frac{\partial x^r}{\partial \theta^i} \cdot \frac{\partial x^r}{\partial \theta^j}; \quad G_{ij} = \frac{\partial y^r}{\partial \theta^i} \cdot \frac{\partial y^r}{\partial \theta^j}; \quad g^{ij} = \frac{\partial \theta^i}{\partial x^r} \cdot \frac{\partial \theta^j}{\partial x^r}; \quad G^{ij} = \frac{\partial \theta^i}{\partial y^r} \cdot \frac{\partial \theta^j}{\partial y^r}; \tag{34}$$

For a homogeneous, isotropic, elastic material, the strain invariants, I_1, I_2 and I_3 are obtained from

$$I_1 = g^{rs}G_{rs}, \;\; I_2 = g_{rs}G^{rs}, \;\; I_3 = \frac{G}{g}; \;\; G = |G_{ij}|, \;\; g = |g_{ij}|. \tag{35}$$

When the body is incompressible,

$$I_3 = 1, \;\; W = W(I_1, I_2) \tag{36}$$

where W is the strain energy function.

The stress-strain relation is

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij} \tag{37}$$

where

$$\Phi = 2\frac{\partial W}{\partial I_1}, \quad \Psi = 2\frac{\partial W}{\partial I_2} \tag{38}$$

and

$$B^{ij} = (g^{ij}g^{rs} - g^{ir}g^{js})G_{rs} \tag{39}$$

where p is the hydrostatic pressure and τ^{ij} is the contravariant stress tensor with respect to the θ^i coordinate system.

The equations of motion are given by

$$\tau^{ij}||_i + \rho F^i = \rho f^j \tag{40}$$

where the double line denotes covariant differentiation with respect to θ^i. F^j denotes the contravariant components of the body force vector $\mathbf{F}$ per unit mass, ρ is the density of the material and f^j represent the contravariant components of the acceleration vector $\mathbf{f}$.

$\tau^{ij}||_i$ is given by

$$\tau^{ij}||_r = \tau^{ij}_{,r} + \Gamma^i_{rm}\tau^{mj} + \Gamma^j_{rm}\tau^{im} \tag{41}$$

$$\Gamma^r_{ij} = g^{rs}\Gamma_{ijs} = \frac{1}{2}g^{rs}(g_{is,j} + g_{js,i} - g_{ij,s}) \tag{42}$$

where the subscript "," denotes partial differentiation.

The boundary conditions are

$$\tau^{ij}n_i = P^j \tag{43}$$

where P^j are the contravariant components of the surface force vector and n_i are the covariant components of the unit outward drawn normal vector to the deformed surface.

The physical components σ_{ij} are given by

$$\sigma_{ij} = \left[G_{jj}/G^{ii}\right]^{1/2}\tau^{ij} \tag{44}$$

The strain tensor E^i_j (and its physical components, e_{ij}) are given by

$$E^i_j = \frac{1}{2}(g^{ik}G_{kj} - \delta^i_j) \tag{45}$$

where $\delta^i_j = 1$ when $i = j$ and $\delta^i_j = 0$ when $i \neq j$.

The principal stretch ratios, λ_i, are given by

$$E_i = \frac{1}{2}(\lambda_i^2 - 1) \quad (i = R, \theta) \tag{46}$$

Thus,

$$\lambda_i = \sqrt{2E_i + 1} \tag{47}$$

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References

[1] Lamé, Lecons Sur la Theorie De Lelasticite. Paris, (1852).

[2] Fung YC, Biodynamics: Circulation. (Springer-Verlag, New York, 1984) 404 pp.

[3] Sandler H, and Dodge HT, Left Ventricular Tension and Stress in Man. *Circulation Research.* **13** No. 2, (1963) 91-104.

[4] Wong AYK, and Rautaharju PM, Stress distribution within the left ventricular wall approximated as a thick ellipsoidal shell. *American Heart Journal.* **75** No. 5, (1968) 649-662.

[5] Mirsky I, Elastic properties of the myocardium: a quantitative approach with physiological and clinical application. *In* Handbook of Physiology. The Cardiovascular System. R. M. Berne, editor. (Washington, DC, American Physiological Society, Sect. 2, Vol. 1, Chapter 14,, 1979) pp. 497-531.

[6] Mirsky I, Ventricular and Arterial Wall Stresses based on Large Deformation Analyses. *Biophysical Journal.* **13** (1973) 1141-1159.

[7] Janz RF, Estimation of local myocardial stress, *American Journal of Physiology.* **242** (1982) H875-H881.

[8] Taber LA, On a Nonlinear Theory for Muscle Shells: Part II – Application to the Beating Left Ventricle. *Journal of Biomechanical Engineering.* **113** (1991) 63-71.

[9] Azhari H, Buchalter M, Sideman S, Shapiro E, and Beyar R, A Conical Model to Describe the Nonuniformity of the Left Ventricular Twisting Motion *Annals of Biomedical Engineering.* **20** (1992) 149-165.

[10] Hollinshead, and Rosse, Textbook of Anatomy, IV Edition. (Harper and Row, New York, 1985). p. 531.

[11] Gray H, Anatomy, Descriptive and Surgical. (Bounty Books, New York, 1977), p. 465.

[12] C. Peskin, Fiber Architecture of the Left Ventricular Wall, Asymptotic Analysis. *Comm. Pure and Appl. Math..* **42** (1989) 79-113.

[13] Green AE and Zerna W, Theoretical Elasticity. (Clarendon Press, Oxford, England, 1968) pp. 53-78.

[14] Guccione JM, McCulloch AD and Waldman LK, Passive Material Properties of Intact Ventricular Myocardium Determined for a Cylindrical Model. *J. Biomech. Eng..* **113** (1991) 42-55.

[15] Ritman EL, Heethaar RM, Robb RA and Pao YC, Finite Element Analysis of Myocardial Diastolic Stress and Strain Relationships in the Intact Heart. *Eur. J. Cardiol..* **7** (1978) 105-119.

[16] Guccione JM and McCulloch AD, Finite Element Modeling of Ventricular Mechanics. *In* Theory of Heart: Biomechanics, Biophysics and Nonlinear Dynamics of Cardiac Function. L. Glass, P. J. Hunter and A. D. McCulloch, editors. (Springer, New York, 1991) pp. 121-144.

[17] McPherson DD, Skorton DJ, Kodiyalam S, Petree L, Noel MP, Kieso R, Kerber RE, Collins SM and Chandran KB, Finite element analysis of myocardial diastolic function using three dimensional echocardiographic reconstructions: Application of a new method for study of acute ischemia in dogs. *Circ. Res..* **60** (1987) 647-682.

[18] Hamid MS, Sabbah HN and Stein PD, Determination of the left ventricular wall stress during isometric contraction using finite elements. *Comput. Structures.* **24** (1986) 589-594.

[19] Hunter PJ, McCulloch AD, Nielsen PMF and Smaill DH, A finite element model of passive ventricular mechanics. em In Computational Methods in Bioengineering. R. L. Spilker, and B. R. Simon, editors. vol. BED-9, (ASME, Chicago, 1988) pp. 387-397.

[20] McCulloch AD, Guccione JM, Rogers JM and Hunter PJ, Three-dimensional finite element analysis of stress and activation in the heart. *In* Mechanics Computing in 1990's and Beyond: Proceedings of the ASCE Engineering Mechanics Specialty Conference. (ASCE, New York, 1991) pp. 514-581.

[21] McCulloch AD and Hunter PJ, Finite element modeling of left ventricular mechanics. *In* Proceedings of the First World Congress of Biomechanics. (La Jolla, California, 1990) p. 32.

[22] Humphrey JD and Yin YCP, A New Constitutive Formulation in Characterizing the Mechanical Behavior of Soft Tissues. *Biophysical J.*. **52** (1987) 563-570.

[23] McCulloch AD and Omens JH, Nonhomogeneous analysis of three-dimensional transmural finite deformations in canine ventricular myocardium. *J. Biomechanics.* **24** (1991) 539-548.

[24] Guccione JM and McCulloch AD, Mechanics of active contraction in cardiac muscle, I. Constitutive relations for fiber stress that describe deactivation. *ASME J. Biomech. Eng.*. **115** No. 1 (1993) 72-81.

[25] Guccione JM and McCulloch AD, Mechanics of active contraction in cardiac muscle, II. Cylindrical models of the systolic left ventricle. *ASME J. Biomech. Eng.*. **115** No. 1, (1993) 82-90.

[26] Thomas CE, The Muscular Architecture of the Ventricles of Hog and Dog Hearts. *Am. J. Anatomy.* **101** (1957) 17-57.

[27] Humphrey JD, and Yin YCP, Constitutive Relations and Finite Deformations of Passive Cardiac Tissue II: Stress Analysis in the Left Ventricle. *Circ. Res.*. **65** (1989) 805-817.

[28] Spotnitz HM, Sonnenblick EH, and Spiro D, Relation of Ultrastructure to Function in the Intact Heart: Sarcomere Structure Relative to Pressure-Volume Curves of Intact Left Ventricles of Dog and Cat. *Circulation Research.* **18** (1966) 49.

[29] Stauffer J-C, and Gaasch WH, Recognition and Treatment of Left Ventricular Diastolic Dysfunction *Progress in Cardiovascular Diseases.* **32** No. 5 (1990) 319-332.

[30] Chuong CJ and Fung YC, Residual Stresses in Arteries. *In* Frontiers of Biomechanics. G. W. Schmid-Schonbein, S. L.-Y. Woo, and B. W. Zweifach, editors. (Springer-Verlag, New York. 1992) pp. 117-129.

[31] Omens JH and Fung YC, Residual Stress in the Left Ventricle. *In* Advances in Bioengineering. A. G. Erdman, editor. BED-Vol. 3, (ASME, 1987) pp. 37-38.

[32] Omens JH, MacKenna DA and McCulloch AD, Measurement of Strain and Analysis of Stress in Resting Rat Left Ventricular Myocardium. *J. Biomechanics.* **26** (1993) 665-676.

Figure Captions

Fig. 1 The ventricle approximated as a truncated conical shell.

Fig. 2 Circumferential stress-strain relation (σ_{22} vs. e_{22}) through the ventricular wall at $P_i = 20$ mm Hg for the spherical model, the conical model with twist and the conical model without twist. The lowest strains are at the epicardial surface and the highest strains are at the endocardial surface.

Fig. 3 Circumferential stress σ_{22} through the ventricular wall at $P_i = 20$ mm Hg.

Fig. 4 Circumferential strain e_{22} through the ventricular wall at $P_i = 20$ mm Hg.

Fig. 5 All stresses through the wall at $P_i = 20$ mm Hg. The thin solid curve represents σ_{11}. The thin dashed curve represents σ_{22}. The thin dotted curve represents σ_{33}. The thick solid curve represents σ_{12}. The thick dashed curve represents σ_{13}. The thick dotted curve represents σ_{23}.

Fig. 6 Myocardial Elastic stiffness vs. stress at $P_i = 20$ mm Hg. Elastic stiffness is defined as the slope of the stress-strain curve [29]. Myocardial stiffness constant, k_m, is defined as the slope of the relation between elastic stiffness and stress.

Fig. 7 Stretch ratios λ_r and λ_θ through the ventricular wall at pressure $P_i = 20$ mm Hg.

Fig. 8 Circumferential stress, σ_{22}, through the ventricular wall at pressure $P_i = 20$ mm Hg.

Fig. 9 Axial stress, σ_{33}, through the ventricular wall at pressure $P_i = 20$ mm Hg.

Fig. 10 Chamber stiffness constant, k_c, determined as the slope of the relation between pressure and chamber stiffness $,\frac{dP}{dV}$.