Honors Ordinary Differential Equations

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B. Bukiet

- **Text**: Differential Equations - A Modeling Perspective by Borrelli and Coleman
- **Times**: Class meets Tuesday, Thursday and Friday 2:30-3:55PM
- **Office Hours**: Tuesday, Thursday 1:30-2:30PM
- **Office**: 518 Cullimore Hall
- **Phone**: (973)-596-8392
- **e-mail**: *bukiet@model.njit.edu*
- **Course Notes:**: available at www.math.njit.edu/b̃ukiet
- **Grading**: There will be 3 in-class examinations occurring approximately during the 4th, 8th, and 12th weeks of the semester. Together, they will account for 60% of your grade. There will be no make-up exams without prior approval or doctor's note. The

Theory and Numerical Methods for First Order ODEs

- Existence and Uniqueness of solutions
- Analyzing solutions without computing them for $y' = f(y)$
- Long-term behavior and steady states
- Numerical methods for ODEs, Euler and Runge-Kutta, when to trust your solver

Second Order Linear ODEs

- The mass-spring system
- Second order ODEs and their properties
- **Constant coefficient linear ODEs -Undriven**
- **Constant coefficient linear ODEs -Driven**
- Method of Undetermined Coefficients
- Using Complex Functions
- General linear ODEs theory and the Wronskian
- Variation of Parameters
- Pendulum and Linearized pendulum, beats and resonance
- Electrical circuits
- Laplace Transforms for Solving ODEs
- Series solutions, Ordinary points and Regular Singular points

Course Notes

Introduction to Modeling and Analyzing ODEs

What are Ordinary Differential Equations (ODEs): Equations relating quantities and rates of change of quantities. The rates of change are with respect to one quantity (e.g., time). Otherwise it's a PDE (partial differential equation).

Many real world applications in finance, biology, engineering, physics, etc.

Mathematical modeling: Taking the real-world and casting in mathematical terms. Real world $\to$ Simplifying Assumptions $\to$ Model Equations $\to$

Solution: A function $y(t)$ defined on a t-interval I is a solution of the ODE if $f(t, y(t))$ is defined and $y'(t) = f(t, y(t))$ for all t in I. (We'll find out later that a unique solution exists when f and f_y are continuous. No two solutions can cross in this circumstance.)

Direction field: drawing slopes tells how to sketch solution. Consider $y' = y - t^2$. Fig. 1.2.2. Consider $y'(t) = f(y)$. Consider $y'(t) = (y-1)(y-2)(y-3)$.

Discuss Fig 1.2.3 for $y' = y - y^2 - 0.2\sin t$.

Nullclines: where slope is zero

Discuss p. 15 6a in class $(y^2)'$.

Finding Solutions by Integration and Integrating Factors

Guessing and using intuition is nice, but it is useful to have methods that can be used for general classes of ODEs

Order: highest derivative

Linear: linear in y, y' etc. not multiplied by each other, only by $g(t)$. General first order linear ODE $y' + p(t)y = q(t)$ this is in **normal linear form**. $t^2y' - e^ty - \sin 3t = 0$ becomes $y' - \frac{e^t}{t^2}y = \frac{\sin 3t}{t^2}$

$y'(t) = f(t)$. Just integrate. Use FTC $y(t) = F(t) + C$. Consider $y' = \cos t$

Integrating factor: Consider $y' - 2y = 2$. Then consider $y' + p(t)y = q(t)$. Use $P(t) = \int^t p(s)ds$. Solution is $y(t) = e^{-P(t)} \int e^{P(s)}q(s)ds + Ce^{-P(t)}$ Check solutions with different C values do not intersect.

For $y(t_0) = y_0$ solution is $y(t) = e^{-P(t)} \int_{t_0}^t e^{P(s)}q(s)ds + y_0e^{P(t_0)}e^{-P(t)}$.

Solution is unique since for any y_0 this is it. OR: If $p(t)(= -f_y)$ and $q(t)(= f)$ are continuous, solution exists and is unique. Consider 2 solutions y_y and y_2. Then $z = y_1 - y_2$ satisfies $z' + p(t)z = 0$ with $z(t_0) = 0$ and plug into solution above $q(t) = y_0 = 0$ so z must be zero.

Discuss influence of driving term (input) q and initial condition y_0. Works this way only because the ODE is linear.

Consider $y' + 2y = 3e^t$ with $y(0) = 3$ and look at long-time behavior.

Discuss p. 23 2a and 4a in class.

Modeling

Natural (physical) variables, Natural (physical) laws, Natural (physical) parameters. State variables, The natural process is a dynamical system.

Net rate of change = Rate in - Rate out

Pollutant problem: y_0 lbs of pollutant in 100 gal. H_2O. Input 10 gal./min. with concentration $c(t)$ and outflow 10 gal./min. Well-mixed. Set up and solve.

Let $c(t) = 0.2 + 0.1 \sin t$. Discuss Fig. 1.4.1 Discuss $\frac{dy}{dt} = 10c(t) - 10\frac{y}{100}$

Let $c(t) = 0.2$ step$(20 - t)$. Discuss Fig. 1.4.2

Solution by method of undetermined coefficients $y(t) = y_u(t) + y_d(t)$ (undriven + any driven solution). Check this. E.g. $y' + y = 17 \sin 4t$.

Some other applications

Newton's Law of Cooling (p. 32 number 14) Rate of change of temperature of a small object is (minus) proportional to difference between object and surroundings.

Radioactive decay: rate of decrease of radioactive nuclei is proportional to number of radioactive nuclei. $N' = -kN$. Half-life.

Vertical motion: $ma = -mg$ without damping.

Viscous damping: $my'' = -mg - ky'$. Longer to rise or fall. Figs. 1.5.3, 1.5.4 Wiffle ball. k/m is the parameter that really matters here, no use varying each one.

Separation of Variables and miscellaneous "tricks"

If $y'(x) = f(x, y)$ can be written $N(y)y'(x) + M(x) = 0$ then multiply by dx and integrate. E.g. $y' = \frac{-x}{y}$. Note problem at $y = 0$ 2 solutions.

Can also write these in the form $N(y)dy = -M(x)dx$ and integrate both sides. E.g. $yy' - x = 0$ with $y(2) = -1$ (4 branches)

E.g. $(1 - y^2)y' + x^2 = 0$. Note solution is easier to write in implicit form.

Solve logistic equation $y' = ay - cy^2$ and analyze.

Read Newtonian damping at home p. 53

Exact Equations: If $H(x, y) = C$, then $dH = H_x dx + H_y dy = 0$. If you recognize this, you can solve the ODE by integrating. The condition for $Mdx + Ndy = 0$ to be exact is $M_y = N_x$. E.g.

E.g. $y'' = -4y$. I.e. $y' = v$ and $v' = -4y$

Reduction ethod: ODEs of the form $y'' = F(t, y')$ (Note: no y) Reduce to first order letting $v = y'$ and integrating to obtain y. E.g., $y'' = y' - t$.

Reduction ethod: ODEs of the form $y'' = F(y, y')$ (Note: no t) Reduce to first order by letting $y' = v(y)$. This works since F is not a function of t. Then $y'' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$. This leads to $v\frac{dv}{dy} = F(y, v)$ which is first order in v. E.g. $y'' = \frac{y'^2}{y} - \frac{y'}{y}$ with $y(0) = 1$ and $y'(0) = 2$. Note that this can get quite messy.

Read Escape velocity and inverse square law p. 61. Optional reading: Combat models

Cold Pills: Compartmental model $x(t)$ amount in GI tract and $y(t)$ amount in blood $\frac{dx}{dt} = -k_1 x$ and $\frac{dy}{dt} = k_1 x - k_2 y$ with $x(0) = A$ and $y(0) = 0$. Fig. 1.8.1 and 1.8.2. Read Falling Asleep in class example at home.

Variation of Parameters for 2nd order linear ODE: $t^2 y'' + 4ty' + 2y = \sin t$ with one solution $z = e^t$. Let $y = uz$ then $y'' = F(t, y')$.

Special form: If $y' = f(y/x)$ (Can test by checking if $f(kx, ky) = f(x, y)$) then let $y = xz$ and the ODE becomes separable. E.g. Goose flying to its nest problem, p. 79-80.

Go over Temperature experiment with Newton's Law of Cooling allowing a power other than 1. Do a wiffle ball experiment. Go over page 45 9c.

Non-dimensionalizing: Newtonian damping $\frac{dv}{dt} = -g - \frac{k}{m}v|v|$ with $v(0) = v_0$. Let $t = c_1\tau$ and $v = c_2 u$ and get equation with coefficients 1 in $\frac{du}{d\tau}$. Parameters go down to 1.

Theory for First Order ODEs

Initial values problems useful for understanding and predicting behavior of natural processes. How do we describe behavior and solution when analytical solution cannot be found?

Key questions:

- Existence: Under what conditions will the IVP have at least one solution?
- Uniqueness: Under what conditions will the IVP have at most one solution?
- Extension and Long-Term Behavior:: How far into the future and past can a solution be extended? How does a solution behave as t gets large?
- Sensitivity: How much does a solution change when y_0 and f change?
- Description: How can a solution be described?

E&U Theorem: Suppose the functions $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on a closed rectangle R of the ty-plane and that (t_0, y_0) is in R. Then the IVP $y' = f(t, y)$, $y(t_0) = y_0$ has a solution $y(t)$ on some t-interval

I containing t_0 in its interior (existence) but no more than one solution in R on any interval containing t_0 (uniqueness).

Idea: $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$.

Solution curves cannot meet if f satisfies E&U theorem. Check f and f_y

Fig. 2.1.3 $y' = 3y \sin y + t$ and Fig. 2.1.4 $ty' - y = t^2 \cos t$, $y(0) = 0$. (No solution if $y(0) = 1$).

Piecewise continuous – one-sided limits exist at all but a finite number of points. Jump discontinuities. Solve $y' + y = \text{step}(t-1)$. On-off functions okay for driving term in t.

Page 95 1c $y = 1$

Extension principle: if $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous on a closed and bounded rectangle R. If (t_0, y_0) is in R then the solution curve can be extended til it hits the boundary of R.

A solution is maximally extended if it can't be extended to a larger interval than I. E.g. $y' = \frac{-t^2}{(y+2)(y-3)}$. Fig. 2.2.1

Autonomous ODE – f does not depend on t. Then direction fields only depend on y and equilibrium solutions can be studied. Solutions can be translated to the left or right. E.g. $y' = y^2$. $y(0) = 1$. Find interval for solution to exist

Analyzing the sign of f can tell which solution an <u>a</u>utonomous ODE approaches. E.g. $y' = (y-3)(y-1)(y+1)$.

Long-term behavior: If $f(y)$ and $\frac{\partial f}{\partial y}$ are continuous for all y, then any solution $y(t)$ which is bounded for all time approaches an equilibrium solution as $t \to \pm\infty$.

Steady-states: $y' = 0$ points are candidates for first order ODEs. e.g., $y' + p_0 y = q_0$ gives $y_{steady} = q_0/p_0$ as t gets large, if $p_0 > 0$.

Read periodic forced oscillations and Will the message get through p. 103 at home. Read comments page 104.

How much will a solution change when we change a parameter in driving function, y_0, etc. E.g., $y' + p(t)y = q(t)$, $y(t_0) = y_0$. First discuss $y' + p_0 y = q_0$ and the influence of p_0 and q_0 on the solution. If $p(t) \geq p_0 > 0$ and $|q(t)| < M$ for all t then analyze exact solution to find $|y(t)| \leq e^{-p_0 t}|y_0| + M/p_0|1 - e^{-p_0 t}|$ for all t. Thus, $|y(t)| \leq |y_0| + M/p_0$ for all t.

If change $y' + p(t)y = q(t)$, $y(t_0) = a$ to $z' + p(t)z = m(t)$, $z(t_0) = b$ then $(y(t) - z(t))' + p(t)(y - z) = q(t) - m(t) < M$ with $(y-z)(0) = a - b$ and $|y(t) - z(t)| \leq e^{-p_0 t}|a-b| + \frac{M}{|p_0|}|1 - e^{-p_0 t}|$.

Ex. 2.3.4

If such limits can be found, the IVP is well-posed. Existence, uniqueness, extension and solution is continuous in the data.
Skip section 2.4

Numerical Methods

Can only do particular cases, not general numerically

Basis of numerical methods for ODEs is approximating slope on short intervals

Euler's method: Connect with short segments Δx or h of slope $f(t,y)$. E.g. $y' = y$, $y(0) = 1$. with $h = 1$ and $h = 0.5$

Derive that error is proportional to Δx or h after multiple steps.

Heun's method averages slope at 2 points (error proportional to h^2). Present derivation of Heun's and Heun's-like methods. Runge-Kutta at 4 points (error proportional to h^4.

Don't just blindly trust computational results, e.g. Euler for $y' = -10y$, $y(0) = 1$, $h = 0.1$.

Think about problems that might arise with $y' = 1 - t \sin y$, $y(0) = 1$, as t gets larger.

Read comments p. 134-135. Skip section 2.7

Second-Order ODEs

The mass-spring-damper system:

- Spring acts to return to equilibrium:
- Hooke's law $-ky$, Hard-spring $-ky - jy^3$, soft-spring $-ky + jy^3$, aging spring $-k(t)y$
- Forces of gravity, damping (proportional to velocity) and driving force.
- $my'' = S(y) - cy' - mg + f(t)$

Change variables (let $y = z - h$ with $h = mg/k$) to consider motion around equilibrium given mass $kh = mg$. **static de ection**

E.g. Hooke's law spring, 1lb weight, static deflection 15.36 in, damping constant 1.30x10^{-4} lb $\cdot$ sec / in. $f(t) = 0.26\sin(5.6t)$ lb yields $z'' + 0.05z' + 25z$. Release from rest.

Equilibrium solution for a Hooke's law spring (y' and y'' must be zero.) $y'' = -\frac{k}{m}y - \frac{c}{m}y' - g$

Equilibrium for soft spring $g = 9.8$ m/sec, $c/m = 0.2$/sec, $k/m = 10\text{sec}^{-2}$, $j/m = 0.2$(m sec)-2. 3 solutions (-1, near 7.5 and -6.5) but only one normal equilibrium. See Fig. 3.1.2

Turning a higher order ODE into a system. Use previous example. Then numerical method can deal with it.

Linearizing around an equilibrium point $F(y - y_E, y' - y'_E) = 0 + (y - y_E)F_y(y_E, y'_E) + (y' - y'_E)F_{y'}(y_E, y'_E)$. Work out around (-1,0) for our problem. Good close by, not far away. See Figs. 3.1.3, 3.1.4

Fundamental Theorem for 2nd Order ODEs: If F, F_y and $F_{y'}$ are continuous in box B in tyy', then the ODE $y'' = F(t, y, y')$, $y(t_0) = y_0,\ y'(t_0) = v_0$ has a unique solution which can be extended to the boundary of B. The solution depends continuously on the data.

Orbits (phase-plane) for $y'' = -25y - 0.5y'$, $y(0) = A,\ y'(0) = 0$. Figs 3.2.1 and 3.2.2. Time-state curves cannot intersect. But $t - y$ plots might intersect (this is a projection- y' will be different). (Skip rest of 3.2)

Properties of autonomous ODEs $y'' = f(t, y, y')$:

- Time shifting
- Orbit depends on time elapsed, not true time
- distinct orbits do NOT intersect
- self-intersecting orbits must be periodic

Solution to $y'' + ay' + by = 0$. Guess Ce^{rt}.

Characteristic polynomial, roots, $C_1e^{r_1t} + C_2e^{r_2t}$ is also a solution.

Solve $y'' + y' - 2y = 0$ and $y'' + y'/2 + y/16 = 0$ test t times only root.

Theorem 3.3.1 p. 169. Solutions exist for all time, $y = 0$ is the only solution if $y(0) = y'(0) = 0$a

Trivial solution is only solution when $y(0) = 0,\ y'(0) = 0$. Solve $y'' + y' - 2y = 0$ with $y(0) = 0,\ y'(0) = 3$.

The D operator $y'' + y' - 2y = (D^2 + D - 2)y$. Check$(D - 1)(D + 2)y$ and $(D + 2)(D - 1)y$. Look at $(D^2 + D - 2)(\sin 3t)$.

Useful info: $P(D)e^{st} = P(s)e^{st} \qquad P(D)(h(t)e^{st}) = e^{st}P(D + s)[h(t)]$

Linearity: $P(D)[C_1y_1 + C_2y_2] = C_1P(D)y_1 + C_2P(D)y_2$.

If y_1 and y_2 are solutions of $P(D)y = 0$ then so is $C_1y_1 + C_2y_2$.

Set up operator approach to solve $y'' + y'/2 + y/16 = 0$ $(D - 1/4)(D - 1/4)y = 0$. Let $v = (D - 1/4)y$, then $(D - 1/4)v = v' - v/4 = 0$ and $(D - 1/4)y = y' - y/4 = v$

Solve $(D^2 + D - 2)y = \sin t$. So $(D+2)(D-1)y = \sin t$. Let $v = (D-1)y$, then $(D+2)v = v'+2v = \sin t$ and $(D - 1)y = y' - y = v$

e^{it} Taylor series is $\cos t + i \sin t$. $D(e^{rt}) = r(e^{rt})$ for complex r.
Show equivalence between $C_1e^{(a+bi)t} + C_2e^{(a-bi)t}$ and $D_1e^{at}\cos(bt) + D_2e^{at}\sin(bt)$

Solve $y'' + y' + 100.25y = 0$.

Theorem 3.4.1

Periodic Functions. Fundamental period or period ($2\pi/\omega for \sin\omega t$), cycle, amplitude (half the difference between max and min), frequency (cycles per unit time), circular frequency (radians per unit time).
Simple harmonic motion $y'' + \omega^2 y = 0$

Aliasing: if sample two few points in the numerical solver. Read p. 187-8 at home.

Solve $y'' + \omega^2 y = 3\sin kt$. If $k/\omega = m/n$ with m, n integers, periodic with period $2\pi m/k = 2\pi n/\omega$. Figs 3.5.5 and 3.5.6. What happens as k goes to ω

Undetermined coefficients: Guess a solution and match coefficients.

- Solve $(D^2 - 2D + 1)y = 3e^{-t}$.
- Solve $(D^2 - 2D + 1)y = 3e^{t}$. Try. Then use $h(t)e^t$.
- Solve $(D^2 - D - 2)y = 4t$.
- Solve $(D^2 - D - 2)y = e^t$.
- Solve $(D^2 - D - 2)y = 4t + e^t$. We can sum particular solutions since the ODE is linear.
- Solve $(D-1)(D-2)y = te^{-t}$.
- When RHS is t^n and $P(D)$ has non-zero roots. Guess $A_n t^n + A_{n-1}t^{n-1} + ... + A_1 t + A_0$.
- When RHS is t^n and $P(D)$ has k zero roots. Guess $A_{n+k}t^{n+k} + A_{n+k-1}t^{n+k-1} + ... + A_{k+1}t^{k+1} + A_k t^k$.
- Solve $(D^2 + 25)y = \sin 4t$. Consider e^{4it} and take imaginary part.
- Solve damped Hooke's Law spring with oscillatory driving force. $(D^2 + 2D + 4)y = -12t^2 e^{-t}\cos 2t$. Use $h(t)e^{(-1+2i)t}$

General Theory of Linear ODEs

If $a(t)$, $b(t)$ and $f(t)$ are continuous, them $y'' + a(t)y' + b(t)y = f(t)$ with $y(t_O) = y_0$ and $y'(t_0) = v_0$ has a unique solution for all t. I.e. solutions do not go to ∞ in finite time.

Corollary: If $y_0 = v_0 = 0$ then the trivial solution $y = 0$ is the unique solution.

$t^2 y'' - 2ty' + 2y = 0$ with $y(t_O) = 0$ and $y'(t_0) = 0$ has ∞ solutions. $y = Ct^2$.

Polynomial operators work even if (for $P(D) = D^2 + aD + b$) $a(t)$ and $b(t)$ are not constant. However, we cannot factor.

Nullspace - Set of solutions to $P(D)y = 0$
Wronskian: For any two functions f and g in $\mathbf{C}^1$, the function $W[f,g](t) = f(t)g'(t) - f'(t)g(t)$ is called the Wronskian of f and g.

Basic solution set: A pair of solutions y_1 and y_2 of the ODE $P(D)y = 0$ is called a basic solution set if $W[y_1, y_2](t) \neq 0$ anywhere in t.

Wronskian satisfies the ODE: $W' + aW = 0$ for second order $y'' + a(t)y' + b(t)y = 0$. So $W(t)$ is never zero or always zero. Check ODE and solve by integrating factor.

Any solution to the undriven ODE can be written as a sum of basic solution set functions. I.e., $y = c_1 y_1 + c_2 y_2$.

In theory we can find 2 solutions with $y(O) = 1$ and $y'(0) = 0$ and $y(O) = 0$ and $y'(0) = 1$. Here $W(0) = 1$ so $W(t)$ is never 0.

Sometimes we can guess a solution. E.g. $t^2 y'' - 2y = 0$ *Cuachy-Euler equation*, let $y = t^\alpha$. Note numerical issues in finding solution for t^2 coefficient of 0. (Here power of t was same as order of y for each term.)

Reduction of order: If know one undriven solution y_1, guess $y_2 = uy_1$. E.g. $t^2 y'' - ty' + y = 0$. One solution $y_1 = t$.

Discuss $t^2 y'' + 2ty' + y = 0$ and complex roots. Work p. 210, 6c in class.

Variation of Parameters: If you have the undriven basic solution set and want to solve $P(D)y = f$, let $y_p = c_1(t)y_1(t) + c_2(t)y_2(t)$ and force all $c'y$ sums to be zero. Works ev

We call $H(i\omega) = \frac{1}{P(i\omega)}$

Let $x_1 = y$,so $y' = x_1'$.
Let $x_2 = x'$

Idea is to take the ODE and "transform it" to algebraic equations. Solve the algebraic equations and then transform back.

If $f(t)$ is defined for $t \geq 0$ the $L[f](s) = \int_0^\infty e^{-st} f(t)dt$. This is a function of s.

Work out for $f(t) = c(\to c/s), t(\to 1/s^2), e^{at}(\to 1/(s-a))$. (Consider s to be positive or bigger than a as needed so the transform exists).

Transform of y'. Use integration by parts. $L[y'] = -y(0) + sL[y]$

Discuss linearity of Laplace Transform. $L[af + bg] = aL[f] + bL[g]$ where a and b are constants.

Two continuous functions with the same Laplace Transform are equal (we won't prove this).

Solve $y' + ay = f(t)$ with $y(0) = y_0$. Take $y(0) = 1$ and $f(t) = 4t^3 e^{-at}$. Use table on p. 337

We can take Laplace transforms of functions that do not grow too fast. I.e. of exponential order $|f(t)| \leq Me^{at}$ for all $t \geq 0$. The would be a problem e.g. with e^{t^2}.

Find Laplace Transform of sin and cos using e^{iat} to find $L[\sin(at)] = \frac{a}{s^2 + a^2}$ and $L[\cos(at)] = \frac{s}{s^2 + a^2}$

Find the Laplace transform of a square wave on [0,1]. $L[square\ wave] = \frac{1-e^{-s}}{s}$

Laplace Transform goes to 0 as $s \to \infty$. Laplace transform has derivatives of all orders.

LT of y'', $y'''...y^{(n)}$ and of $t^n f(t)$ in terms of derivative of Laplace transform of f. $L[t^n f(t)] = \frac{d^{(n)}L[F]}{ds}$. So $L[t^n] = \frac{n!}{s^{n+1}}$. Consider $t\cos t$.

$y'' - y = 1$ with $y(0) = 0$ and $y'(0) = 1$.

$L[\int_a^t f(x)dx] = \frac{1}{s}L[f] - \frac{1}{s}\int_0^a f(x)dx$. Solve $x(t) = \frac{1}{2}\int_0^t x(s)ds + 1$.

Shifting theorems and the Heaviside function. $L[e^{at}f(t)](s) = F(s-a)$ $L[f(t-a)step(t-a)] = e^{-as}F(s)$ $L[f(t)step(t-a)] = e^{-as}L[f(t+a)]$. Find $L[e^{-2t}\cos(3t)]$ and $L^{-1}\frac{2s+3}{s^2-4s+20}$ Work out examples 6.2.5 and 6.2.6.

Transform of a periodic function, square wave. $L[f] = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$ where the perild is T. P. 318-320.

Find Laplace transform of $\sin^2 t$ and $\cos 3t$
Define convolution $(f * g)(t) = \int_0^t f(t-u)g(t)dt$. Properties: $f * g = g * f$; $(f * g) * h = f * (g * h)$ and $(f + g) * h = f * h + g * h$

Convolution theorem $L^{-1}[FG] = f * g$ and $L[f * g] = FG$. Ex. 6.4.1 and 6.4.2.

Discuss Theorem 6.4.3.

Solving ODEs with impulse functions. The Dirac delta function. $L[\delta(t)] = 1$. $L[\delta(t-u)] = e^{-us}$. Work out impulse in oscillating spring ex. 6.5.1.

Systems of Linear ODEs

Some simple matrix operations so that we can handle 2x2 linear systems of ODEs

Adding matrices termwise. Multiplying by a scalar. Multiplying a matrix times a vector and by another matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \qquad c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = c\begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Solving $\mathbf{Ax} = \mathbf{b}$ using Gaussian Elimination. Consider

$$\begin{aligned} 2x+3y &= -2 \\ 4x+y &= 6 \end{aligned} \qquad \begin{aligned} 2x+3y &= 2 \\ 4x+6y &= -4 \end{aligned} \qquad \begin{aligned} 2x+3y &= -2 \\ 4x+6y &= -5 \end{aligned}$$

Solving $\mathbf{Ax} = \mathbf{b}$ using Gaussian Elimination. Consider

$$\begin{aligned} ax+by &= e \\ cx+dy &= f \end{aligned}$$

Note that cannot solve uniquely unless $ad - bc \neq 0$

Eigenvalues of 2x2 matrix $\mathbf{Ax} = \lambda\mathbf{x}$ or $\mathbf{A} - \lambda\mathbf{Ix} = \mathbf{0}$.

Find and solve the characteristic polynomial for

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

Consider $\mathbf{x}' = \mathbf{Ax}$. If guess solutions $\mathbf{v}exp^{\lambda t}$ then $\lambda\mathbf{v}exp^{\lambda t} = \mathbf{Av}exp^{\lambda t}$. So $\mathbf{Av} = \lambda\mathbf{v}$ and λ is an eigenvalue and $\mathbf{v}$ is an eigenvector.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

use initial conditions $x(0) = 0$ and $y(0) = -2$.

Complex eigenvalues similar but need only real and imaginary parts for one eigenvalue/eigenvector. Work through

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Repeated eigenvalues with a full set of eigenvectors. $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If not a full set of eigenvectors, we need generalized eigenvectors. This is beyond the scope of this course.

Driven systems – Use undetermined coefficients. Use all terms you'd want for right hand side of any of the equations in guess for both. Work these examples:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} e^{4t} + e^{2t} \\ e^{4t} + e^{2t} \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix}$$

Series Solutions

Discuss aging spring $my'' + ke^{-\epsilon t}y = 0$.

Heat Equation in a cylinder 11.1.2 and separation of variables $u = R(r)Z(z)$ gives $rR'' + R' + \lambda rR = 0$

Review convergence of Power series (ratio test) and interval of convergence. Discuss Taylor series for $\sin x$ as derivative of $\cos x$, Also [1]