Dynamic Stresses and Strains in the Left Ventricular Wall based on Large Deformation Theory

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Abstract

In this paper, elastodynamic large deformation theory is applied to calculate dynamic stresses and strains in the left ventricular wall. Researchers usually neglect inertial forces and consider the wall to be thin. In this work, we do *not* make these simplifying assumptions, although we consider the ventricle to be spherical. We examine the influence the inertial term has on stresses, strains, stretch ratios, wall thickness and myocardial stiffness, which we have computed using available data for the heart of a dog. During one complete cardiac cycle, wall thickness, stretch ratios and stress at the inner wall are, of course, constant when using a static analysis and vary in time using a dynamic analysis. The effect of the inertia term on the difference between static and dynamic circumferential stress is a maximum (1%) at the endocardium (inner wall) and zero at the epicardium (outer wall).

1 Introduction

Accurate evaluation of stresses and strains in the left ventricular wall is important in understanding myocardial oxygen supply and demand. For example, as stresses increase, strain energy increases, resulting in an increase in the energy and consequently oxygen demand [1]. The increased stresses also reduce the oxygen supply by collapsing the intraventricular arteries and thus causing flow resistance [2]. In addition, an increase in tension or stress and the duration of such tension within the left ventricular wall is responsible for a decrease in cardiac efficiency.

The magnitude of strains is needed to determine the elastic stiffness of the ventricular wall [3]. It is essential in determining how far the cardiac muscle shortens during contraction of the left ventricle. It also can lead to a better understanding of the stroke volume in normal and diseased patients [4]. The pathophysiology of the myocardium undergoing hypertrophy or aneurysm formation may be better defined using dynamic stress and strain information.

Traditionally, stresses and strains in the left ventricular wall were evaluated using Hooke's Law or the Laplace Law under the assumptions that deformations are small and the thickness of the wall is also small [5, 6, 7, 8, 9]. However, the use of the simple Laplace law to describe such thick-walled structures as the left ventricle is not appropriate [5]. A more complex description is required.

Mirsky [10], was the first to apply large deformation elasticity theory (modeling the ventricle as a thick-walled structure) to this problem. He found that the stresses at the endocardial surface calculated using this theory and assuming a spherical shape for the left ventricle were ten times as high as the values calculated using the Laplace Law assumption. The wall thickness was not assumed to be small. However, the material was assumed to be isotropic and homogeneous.

Many other studies relevant to investigation of stresses in the left ventricular wall have been published [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. Some of these authors have employed the finite element method successfully for three-dimensional analysis of stress distribution in the left ventricular wall. The effects of anisotropy due to muscle fiber orientation on the wall stresses have also been examined.

All the aforementioned analyses of stresses and strains were developed considering the ventricular wall to be in static or quasi-static equilibrium, i.e., only measurements at a particular time (for the geometry and left ventricular pressure) are used in a static analysis. The influence of inertial forces and the motion of the myocardial surface on stresses and strains has been ignored assuming the effect to be small. Since, in reality, pressures are oscillatory during both expansion and contraction of the left ventricle, the deformation of the endocardial and epicardial surfaces and the consequential stresses and strains so produced are also oscillatory. Much experimental work has been performed concerning force-velocity-length-time relations of contractile material in heart muscle [25], [26], [27], [28], [29] but the effect of the inertial term has been ignored. This paper extends the work of Mirsky using large deformation theory [10] by incorporating inertial forces to examine how small the influence of inertia is. In this way, we can quantify the influence of the inertial forces and also calculate a variety of time dependent parameters describing the mechanics of the heart. Such *dynamic* stresses (i.e., stress vs. time) are needed to evaluate preload (end diastolic stress) and afterload (wall stress at the aortic valve opening or peak systolic stress) of ventricular muscle ([30], p. 505). Dynamic stresses are also useful in the evaluation of force-velocity relations as relates to myocardial oxygen consumption and in the assessment of muscle elasticity [30].

In this paper, we develop a method for calculating the dynamic stresses and strains in the left ventricular wall of a dog using large deformation analysis and without considering the wall thickness to be small. Although it is known that the ventricle is not homogeneous and isotropic, we neglect the anisotropy and nonhomogeneity of the wall material to make the mathematics tractable and focus mainly on the effect of inertia on the ventricle. We also calculate stretch ratios, time dependent wall thickness and myocardial elastic stiffness. During one complete cardiac cycle, wall thickness, stretch ratios and stress at the inner wall vary in time using a dynamic analysis and are, of course, constant when using a static analysis. We also observe that the effect of the inertia term on the difference between static and dynamic circumferential stress is very small, a maximum of 1%.

2 Method

We approximate the left ventricle as a thick spherical shell made of an incompressible, homogenous, isotropic, elastic material. We assume that the ventricular wall material is isotropic, homogeneous and spherical in order to examine quantitatively the effect of interia terms on the myocardial properties. The assumption of a spherical shape has been made to reduce the mathematical complexity. We also assume that as a result of time-varying forces at the endocardial (inner) surface, the oscillations of the ventricular wall are spherically symmetric. The inner and outer radii of the shell in the undeformed state are denoted by r_1 and r_2, respectively. The epicardial (outer) surface is considered to be free of applied pressure and the endocardial surface is subjected to blood pressure

$P(t)$ where t is time. We identify $(\theta_1,\ \theta_2,\ \theta_3)$ of standard elastic large deformation theory notation (see appendix, equation (23)) with the spherical coordinates $(R(r,t),\ \theta,\ \phi)$ in the deformed state and with the coordinates (r, θ, ϕ) in the undeformed state, so that the rectangular coordinates (y_1, y_2, y_3) and (x_1, x_2, x_3) become

$$\begin{aligned} y_1 &= R(r,t)sin\theta cos\phi, \quad y_2 = R(r,t)sin\theta sin\phi, \quad y_3 = R(r,t)cos\theta \\ x_1 &= rsin\theta cos\phi, \qquad x_2 = rsin\theta sin\phi, \qquad x_3 = rcos\theta \end{aligned} \tag{1}$$

We further assume that a shell element is distorted by a displacement which is a function of R and t, where R is the radial coordinate in the deformed state of any material point r. Thus, any point $(R(r,t), \theta, \phi)$ was originally at the point $(R(r,t)Q(R,t), \theta, \phi)$, i.e., $r = R(r,t)Q(R,t)$. In view of this, equation (1) can be written as

$$\begin{aligned} y_1 &= R(r,t)sin\theta cos\phi, \quad y_2 = R(r,t)sin\theta sin\phi, \quad y_3 = R(r,t)cos\theta \\ x_1 &= R(r,t)Q(R,t)sin\theta cos\phi, \quad x_2 = R(r,t)Q(R,t)sin\theta sin\phi, \\ x_3 &= R(r,t)Q(R,t)cos\theta \end{aligned} \tag{2}$$

From here on, we shall simplify the notation for $R(r,t)$ and $Q(R,t)$ as simply R and Q, respectively. The stresses and strains in a spherical shell under static pressure have been evaluated using large deformation theory [31]. We, therefore omit the details of the kinematics and convert the equilibrium equations into equations of motion (30, in the appendix) by treating all the variables as functions of time. We present only those results useful for our dynamic analysis.

That the ventricular wall is incompressible implies that

$$R^3 - r^3 \;=\; R_1^3 - r_1^3 \;=\; R_2^3 - r_2^3 \tag{3}$$

or

$$R^3 \;=\; r^3 \;+\; R_1^3 - r_1^3 \tag{4}$$

Thus,

$$Q(R,t) \;=\; \frac{r}{R} \;=\; \left[1 \;+\; \frac{r_1^3 - R_1^3}{R^3}\right]^{1/3}; \qquad \frac{\partial Q}{\partial R} \;=\; \frac{1}{R}\left(\frac{1}{Q^2} - Q\right) \tag{5}$$

The incompressibility condition also reduces the metric tensors to (see (24), in Appendix)

$$g_{ij} \;=\; \begin{pmatrix} Q^{-4} & 0 & 0 \\ 0 & (RQ)^2 & 0 \\ 0 & 0 & (RQsin\theta)^2 \end{pmatrix}; \quad g^{ij} \;=\; \begin{pmatrix} Q^4 & 0 & 0 \\ 0 & (RQ)^{-2} & 0 \\ 0 & 0 & (RQsin\theta)^{-2} \end{pmatrix} \tag{6}$$

$$G_{ij} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & (Rsin\theta)^2 \end{pmatrix}; \quad G^{ij} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^{-2} & 0 \\ 0 & 0 & (Rsin\theta)^{-2} \end{pmatrix}; \tag{7}$$

In our present analysis, the strain energy is assumed to be a function of the strain invariant I_1 only (see (25), in Appendix and [10], p. 143), where

$$I_1 \;=\; Q^4 \;+\; \frac{2}{Q^2} \;=\; \left(\frac{r}{R}\right)^4 \;+\; 2\left(\frac{R}{r}\right)^2 \tag{8}$$

In what follows, I_1 will be denoted by I. Using (27), (28), (31), (32) and (6) in (30) see Appendix and setting $F^i \;=\; 0$ in the absence of body forces, then the only equation of motion is given by

$$\frac{\partial}{\partial R}\left[p \;+\; 2\left(\frac{r}{R}\right)^4 \frac{\partial W}{\partial I}\right] \;+\; \frac{4}{R}\left[\left(\frac{r}{R}\right)^4 \;-\; \left(\frac{R}{r}\right)^2\right]\frac{\partial W}{\partial I} \;=\; -\rho \ddot{R} \tag{9}$$

where $\ddot{R} = \frac{d^2R}{dt^2}$. The other equations of motion are satisfied identically if $p = p(R,t)$, where p is hydrostatic pressure. See [31], p. 104 for these calculations.

At this point, we note that the acceleration computed from (4) has a potential [32]

$$\zeta = R^{-1}(2R_1\dot{R}_1^{\,2} + R_1^2\ddot{R}_1) - \frac{1}{2}R^{-4}R_1^4\dot{R}_1^{\,2} \tag{10}$$

such that

$$\ddot{R} = -\frac{\partial \zeta}{\partial R} \tag{11}$$

Thus, (9) can be rewritten as

$$\frac{\partial}{\partial R}\left[p + 2\left(\frac{r}{R}\right)^4\frac{\partial W}{\partial I}\right] + \frac{4}{R}\left[\left(\frac{r}{R}\right)^4 - \left(\frac{R}{r}\right)^2\right]\frac{\partial W}{\partial I} = -\rho\frac{\partial \zeta}{\partial R} \tag{12}$$

Integrating this equation and using the boundary conditions (see (33), in Appendix), the physical components of the stresses become

$$\begin{aligned}
\sigma_{RR} &= -\rho(\zeta - \zeta_2) + 4\int_{R_2}^{R}\left[\left(\frac{R}{r}\right)^2 - \left(\frac{r}{R}\right)^4\right]\frac{\partial W}{\partial I}\frac{\partial R}{R} \\
\sigma_{\theta\theta} &= 2\frac{\partial W}{\partial I}\left[\left(\frac{R}{r}\right)^2 - \left(\frac{r}{R}\right)^4\right] + \sigma_{RR}
\end{aligned} \tag{13}$$

Using (5), the stresses can be expressed in the form

$$\begin{aligned}
\sigma_{RR} &= -\rho(\zeta - \zeta_2) + 4\left[\int_{Q_1}^{Q}\left(1 + Q^3\right)\frac{\partial W}{\partial I}dQ - \int_{Q_1}^{Q_2}\left(1 + Q^3\right)\frac{\partial W}{\partial I}dQ\right] \\
\sigma_{\theta\theta} &= 2\frac{\partial W}{\partial I}\left[\frac{1}{Q^2} - Q^4\right] + \sigma_{RR}
\end{aligned} \tag{14}$$

We note that these stresses reduce to those obtained by Mirsky [10] for the static case.

The inner surface force $P(t)$ satisfies the relation

$$P(t) = \rho(\zeta_1 - \zeta_2) + 4\int_{Q_1}^{Q_2}\left(1 + Q^3\right)dQ\frac{\partial W}{\partial I} \tag{15}$$

One could derive an equation for the motion of the ventricular wall using the method described in [32], letting $x(t) = \frac{R_1(t)}{r_1}$, and $\gamma = \left(\frac{r_2}{r_1}\right)^3 - 1$ and using (4), to obtain from equation (15)

$$2x\ddot{x}\left(1 - \left[1 + \frac{\gamma}{x^3}\right]^{-1/3}\right) + \dot{x}^2\left(3 - 3\left[1 + \frac{\gamma}{x^3}\right]^{-1/3} - \frac{\gamma}{x^3}\left[1 + \frac{\gamma}{x^3}\right]^{-4/3}\right) + f(x,\gamma) = \frac{2}{\rho r_1^2}P(t) \tag{16}$$

where

$$f(x,\gamma) = \frac{8}{\rho r_1^2}\int_{Q_1}^{Q_2}\left(1 + Q^3\right)\frac{\partial W}{\partial I}dQ \tag{17}$$

The initial conditions for (16) are

$$x(0) = \frac{R_1(0)}{r_1} \quad \text{and} \quad \dot{x}(0) = 0 \tag{18}$$

Equation (16) can then be solved in conjunction with these initial conditions to determine $x(t)$ and therefore $R_1(t)$ for a given r. Then $R(r,t)$ can be determined for any r from (4) and the radial displacement is given by $u_r = R - r$. However, since experimental data for volume vs. time is known, $R_1(t)$ can be calculated. In the following section, we demonstrate how to use $R_1(t)$ and equation (14) to calculate stresses and strains in the ventricular wall at any time t. One then can use (15) to find $P(t)$.

3 Results

To determine quantitatively the time-dependent stresses $\sigma_{\theta\theta}$ (see equation (14)) of the ventricular wall, we need to obtain $\frac{\partial W}{\partial I}$. To obtain $\frac{\partial W}{\partial I}$ we use the (static) pressure-radius data for the left ventricle of a dog, from [33], see Table 1. This data was taken from fresh hearts which were removed from 27 anesthetized dogs and averaged.

Table 1: Pressure - Volume - Radius Data for the Left Ventricle of the Dogs

Pressure *(mm Hg)*	**Volume** *ml*	R_1 *cm*	R_m *cm*	R_2 *cm*	I_M
0	12.0	1.42	2.21	2.99	
5	31.5	1.96	2.56	3.15	3.56
10	40.0	2.12	2.67	3.22	3.86
15	46.7	2.23	2.74	3.27	4.1
20	52.0	2.31	2.80	3.31	4.28
25	56.5	2.38	2.85	3.34	4.44
30	60.0	2.43	.290	3.37	4.56

Assuming $\frac{\partial W}{\partial I}$ to be constant and equal to the value at the middle of the range of integration in equation (15) (with $\zeta_1 = \zeta_2 = 0$) as in [10], we can evaluate $\frac{\partial W}{\partial I}$ using Table 1 for various values of I_M. This assumptions yields a value for pressure P to within 10% [10]. Then, we use this data to obtain the best fit for $\frac{\partial W}{\partial I}$ of the form

$$\frac{\partial W}{\partial I} = A + Be^{CI} \tag{19}$$

The best fit is obtained (using least squares) as

$$\frac{\partial W}{\partial I} = -3.28 + 0.132e^{1.108\ I} \tag{20}$$

Now, from the pulse rate, which we take to be 110 beats per minute, we approximate the volume function $V(t)$ based on experimental data [34, 35] as sinusoidal, i.e. we let

$$V(t) = \frac{V_{min} + V_{max}}{2} + \frac{V_{max} - V_{min}}{2} cos(2\pi\omega\ t/60) \tag{21}$$

where ω is the pulse rate in beats per minute and t is in seconds. V_{max} denotes the maximum volume (i.e., end diastole) and V_{min} demotes the minimum volume (i.e. end systole). In our sample computations described below, we take V_{max} to be 75 ml and

V_{min} to be 37.5 ml. The end of diastole occurs at $t = 0$. Taking derivatives and using $V = \frac{4\pi}{3}R^3$ yields the following expressions for R, $\dot{R}$ and $\ddot{R}$.

$$R = \left[\frac{3V}{4\pi}\right]^{1/3}; \qquad \dot{R} = \frac{dV}{dt}\frac{1}{4\pi R^2}; \qquad \ddot{R} = \frac{d^2V}{dt^2}\frac{1}{4\pi R^2} - 2\frac{\dot{R}^2}{R} \tag{22}$$

Substituting (21) into (22) and the result into equation (10) gives ζ, and therefore ζ_1 and ζ_2. Using ζ, ζ_2 and $\frac{\partial W}{\partial I}$ (from (20)) in (14), we calculate the stresses in the ventricular wall at any time, t. Equations (6), (7) and (35) yield the strains, while (37) gives the stretch ratios and (22) is used to calculate the wall thickness.

4 Conclusion

Computations were performed to determine the effect of inertial forces on stresses, strains, myocardial elastic stiffness, stretch ratios and wall thickness. In addition, the time dependent behavior of stress, wall thickness and stretch ratios were studied. For the static computations, the volume is set to V_{max}, whereas, in the dynamic computations the volume in equation (21) is used.

In Fig. 1, we present the nonlinear (circumferential) stress-strain relation through the left ventricular wall at the end of diastole for the dynamic and static cases. The curves are almost identical. Because the lowest stresses and strains occur at the epicardial surface and the highest stresses and strains occur at the endocardial surface, the curve goes from the epicardial to the endocardial surface from left to right. The elastic stiffness is defined as the slope of the stress-strain relation. Elastic stiffness versus stress is plotted in Fig. 2. The myocardial stiffness constant, k_m, can be determined as the slope of the relation between elastic stiffness and stress [3]. In Fig. 3, we present a plot of the stretch ratios λ_r

and λ_θ in the radial and circumferential directions, respectively, through the wall at the end of diastole. Stretch ratio is defined as the stretched length divided by the unstretched length.

In Fig. 4, the time course of circumferential stress at the endocardial surface is presented through one period (one heartbeat). For the dynamic case, it is a minimum at the end of systole and a maximum at the end of diastole. For the static case, of course, this stress does not vary. In Fig. 5, the time course of wall thickness is shown for one period. The result is in qualitative agreement with experiment [36]. For the static case the wall thickness does not vary.

The variation of circumferential stress through the wall of the ventricle at the end of diastole is presented in Fig. 6 for the dynamic case. This reduces to the results of Mirsky [10] for the static case. The high stress concentration at the endocardial surface may be reduced by considering residual stresses in the "reference" (zero pressure but intact) ventricle [37], [38]. However, experimental data that would make possible calculation of an appropriate strain energy function for the residual stress model is not available. The difference between the circumferential dynamic and static stress through the ventricular wall is presented in Fig. 7. The difference (although small, about 1%, compared to the total stresses shown in Fig. 6) is a maximum at the endocardial surface and vanishes at the epicardial surface. Finally, in Fig. 8, we present the time course of stretch ratios λ_r and λ_θ. For the static case ($V = V_{max}$) the stretch ratios remain constant while in the dynamic case the circumferential stretch ratio is a minimum at the end of systole, the radial stretch ratio is minimized at the end of diastole and maximized at the end of systole.

5 Discussion

We have presented a method for the computation of time dependent parameters in the heart including dynamic stresses and strains, stretch ratios and wall thickness in the left ventricle by including inertial forces and applying large deformation theory. Although we have included results for both the diastolic and systolic phases of the heart beat, we note that the activation strain present during systole has not been taken into account in our analysis. In addition, the data used includes pressures up to only 30 mm-Hg so results at higher pressures may not be valid. The present analysis does not assume that the wall of the ventricle is thin. Spherical geometry was assumed in order to reduce the mathematical complexity. It is found that the influence of inertial forces is small. The dynamic analysis presented in this paper models the time dependence of wall thickness, stretch ratios and stress at the inner wall during a cardiac cycle.

It is hoped that the dynamic stress and strain model presented in this paper will form a basis for future studies of diastolic and systolic function by clinical cardiologists for normal and diseased humans. This work can be further extended by considering the left ventricle as a conical shell, which of course greatly increases the mathematical complexity. We hope to address this issue in our future work.

6 Appendix

We adopt the notation and method of Green and Zerna [31], wherein we suppose a body of perfectly elastic material is deformed such that a point initially at x_i with respect to a fixed rectrangular Cartesian coordinate system moves to y_i in the same coordinate system. The curvilinear convected coordinates θ_i are related to x_i and y_i by

$$x_i \;=\; x_i(\theta_1, \theta_2, \theta_3); \qquad y_i \;=\; y_i(\theta_1, \theta_2, \theta_3) \tag{23}$$

The metric tensors g_{ij} and G_{ij} in the undeformed and deformed states, respectively, are given by

$$g_{ij} \;=\; \frac{\partial x^r}{\partial \theta^i} \cdot \frac{\partial x^r}{\partial \theta^j}; \quad G_{ij} \;=\; \frac{\partial y^r}{\partial \theta^i} \cdot \frac{\partial y^r}{\partial \theta^j}; \quad g^{ij} \;=\; \frac{\partial \theta^i}{\partial x^r} \cdot \frac{\partial \theta^j}{\partial x^r}; \quad G^{ij} \;=\; \frac{\partial \theta^i}{\partial y^r} \cdot \frac{\partial \theta^j}{\partial y^r}; \tag{24}$$

For a homogeneous, isotropic, elastic material, the strain invariants, I_1, I_2 and I_3 are otianed from

$$I_1 \;=\; g^{rs} G_{rs}, \;\; I_2 \;=\; g_{rs} G^{rs}, \;\; I_3 \;=\; \frac{G}{g}; \;\; G \;=\; |G_{ij}|, \;\; g \;=\; |g_{ij}|. \tag{25}$$

When the body is incompressible,

$$I_3 \;=\; 1, \;\; W \;=\; W(I_1, I_2) \tag{26}$$

where W is the strain energy function.

The stress-strain relation is

$$\tau^{ij} \;=\; \Phi g^{ij} \;+\; \Psi B^{ij} \;+\; p G^{ij} \tag{27}$$

where

$$\Phi \;=\; 2\frac{\partial W}{\partial I_1}, \quad \Psi \;=\; 2\frac{\partial W}{\partial I_2} \tag{28}$$

$$B^{ij} \;=\; (g^{ij} g^{rs} \;-\; g^{ir} g^{js}) G_{rs}, \tag{29}$$

where p is the hydrostatic pressure and τ^{ij} is the contravariant stress tensor with respect to the θ^i coordinate system.

The equations of motion are given by

$$\tau^{ij}||_i \ + \ \rho F^i \ = \ \rho f^j \tag{30}$$

where the double line denotes covariant differentiation with respect to θ^i. F^j denotes the contravariant components of the body force vector $\mathbf{F}$ per unit mass, ρ is the density of the material and f^j represent the contravariant components of the acceleration vector $\mathbf{f}$.

$\tau^{ij}||_i$ is given by

$$\tau^{ij}||_r \ = \ \tau^{ij}_{,r} \ + \ \Gamma^i_{rm}\tau^{mj} \ + \ \Gamma^j_{rm}\tau^{im} \tag{31}$$

$$\Gamma^r_{ij} \ = \ g^{rs}\Gamma_{ijs} \ = \ \frac{1}{2}g^{rs}(g_{is,j} \ + \ g_{js,i} \ - \ g_{ij,s}) \tag{32}$$

where the subscript "," denotes partial differentiation.

The boundary conditions are

$$\tau^{ij}n_i \ = \ P^j \tag{33}$$

where P^j are the contravariant components of the surface force vector and n_i are the covariant components of the unit outward drawn normal vector to the deformed surface.

The physical components σ_{ij} are given by

$$\sigma_{ij} \ = \ \left[G_{jj}/G^{ii}\right]^{1/2}\tau^{ij} \tag{34}$$

The strain tensor E^i_j (also known as the physical components, e_{ij}) is given by

$$e_{ij} \ = \ E^i_j \ = \ \frac{1}{2}(g^{ik}G_{kj} \ - \ \delta^i_j) \tag{35}$$

where $\delta^i_j \ = \ 1$ when $i \ = \ j$ and $\delta^i_j \ = \ 0$ when $i \ \neq \ j$.

The principal stretch ratios, λ_i, are given by

$$E_i \ = \ \frac{1}{2}(\lambda_i^2 \ - \ 1) \quad (i \ = \ R, \theta) \tag{36}$$

Thus,

$$\lambda_i \;=\; \sqrt{2E_i \;+\; 1} \tag{37}$$

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Figure Captions

Fig. 1 The non-linear stress-strain relation at the endocardial surface at the end of diastole in the dynamic (dashed curve) and static (dotted curve) cases.

Fig. 2 Circumferential elastic stiffness vs. stress for the dynamic (dashed) and static (dotted) cases.

Fig. 3 Variation of stretch ratios λ_r (lower curve) and λ_θ (upper curve) through the ventricular wall at end diastole.

Fig. 4 Time course of stress through one heartbeat at the endocardial surface for the dynamic (dashed) and static (dotted) ventricle. Time = 0 corresponds to the end of diastole. Stress is a minimum at the end of systole. The shape of the curve resembles the one found experimentally between dynamic tension and time ([30], p. 510) for papillary muscle. The units are 10^6 dynes/cm^2.

Fig. 5 Wall thickness vs. time for the dynamic (dashed) and static (dotted) ventricle. Time = 0 corresponds to the end of diastole.

Fig. 6 Variation of circumferential stress through the ventricular wall at end diastole.

Fig. 7 Difference between dynamic and static circumferential stress through the ventricular wall. The difference, which is small compared to the total circumferential stress, is a maximum at the endocardial surface and vanishes at the epicardial surface.

Fig. 8 Time course of stretch ratios λ_r (lower curves) and λ_θ (upper curves) at the endocardial surface for the dynamic (dashed) and static (dotted) cases. Time $=$ 0 corresponds to the end of diastole.