REVISION
Stresses and Strains in the Passive Left Ventricle

H. R. Chaudhry
Department of Mathematics
Center for Applied Mathematics and Statistics
New Jersey Institute of Technology
Newark, NJ 07102
Department of Physical Medicine and Rehabilitation
UMDNJ-New Jersey Medical School
Newark, NJ 07102
and
Research Department
Kessler Institute for Rehabilitation
West Orange, New Jersey 07052
B. Bukiet
Department of Mathematics
Center for Applied Mathematics and Statistics
New Jersey Institute of Technology
Newark, NJ 07102
T. Findley, Director
Research Department
Kessler Institute for Rehabilitation
West Orange, New Jersey 07052
Department of Physical Medicine and Rehabilitation
UMDNJ-New Jersey Medical School
Newark, NJ 07102
and
Department of Mathematics
Center for Applied Mathematics and Statistics
New Jersey Institute of Technology
Newark, NJ 07102
A. B. Ritter
Department of Physiology
UMDNJ – New Jersey Medical School
Newark, New Jersey 07103
(send all correspondence to B. Bukiet, Tel: (201)-596-8392, Fax: (201)-596-6467)

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Stresses and Strains in the Passive Left Ventricle

H. R. Chaudhry [1, 2] B. Bukiet [2] T. Findley [1, 2, 3] and A. B. Ritter [4]

(send all correspondence to B. Bukiet, Tel: (201)-596-8392, Fax: (201)-596-6467)

Abstract

In this paper, we estimate the stresses and strains from the equatorial region down to the apex of the heart by modeling the passive left ventricle as a frustrum of a thick hollow cone. Large deformation theory has been employed in this analysis. Furthermore, the effects of residual stresses and the anisotropy due to muscle fiber orientation have been included. It is observed that circumferential stress, which is the most important physiologically, decreases considerably at the endocardium and is more evenly distributed through the wall when residual stresses are taken into account. The stresses also decrease as we go from the equatorial region to the apex. Because heart muscles physically have residual stresses, the consequent lower stress gradient through the wall enhances the diastolic function of the left ventricle.

[1]Department of Physical Medicine and Rehabilitation, UMDNJ-New Jersey Medical School, Newark, NJ 07102

[2]Center for Applied Mathematics and Statistics, New Jersey Institute of Technology, Newark, NJ 07102

[3]Kessler Institute for Rehabilitation, West Orange, NJ 07052

[4]Department of Physiology, UMDNJ-New Jersey Medical School, Newark, NJ 07102

1 Introduction

During a single heart beat, coronary ventricles are subject to pressure and volume changes which induce changes in stress and strain within the ventricular wall. The distribution of stresses and strains in the ventricular wall during the diastolic phase of the cardiac cycle effects the diastolic filling function [29] and influences an important regulatory mechanism [16].

Preload is defined as the stress in the ventricular wall at the end of passive filling [10]. Ventricles respond reflexively to changes in preload by altering the force of contraction. This response is independent of neural input and is called "heterometric autoregulation" or more commonly, the Frank-Starling effect. The Frank-Starling effect allows ventricles to accommodate beat to beat variations in venous return (changes in ventricular deformation) without invoking autonomic reflex regulatory mechanisms. This mechanism is prevented from operating if the ventricular muscle is stretched beyond the optimal actin-myosin cross-bridge formation length (the sarcomere length-reserve is exhausted) or if the ventricular muscle cannot stretch adequately due to increased stiffness of the ventricular wall (e.g., ventricular hypertrophy) [16].

The stress which remains in the ventricle in the unloaded state is called the residual stress. Associated with this stress is the residual strain. Fung [5] has pointed out the significance of residual stresses and strains on the functioning of the arteries. These residual stresses and strains depend upon the opening angle ($2\pi - 2\Theta_0$) (See Fig. 1). The residual stresses also effect functioning of the left ventricle [24]. In particular, they influence diastolic filling time, heterometric autoregulation and coronary blood flow.

In the analysis of transmural stresses and strains in the ventricular wall, four major aspects must be taken into account. These are (1) the geometry of the left ventricle; (2) the anisotropy introduced by muscle fiber orientation in the heart; (3) the large deformations that occur in heart tissue and (4) the presence of residual stresses in the undeformed state. If these four aspects, coupled with material non-homogeneity and solid-fluid interaction are considered, the mathematics of the problem becomes intractable. For a cylindrical model of the left ventricle, Guccione, et. al. [9] found that "torsion, residual stress and material anisotropy associated with fiber architecture all can act to reduce endocardial stress gradients in the passive left ventricle."

In the finite element method approach see, (e.g. [8], [11]. [15], [18], [19], [20], [27]) many of these aspects have been taken into account. Humphrey and Yin [13] have noted that this approach is somewhat expensive due to the efforts involved in calculation and acquisition of complete data. Therefore, the closed form analytic solutions such as the one employed by Humphrey and Yin [13] (and the one presented in this paper) can continue to provide valuable insight into cardiac mechanics [13]. Both the analytical and the finite element approaches have their own advantages and disadvantages.

The geometry of the left ventricle is an important factor in accurate estimation of stresses and strains in the heart. Most researchers have employed thick-walled spherical and cylindrical geometries in order to avoid mathematical complexities. Although, the solution for a spherical shell is straightforward it is valid only for isotropic materials. Furthermore, spherical models require radial symmetry and do not closely approximate the shape of the heart. Assuming the ventricle to be cylindrical is also an oversimplification [9], [13], and solutions for such a model are valid only for a small portion of the heart, i.e., in the equatorial region. Prolate spheroids resemble the geometry of the left ventricle but the mathematics becomes intractable, particularly when the large deformations that actually occur in the heart are considered (see also [13]). Some researchers [6], [12], [26] have noted that the left ventricle can be described more accurately as conical with a circular cross-section rather than as spherical or cylindrical. Azhari, et. al. [1], noting that a cylindrical model of the left ventricle does not accurately predict the nonuniformity of its twisting motion, developed a conical model to help understand this nonuniformity. The conical model takes into consideration the gradual decrease of the endocardial and epicardial diameter and myocardial thickness from base to apex, which better resembles the actual left ventricle geometry (see also [1]). However, they [1] used linear elasticity theory and assumed that the ventricular wall material is isotropic. Apart from the above justifications for using a conical model, the mathematics of the conical geometry becomes tractable and covers almost the whole left ventricle, from the equatorial region down to the apex. In a recent paper, we used the conical model in the absence of residual stresses and for the isotropic case [2] and compared the results to those obtained using a spherical model. The result was that the conical model leads to more physiologically realistic results than the spherical model.

In this paper, we take into account all four aspects noted above. We use the frustrum of a hollow cone to approximate the left ventricle. We apply large deformation theory and include the effects of residual stresses as well as the influence of muscle fibers in the heart and extend the work of [13] and [2] to enable calculation of the stresses from the equatorial region down to the apex of the ventricle instead of only in the equatorial region. We apply our model using the polynomial form of the strain energy function proposed in [14].

In the present study, it is hypothesized that the presence of residual stresses in the passive left ventricle results in a lower stress gradient and a more uniform stress distribution across the ventricular wall than would be the case if the residual stresses were not present. Thus, the presence of residual stresses may increase the apparent compliance of the ventricle during diastole, allowing for a greater sarcomere-length reserve and enhancing the diastolic function of the left ventricle.

2 Conical Model of the Ventricle

We approximate the left ventricle of the canine as the frustrum of a conical shell with circular cross-section (see Fig. 2, reproduced from [1]) in which the thickness at a point is proportional to the radius of the endocardium at that point. We use circular cylindrical coordinates to describe the deformation. Since the $(R,\ Z)$ coordinates at the endocardial surface of the canine heart are not available, we use experimental data of $(R,\ Z)$ coordinates obtained by Azhari, et. al. (Table 2 of [1]) for the human heart. We assume for this purpose that the shape of the human and canine hearts are identical and find the corresponding points $(R,\ Z)$ of the canine heart from the given $(R,\ Z)$ points of the human heart. Azhari used the hollow conical model with decreasing thickness from base to apex to represent the left ventricle of the human heart. Using least square linear regression analysis, the equation of the cone fitting the given $(R,\ Z)$ coordinates at the endocardium is obtained as $R = 0.1428Z + 0.771$. Further, assuming that the thickness at any point of the endocardium is proportional to the radius at that point and scaling the human heart to that of the canine, such that the values of the radii at the equator are the same as in [13], we find that the internal and external radii at the equator, apex and midway between them are as follows:

$$\begin{array}{llll} R_i = 1.5\text{cm} & R_o = 2.55\text{cm} & r_i = 1.77\text{cm} & \text{At the equator} \\ R_i = 1.249\text{cm} & R_o = 2.124\text{cm} & r_i = 1.473\text{cm} & \text{At the midpoint} \\ R_i = 1.00\text{cm} & R_o = 1.70\text{cm} & r_i = 1.18\text{cm} & \text{At the apex} \end{array}$$

where R and r are the radii in the undeformed and deformed states, respectively (see appendix, equation (2)).

3 Results

Using the relevant equations from the Appendix, we present plots of stresses and strains through the wall for the polynomial strain energy function described by (37) in the Appendix. For all computations presented, we use the values of the parameters given in Table 1.

The results in the absence of residual stresses ($\alpha_1 = 1$, $\lambda = 1$, $b = 0$) in the equatorial region are presented in Figures 3 and 4. The effects of including residual stresses and strains (i.e., taking $\alpha_1 = 1.145$, $\lambda = 1.0075$, $b = 0.12^0$ per cm) in the equatorial region are presented in Figures 5 and 6. Since our analysis allows the computation of stresses and strains at any location, we present in Figures 7 and 8 the stresses and strains through the wall at the apex of the heart. In these plots, we include the effects of residual stresses and strains.

From these figures, we observe that circumferential stresses, stress gradients, strains and strain gradients decrease at the endocardium when residual stresses and strains are taken into account. The shape of the $t_{\theta\theta}$ curve changes noticeably and has a much wider and flatter hump when residual stresses are included. In the equatorial region, the circumferential stress $t_{\theta\theta}$, which is the most important physiologically, [5], [21], decreases

Table 1: Reported values of parameters used and their sources

Parameters	Source	Values Used
Λ	[13] Fig. 2	1.015
ψ	[13] Fig. 2	1.7^0 per cm
P_0	[13] Fig. 2	0.98 kPa
$\frac{dw}{dR}\vert_{R=R_i}$	[13] Fig. 2	0.2
$\frac{d\omega}{dR}\vert_{R=R_i}$	[13] Fig. 2	0.1
R_i	[13] Fig. 2	1.5 cm
R_o	[13] Fig. 2	2.55 cm
r_i	[13] Fig. 2	1.77 cm
Φ_0	[30] range -60^0 to -90^0	-80^0 as in [23]
c_1	[14], Table 1, p. 343	15.188 kPa (see also (37))
c_2	[14], Table 1, p. 343	73.432 kPa (see also (37))
c_3	[14], Table 1, p. 343	1.4016 kPa (see also (37))
c_4	[14], Table 1, p. 343	-19.482 kPa (see also (37))
c_5	[14], Table 1, p. 343	18.887 kPa (see also (37))
(R, Z) coordinates of the human heart	[1], Table 2	$(1.8, 7.2)$, $(1.6, 5.8)$, $(1.4, 4.4)$, $(1.2, 3.0)$
Θ_0	[22]	$\pi - 0.4$ radians
λ	this paper (Appendix)	1.0075
b	this paper (Appendix)	0.12^0 per cm

by 50% at the endocardium, increases by 91% at the epicardium and increases by 14% at the midwall (see Figs. 3 and 5). The circumferential strain ($E_{\theta\theta}$) curve also flattens considerably when residual stresses are included (see Figs. 4 and 6). We also note that circumferential stresses, stress gradients, strains and strain gradients decrease as one goes from the equator to the apex (see Figs. 5-8). t_{zz} and t_{rr} become negative at the endocardium and epicardium whether or not residual stresses are considered (see Figs. 3 and 5).

Finally, in Fig. 9, we present a plot of fiber stretch ratio, α through the wall with and without residual stresses and at different locations. It is interesting that all six of these curves have a common intersection point in the sub-endocardium region.

It has been verified for the exponential form of the strain energy function that our results for stresses and strains in the equatorial region in the absence of residual stresses are identical to those reported in [13] for a cylindrical model by taking $\Phi_0 = -90^0$ as in [13].

4 Discussion

(i) Our observation that the quantitative effect of residual stresses, torsion and material anisotropy associated with fiber architecture is to reduce the stress gradient in the equatorial region is identical to that of Guccione [9] (found by a different method). This confirms the validity of our conical model.

(ii) In the absence of results based on directly measured stresses through the ventricular wall, it is not possible to predict the accuracy of the evaluated stresses.

(iii) In the absence of residual stresses, the stresses and strains computed using the polynomial form of the strain energy function are similar qualitatively to those obtained by Humphrey and Yin [13] using the exponential form of the strain energy function.

(iv) It has been verified that the total twisting moment (the sum of the twisting moments produced by the shear stresses at the lateral surfaces and the apex) is larger at the endocardium than at the epicardium, in agreement with experimental observation [1]. This further confirms the validity of our conical model. We note that the conclusion in [1] p. 164-5 that the twisting moment in a cylindrical model at the epicardium is larger than at the endocardium is incorrect because the authors [1] ignored shear stresses [3]. However, the results of their conical model calculations [1] are in agreement with experimental observation.

(v) Since transmural stresses cannot be measured directly, the stresses evaluated in this paper can be useful to physiologists and experimentalists. Further, pathological conditions such as ischemic heart disease, ventricular hypertrophy and congestive heart failure which depend on wall stresses and myocardial stiffness [29], [21] can be better analyzed by cardiologists.

(vi) Physiologically, $t_{\theta\theta}$ is the most important stress component since it represents the wall tension at any time including the preload at end diastole. Fig. 5 shows the changes in distribution of $t_{\theta\theta}$ when residual stresses are present in comparison to Fig. 3 when residual stresses are not present. It is observed that $t_{\theta\theta}$ becomes more uniform with a decreased stress gradient. We conclude, therefore, that the residual stresses provide a greater reserve to accomodate additional stresses which might occur due to increased ventricular stiffness in diseased states.

(vii) We observe, from Figs. 3 and 5, that the computed radial stress t_{rr} changes very little (about 5%) at the inner wall from equator to the apex and not at all at the outer wall in the presence as well as absence of residual stresses. The uniformity of t_{rr} at the outer wall is based on the assumption that the external pressure P_0 on the epicardium by the pericardial sac is constant and has the value of 0.98 kPa. The small nonuniformity in P_i (i.e., t_{rr} at the endocardium) is expected in a conical model and also physiologically. It may be noted that if P_i is taken to be uniform, then P_0 will become nonuniform, as can be seen from (30).

(viii) We have taken the opening angle at the apex to be the same as at the equator because the actual opening angle at the apex is not known. This is likely to be different, in view of the observation made for arteries that opening angle varies along the arterial tree [17]. Therefore, our computed results at the apex may not be entirely correct.

(ix) It is observed from Table 1, of [14] that the range of estimated parameters c_1 through c_5 is very wide. That is, these values vary significantly for the various specimens. Therefore, the appropriate values of c_1 through c_5 can be chosen to approximate the computed stresses and strains with those observed physiologically for a given specimen.

(x) Finally, an interesting observation is that in the sub-endocardial region of the conical ventricular wall, the stretch ratio is independent of the existence of residual stresses and the location from the equator to the apex. This observation may be useful to cardiologists and experimentalists.

5 Conclusions

This study attempts to present a more realistic analysis of the effects of residual stresses and strains on the diastolic function of the left ventricle. The model incorporates a conical geometry with circular cross-section, anisotropy of muscle fiber orientation, large deformations, such as occur physiologically and the presence of residual stresses in the undeformed state. This model allows calculation of stresses and strains from the equatorial region down to the apex of the heart in comparison to the cylindrical model which allows calculations valid for the equatorial region only. The model yields results for transmural distributions of stresses and strains with and without residual stresses and strains and allows comparison between the two cases. As with earlier cylindrical models, [9], [13] and [22], the present analysis predicts that residual stresses and strains give rise to more uniform stress and strain distributions with smaller endocardial-epicardial gradients across the ventricular wall. The pressure (30.75 mm-Hg) at the endocardial surface calculated from the radial stress is within the physiological range for end diastolic pressure. The calculated mean value of the circumferential stress decreases from the equatorial region down to apex, in line with physiological expectation. The model predicts that the most physiologically important of the stress components, i.e., the circumferential stress, known also as the wall tension, has a smaller, more uniform endocardial-epicardial gradient in the presence of residual stress. In terms of the diastolic function of the left ventricle, the presence of residual stresses increases the apparent compliance of the ventricle and allows a greater sarcomere-length reserve. These factors allow the passive left ventricle to better accommodate for increases in ventricular stiffness caused by diseased states, such as hypertension induced hypertrophy.

6 Appendix (Kinematics and Constitutive Relations)

In this section, we present a derivation of the equations needed to calculate the stresses ($\mathbf{t}$) and strains ($\mathbf{E}$) through the ventricular wall. These stresses must satisfy the differential equations of equilibrium subject to prescribed boundary conditions..

In order to extend the work of [13], we employ the kinematics, constitutive relations and method of solution as given in [13] and [28]. We assume as in [13] that the noncontracting myocardium is incompressible and transversely isotropic with respect to the direction of the muscle fibers. We also include the effect of residual stresses in our analysis. Thus, we first consider the material particles of blood vessels in a stress-free configuration (State 0) (ρ, ϕ, ζ) to go to (R, Θ, Z) in an unloaded but residually stressed state (State 1) (see Fig. 1). Although for arteries, Chuong and Fung [4] (see their Eq. 5) assumed that the axial extension ratio $\lambda_z = 1$ (see their Results section) and ignored torsion, we do not make these assumptions for the left ventricle and assume that the deformation is given by

$$R = R(\rho); \quad \Theta = \alpha_1 \phi + b\zeta; \quad Z = \lambda \zeta \tag{1}$$

where $\alpha_1 = \pi/\Theta_0$ and Θ_0 is as shown in Fig. 1, b is the torsional constant and λ is the extension ratio in the axial direction. The relation between Z and ζ is taken to be linear as in ([22]) to simplify the already complicated mathematics, especially when the type of nonlinearity involved, if any, is not known. It may be noted here that ρ is also a function of Z implicitly, since R is a function of Z in our conical model (see Section 2).

Next we assume, consistent with physiological observation, that as a result of passive filling, the left ventricle experiences the following deformations from the unloaded to the loaded state:

1. A radial inflation depending on radial and axial position. It may, however, be noted that unlike a cylindrical model, the radial coordinate R is a function of the axial coordinate Z given by the equation $R = 0.1428Z + 0.771$ (see Section 2) for a conical model.

2. Simple torsion in which the planes perpendicular to the axis of the conical tube are rotated about that axis through an angle proportional to undeformed axial coordinate, the constant of proportionality being ψ.

3. An azimuthal shear of the conical tube in which each point moves about the longitudinal axis through an angle which is a function of radius R at that point, i.e., $\omega(R)$.

4. Extension with an extension ratio Λ denoting change in axial length.

5. Axial shear in which each point moves parallel to the longitudinal axis through a distance w which is a function of R, i.e. $\mathrm{w}(R)$.

Thus, the material particle in the undeformed state (R, Θ, Z) goes to (r, θ, z) in the deformed state (State 2) by the deformation given by

$$r = r(R); \quad \theta = \Theta + \psi Z + \omega(R); \quad z = \Lambda Z + \mathrm{w}(R) \tag{2}$$

where ψ is the twist per unit unloaded length, $\omega(R)$ is the radially dependent portion of circumferential displacement, Λ is an extension ratio and w(R) is a radially dependent part of axial displacement. As noted above, r is also a function of Z implicitly since R is a function of Z in our conical model (see Section 2). The values of ψ and Λ are taken from the experimental data [13] and $w(R)$ and $\omega(R)$ are to be determined from the differential equations of equilibrium and the boundary conditions. The physical components of the deformation gradient **F** (required to determine the stress-strain relation (9)) in cylindrical coordinates are given by ([28], p. 158)

$$F_{mM} = \begin{pmatrix} \partial r/\partial R & \partial r/R\partial\Theta & \partial r/\partial Z \\ r\partial\theta/\partial R & r\partial\theta/R\partial\Theta & r\partial\theta/\partial Z \\ \partial z/\partial R & \partial z/R\partial\Theta & \partial z/\partial Z \end{pmatrix} \tag{3}$$

The physical components of left and right Cauchy-Green deformation tensors $(\mathbf{B}, \mathbf{C})$ are

$$B_{mn} = F_{mM}F_{nM} \text{ and } \mathrm{C_{MN}} = \mathrm{F_{mM}F_{mN}} \text{ where m, n, M and N} = 1,2,3. \tag{4}$$

The Green strain measure **E** is given by

$$\mathbf{E} = 0.5(\mathbf{C} - \mathbf{I}) \tag{5}$$

where **I** is the identity tensor. The first principal strain invariant I_1 is given by

$$I_1 = tr(\mathbf{C}) = \mathbf{tr}(\mathbf{B}) \tag{6}$$

The stretch ratio α in the direction of muscle fibers needed for finding stresses from (9) is determined from the relation

$$\alpha^2 = \mathbf{N} \cdot \mathbf{C} \cdot \mathbf{N} = \mathbf{C_{MN}N_M N_N} \tag{7}$$

where N is a unit vector in the direction of a muscle fiber in the unloaded configuration and

$$[N_1, N_2, N_3] = [0, cos\Phi(R), sin\Phi(R)] \tag{8}$$

where $\Phi(R)$ describes the transmural distribution of muscle fibers through the heart wall. The Cauchy stress (force/ current area) for a material described by strain energy function $W = W(I_1, \alpha)$ is given by

$$\mathbf{t} = -p\mathbf{I} + 2W_1\mathbf{B} + (W_\alpha/\alpha)\mathbf{F} \cdot \mathbf{N} \text{ x } \mathbf{N} \cdot \mathbf{F}^T \tag{9}$$

where $\mathbf{t}$ is the Cauchy stress, p is the hydrostatic pressure for an incompressible material, $W_1 = \partial W/\partial I_1$, $W_\alpha = \partial W/\partial \alpha$, x denotes tensor product and the superscript T denotes transpose. The first two terms in (9) correspond to stresses obtained without considering fibers in the myocardium and the last terms represents the effect of including the fiber orientation. The differential equations of equilibrium in cylindrical coordinates in the absence of body forces are given by ([28], p. 160)

$$\begin{array}{lll} \frac{\partial}{\partial r}(t_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}t_{r\theta} + \frac{\partial}{\partial z}t_{rz} - \left(\frac{t_{\theta\theta} - t_{rr}}{r}\right) & = & 0 \\ \frac{\partial}{\partial r}(t_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}t_{\theta\theta} + \frac{\partial}{\partial z}t_{\theta z} + \frac{2t_{r\theta}}{r} & = & 0 \\ \frac{\partial}{\partial r}(t_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}t_{\theta z} + \frac{\partial}{\partial z}t_{zz} + \frac{t_{rz}}{r} & = & 0 \end{array} \tag{10}$$

The components of Cauchy stress, t^*_{mn}, in the direction of the muscle fibers are given by

$$t^*_{mn} = Q_{mi}Q_{nj}t_{ij} \tag{11}$$

where Q_{mi} are the direction cosines for the transformation.

The method of solution using the theory of elasticity consists of solving these partial differential equations after finding the stresses in terms of other unknown functions (in our case w(R) and $\omega(R)$) occurring in (9) subject to prescribed boundary conditions.

In this particular problem, the deformation described from the stress-free state to the unloaded state is given by (1) in which we need to know Θ_0 for the canine. This is obtained from [22] by writing $\Theta_0 = \pi - \delta$ where $\delta = 0.4$ radians is half of the opening angle of a thin equatorial slice cut from a dog's left ventricle. Thus, the value of $\pi/\Theta_0 = \alpha_1 = 1.145$. Now, using (1) and (3), we find ${}^R\mathbf{F}$, the residual deformation gradient, to be

$${}^R\mathbf{F} = \begin{pmatrix} dR/d\rho & 0 & 0 \\ 0 & \alpha_1 R/\rho & bR \\ 0 & 0 & \lambda \end{pmatrix} \tag{12}$$

The incompressibility condition $det\ \mathbf{F} = 1$ becomes

$$\frac{dR}{d\rho}\left(\frac{\alpha_1 R\lambda}{\rho}\right) = 1 \tag{13}$$

Now integrating (13) we obtain the relation between R and ρ given by

$$\lambda\alpha_1(R^2 - R_i^2) = \rho^2 - \rho_i^2 \quad \text{or} \quad \lambda\alpha_1(\mathrm{R_o^2} - \mathrm{R_i^2}) = \rho_{\mathrm{o}}^2 - \rho_{\mathrm{i}}^2 \tag{14}$$

where subscripts i and o denote the inner and outer radii, respectively.

Using (14), we express ρ_i, ρ_o in terms of R_i, R_o and q given by

$$\begin{aligned} \rho_i &= qR_i[\alpha_1\lambda(R_o^2 - R_i^2)/(R_o^2 - q^2R_i^2)]^{1/2} \\ \rho_o &= \frac{\rho_i}{q}\frac{R_o}{R_i} \end{aligned} \tag{15}$$

where $q = \frac{R_o}{R_i}/\frac{\rho_o}{\rho_i}$, is the ratio of external and internal radii in the unloaded and reference state. Then, using (12), (15) and (10) with boundary conditions consisting of vanishing load both at the epicardium and endocardium in the unloaded state 1, we find $\lambda = 1.0075$, $b = 0.12$ degree/cm and $q = 1.063$. This means that due to residual stresses, the unloaded cone becomes slightly longer ($\lambda = 1.0075$) and thicker ($q = 1.063$).

Now in order to evaluate the stresses in the loaded state, we use the deformation described by (2). Then using (2) and (3), the deformation gradient matrix, ${}^L\mathbf{F}$, for this specific deformation for the loaded state is given by

$${}^L\mathbf{F} = \begin{pmatrix} dr/dR & 0 & 0 \\ rd\omega/dR & r/R & r\psi \\ d\mathrm{w}/dR & 0 & \Lambda \end{pmatrix} \tag{16}$$

The deformation gradient tensor for the combined deformation of (12) and (16) is given by ${}^L\mathbf{F}\cdot{}^R\mathbf{F}$ [9]

$$\mathbf{F} = \begin{pmatrix} dr/d\rho & 0 & 0 \\ rd\omega/d\rho & \alpha_1 r/\rho & (\psi\lambda + b)r \\ d\mathrm{w}/d\rho & 0 & \Lambda\lambda \end{pmatrix} \tag{17}$$

The incompressibility condition, i.e., $det\ \mathbf{F} = 1$ becomes

$$\frac{dr}{d\rho} = \frac{\rho}{\alpha_1 r \Lambda \lambda} \tag{18}$$

Integrating (18), we find a relation between r and ρ

$$\alpha_1 \Lambda\lambda (r^2 - r_i^2) = (\rho^2 - \rho_i^2) \text{ or } \alpha_1 \Lambda\lambda (r_o^2 - r_i^2) = (\rho_o^2 - \rho_i^2) \tag{19}$$

Using (4) and (17), the left and right Cauchy-Green deformation tensor $(\mathbf{B}, \mathbf{C})$ are evaluated:

$$B_{mn} = \begin{pmatrix} (\frac{dr}{d\rho})^2 & \frac{dr}{d\rho}(r\frac{d\omega}{d\rho}) & \frac{dr}{d\rho}\frac{d\mathrm{w}}{d\rho} \\ \frac{dr}{d\rho}\left(r\frac{d\omega}{d\rho}\right) & \left(\frac{\alpha_1 r}{\rho}\right)^2 + \left(r\frac{d\omega}{d\rho}\right)^2 + (r\psi_1)^2 & r\psi_1\lambda\Lambda + r\frac{d\mathrm{w}}{d\rho}\frac{d\omega}{d\rho} \\ \frac{dr}{d\rho}\frac{d\mathrm{w}}{d\rho} & r\psi_1\lambda\Lambda + r\frac{d\mathrm{w}}{d\rho}\frac{d\omega}{d\rho} & \left(\frac{d\mathrm{w}}{d\rho}\right)^2 + (\lambda\Lambda)^2 \end{pmatrix} \tag{20}$$

$$C_{MN} = \begin{pmatrix} \left(\frac{dr}{d\rho}\right)^2 + \left(r\frac{d\omega}{d\rho}\right)^2 + \left(\frac{d\mathrm{w}}{d\rho}\right)^2 & \frac{\alpha_1 r^2}{\rho}\frac{d\omega}{d\rho} & \left(r^2\frac{d\omega}{d\rho}\psi_1\right) + \lambda\Lambda\frac{d\mathrm{w}}{d\rho} \\ \frac{\alpha_1 r^2}{\rho}\frac{d\omega}{d\rho} & \left(\frac{\alpha_1 r}{\rho}\right)^2 & \frac{\alpha_1\psi_1 r^2}{\rho} \\ \left(r^2\frac{d\omega}{d\rho}\psi_1\right) + \lambda\Lambda\frac{d\mathrm{w}}{d\rho} & \frac{\alpha_1\psi_1 r^2}{\rho} & r^2\psi_1^2 + (\lambda\Lambda)^2 \end{pmatrix} \tag{21}$$

where $\psi_1 = \psi\lambda + b$.

Using (21), (6), (7) and (8), we evaluate I_1 and α^2

$$I_1 = \left(\frac{dr}{d\rho}\right)^2 + \left(r\frac{d\omega}{d\rho}\right)^2 + \left(\frac{d\mathrm{w}}{d\rho}\right)^2 + \left(\frac{\alpha_1 r}{\rho}\right)^2 + (r\psi_1)^2 + (\lambda\Lambda)^2 \tag{22}$$

$$\alpha^2 = \left(\frac{\alpha_1 r}{\rho}\right)^2 cos^2(\Phi(R)) + 2\left(\frac{\alpha_1 r^2\psi_1}{\rho}\right) sin(\Phi(R))cos(\Phi(R)) + \left[r^2\psi_1^2 + (\lambda\Lambda)^2\right] sin^2(\Phi(R)) \tag{23}$$

where $\Phi(R)$ is given by [13]

$$\Phi(R) = \Phi_0\left(\frac{2R - (R_o + R_i)}{R_o - R_i}\right)^3 \tag{24}$$

Here, Φ_0 is the value of Φ at the epicardium and varies from -60^0 to -90^0 and R_i and R_o denote unloaded endocardial and epicardial radii respectively. Using (17), (20) and (9), the Cauchy stresses become

$$
\begin{aligned}
t_{rr} &= -p(r) + 2W_1\left(\tfrac{dr}{d\rho}\right)^2 \\
t_{\theta\theta} &= -p(r) + 2W_1\left[\left(\tfrac{\alpha_1 r}{\rho}\right)^2 + \left(r\tfrac{d\omega}{d\rho}\right)^2 + r^2\psi_1^2\right] + \tfrac{W_\alpha}{\alpha}\left[\alpha^2 - (\lambda\Lambda)^2 sin^2\Phi\right] \\
t_{zz} &= -p(r) + 2W_1\left[\left(\tfrac{d\mathrm{w}}{d\rho}\right)^2 + (\lambda\Lambda)^2\right] + \tfrac{W_\alpha}{\alpha}\left[(\lambda\Lambda)^2 sin^2\Phi\right] \\
t_{r\theta} &= 2W_1 r\tfrac{dr}{d\rho}\tfrac{d\omega}{d\rho} \\
t_{rz} &= 2W_1\tfrac{dr}{d\rho}\tfrac{d\mathrm{w}}{d\rho} \\
t_{\theta z} &= 2W_1\left[r\psi_1\lambda\Lambda + r\tfrac{d\mathrm{w}}{d\rho}\tfrac{d\omega}{d\rho}\right] + \tfrac{W_\alpha}{\alpha}\left[\lambda\Lambda r\psi_1 sin^2\Phi + \tfrac{\alpha_1\lambda\Lambda r}{\rho} sin(\Phi)cos(\Phi)\right]
\end{aligned}
\tag{25}
$$

Equations of Equilibrium and Boundary Conditions

Using (25), and (10), the equilibrium equations reduce to

$$
\begin{aligned}
\tfrac{d}{dr}(t_{rr}) - \left(\tfrac{t_{\theta\theta} - t_{rr}}{r}\right) &= 0 \\
\tfrac{d}{dr}(t_{r\theta}) + \tfrac{2t_{r\theta}}{r} &= 0 \\
\tfrac{d}{dr}(t_{rz}) + \tfrac{t_{rz}}{r} &= 0
\end{aligned}
\tag{26}
$$

Integrating the second and third of (26), we have

$$
\begin{aligned}
r^2 t_{r\theta} &= H(\text{a constant}) \\
r t_{rz} &= G(\text{a constant})
\end{aligned}
\tag{27}
$$

Integrating the first of (26), we get

$$
p = 2W_1\left(\frac{dr}{d\rho}\right)^2 - \int_{r_i}^{r} Q(r)dr - K \tag{28}
$$

where K is the constant of integration and

$$Q(r) = \frac{t_{\theta\theta} - t_{rr}}{r} \tag{29}$$

Using the boundary conditions at any cross-section, i.e., $t_{rr}(r_i) = -P_i$ and $t_{rr}(r_o) = -P_o$, where P_i and P_o are pressures at the inner and outer surfaces, respectively, we find that

$$P_i - P_o = \int_{r_i}^{r_o} Q(r)dr \tag{30}$$

$$p(r) = 2W_1 \left(\frac{dr}{d\rho}\right)^2 + \int_r^{r_o} Q(r)dr + P_o \tag{31}$$

The Cauchy stresses given by (25) are thus completely determined using (31).

In order to find quantitatively the stresses, we need to know $\frac{d\omega}{d\rho}$ and $\frac{d\mathrm{w}}{d\rho}$. We use the values $\frac{d\mathrm{w}}{dR} = 0.2$ and $\frac{d\omega}{dR} = 0.1$ at R_i [13]. For this purpose, using (13), we find

$$\frac{d\omega}{d\rho}|_{\rho_i} = \frac{.1}{1.145}\frac{\rho_i}{R_i\lambda} \quad \text{and} \quad \frac{d\mathrm{w}}{d\rho}|_{\rho_i} = \frac{.2}{1.145}\frac{\rho_i}{R_i\lambda} \tag{32}$$

We now use the method proposed by Humphrey and Yin [13] to evaluate $\frac{d\mathrm{w}}{d\rho}$ and $\frac{d\omega}{d\rho}$ at any radial location. Now using (27) and equations (25), we find that

$$\begin{aligned} (2W_1\mathrm{w}'r'r)|_{\rho_i} &= G \\ (2W_1\omega'r'r^3)|_{\rho_i} &= H \end{aligned} \tag{33}$$

where from (18), $r' = \frac{dr}{d\rho} = \frac{\rho}{\alpha_1\lambda\Lambda r}$; $\mathrm{w}' = \frac{d\mathrm{w}}{d\rho}$ and $\omega' = \frac{d\omega}{d\rho}$.

Using (32) in (33), the values of G and H become known. From (33), we find that

$$r\omega' = \left(\frac{H}{Gr}\right)\mathrm{w}' \tag{34}$$

Using (34), in (22), I_1 becomes

$$I_1 = r'^2 + \left(\frac{\alpha_1 r}{\rho}\right)^2 + r^2\psi_1^2 + (\lambda\Lambda)^2 + \left[1 + \left(\frac{H}{Gr}\right)^2\right]x^2 \tag{35}$$

where $x = \mathrm{w}'$. Then the first of (33) becomes

$$f(x) = 2W_1 r r' x - G = 0 \tag{36}$$

where $\frac{df}{dx} = 2rr'(W_1 + x dW_1/dx)$.

Now we use a polynomial form for the strain energy function $W(I_1, \alpha)$ [14], valid for the canine left ventricle, i.e.,.

$$W = c_1(\alpha - 1)^2 + c_2(\alpha - 1)^3 + c_3(I_1 - 3) + c_4(I_1 - 3)(\alpha - 1) + c_5(I_1 - 3)^2 \tag{37}$$

with material parameter values $c_1 = 15.188$, $c_2 = 73.432$, $c_3 = 1.4016$, $c_4 = -19.482$ and $c_5 = 18.887$ all in kPa found by averaging the specimens of Table 1 of [14] that satisfy the required inequality condition. We note that

$$dW_1/dx = 4c_5[1 + (H/Gr)^2]x \tag{38}$$

Now using the Newton-Raphson method, we solve the uncoupled nonlinear algebraic equation (36) for x, i.e., w', at any radial location. Then, ω' is determined from (34). With known values of w' and ω', the stresses given by the (25) can be evaluated at any location. The strains can be evaluated using (5) and (21).

The externally applied loads to maintain this deformation can be determined. For example, the axial load L and moment M at the apex are given by the equations

$$\begin{aligned} L &= 2\pi \int_{r_i}^{r_o} (t_{zz})\ r\ dr - P_i \pi r_i^2 + P_o \pi r_o^2 \\ M &= 2\pi \int_{r_i}^{r_o} (r^2\ t_{\theta z})\ dr \end{aligned} \tag{39}$$

The non-vanishing stresses $(t_{r\theta},\ t_{rz})$ acting on both the surfaces are equivalent to the following forces (See [7], p. 94-99 for a similar situation).

1. An axial couple M_0 acting per unit length of the deformed cone on each of the two surfaces. For example, M_0 at any point $r = r_0$ on the outer surface is given by $M_0 = 2\pi r_0^2 (t_{r\theta})_{r=r_0}$.

2. A longitudinal force per unit length of the deformed cone acting on each of the two surfaces. For example, L_0 at any point $r = r_0$ becomes $L_0 = 2\pi r_0 (t_{rz})_{r=r_0}$.

These forces are required even in a cylindrical model due to assumption of azimuthal and axial shear $\omega(R)$ and $w(R)$, respectively. (See [13], p. 814-5). These are ignored in [22]. We note that these forces are required to maintain the prescribed deformation, although they cannot be measured. Therefore, they are not prescribed, but they do exist. These are supplied by the tissues in contact with the two surfaces. For example, at the epicardium, these are supplied by the pericardium and right ventricle [13] because heart muscle is not in an isolated state.

The components of stresses in the directions of muscle fibers can be evaluated by using (11).

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Figure Captions

Fig. 1 Cross-sectional representation of left ventricle at the stress free reference state under the hypothesis that the unloaded intact left ventricle has residual stresses (State 0); for the hypothesis that the unloaded intact left ventricle is stress free (State 1); and for the subsequent loaded state under transmural pressure (State 2).

Fig. 2 The ventricle approximated as a truncated conical shell.

Fig. 3 Stresses through the wall with polynomial strain energy function, without residual stresses, in the equatorial region of the canine heart. The thin solid line represents t_{rr}. The thick solid line represents $t_{\theta\theta}$. The thin dashed line represents t_{zz}. The thick dashed line represents $t_{r\theta}$. The thin dotted line represents t_{rz}. The thick dotted line represents $t_{\theta z}$.

Fig. 4 Strains through the wall with polynomial strain energy function, without residual stresses, in the equatorial region of the canine heart. The thin solid line represents E_{rr}. The thick solid line represents $E_{\theta\theta}$. The thin dashed line represents E_{zz}. The thick dashed line represents $E_{r\theta}$. The thin dotted line represents E_{rz}. The thick dotted line represents $E_{\theta z}$.

Fig. 5 Stresses through the wall with polynomial strain energy function, with residual stresses, in the equatorial region of the canine heart.

Fig. 6 Strains through the wall with polynomial strain energy function, with residual stresses, in the equatorial region of the canine heart.

Fig. 7 Stresses through the wall with polynomial strain energy function, with residual stresses, at the apex of the canine heart.

Fig. 8 Strains through the wall with polynomial strain energy function, with residual stresses, at the apex of the canine heart.

Fig. 9 Stretch ratio, α through the wall with and without residual stresses and at different locations. The thin solid curve represents α in the equatorial region, without considering residual stresses. The thick solid curve represents α in the equatorial region, considering residual stresses. The thin short-dashed curve represents α in the midway between the equator and the apex, without considering residual stresses. The thick short-dashed curve represents α in the midway between the equator and the apex, considering residual stresses. The thin dotted curve represents α at the apex, without considering residual stresses. The thick dotted curve represents α at the apex, considering residual stresses. (We note that the curves without residual stresses at the base, apex and midway between them are virtually identical).