ECE436 - 698 Mid-term Solutions

1) $\dot{x}_1 = 0 \Rightarrow x_2 = 0$

$\dot{x}_2 = 0 \Rightarrow -x_1 + \mu(1 - x_1^2)x_2 = 0$ or $x_1 = 0$

$x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2\mu & \mu(1-x_1^2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

characteristics polynomial: $s(s-\mu)+1 = 0$

$s^2 - \mu s + 1 = 0$

1) $\mu > 0$ unstable

2) $\mu = 0$ marginally stable

3) $\mu < 0$ stable

For oscillation, system must be under / undamped

$$\left.\begin{matrix} s^2 - \mu s + 1 \\ s^2 + 2\zeta\omega_n s + \omega_n^2 \end{matrix}\right\} \quad \begin{matrix} \omega_n = 1 \\ \mu = -2\zeta \end{matrix}$$

$0 \leq \zeta < 1$

or $0 \leq \frac{\mu}{-2} < 1$ or $\underline{0 \geq \mu > -2}$

frequency $=$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$= \omega_n\sqrt{1-\left(\frac{\mu}{2}\right)^2} = \sqrt{1-\left(\frac{\mu}{2}\right)^2}$ rad/s

2) $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Laplace

$$e^{At} = \mathcal{L}^{-1}\left[\begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}^{-1}\right]$$

$$= \mathcal{L}^{-1} \frac{\begin{pmatrix} s+2 & 1 \\ -1 & s \end{pmatrix}}{s^2+2s+1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+2}{s^2+2s+1} & \frac{1}{s^2+2s+1} \\ \frac{-1}{s^2+2s+1} & \frac{s}{s^2+2s+1} \end{bmatrix}$$

$$= \begin{pmatrix} 2te^{-t}+1e^{-t}-1te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}+e^{-t}t & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{pmatrix}$$

the same

Caley Hamilton

$s^2+2s+1=0$

$e^{At} = \alpha_1 A + \alpha_0 I$

$e^{-t} = -\alpha_1 + \alpha_0$

$te^{\lambda t} = \alpha_1 = te^{-t}$

$\therefore \alpha_0 = e^{-t}+te^{-t}$

$e^{At} = (e^{-t}+te^{-t})\cancel{A}^{\,I} + te^{-t}A$

$$= \begin{bmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{bmatrix}$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$e^{-At} = \begin{bmatrix} e^{t}-te^{t} & -te^{t} \\ te^{t} & e^{t}+te^{t} \end{bmatrix}$$

$$= \begin{pmatrix} e^{-t}+te^{-t} \\ -te^{-t} \end{pmatrix} + \begin{pmatrix} e^{-\tau} \\ -e^{-t} \end{pmatrix}\Bigg|$$

$$= \begin{pmatrix} 2e^{-t}+te^{-t} \\ -te^{-\tau}-e^{-t} \end{pmatrix}$$

3) $f_1 = x_2 + \alpha x_1 (\beta^2 - x_1^2 - x_2^2)$
$f_2 = -x_1 + \alpha x_2 (\beta^2 - x_1^2 - x_2^2)$

$$\nabla f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$
$$= 2\alpha\beta^2 - 4\alpha(x_1^2 + x_2^2) = 2\alpha\left[\beta^2 - \frac{x_1^2 + x_2^2}{2}\right]$$

$\therefore$ $\nabla f > 0$ when $x_1^2 + x_2^2 < \frac{\beta^2}{2}$

no limit cycle inside $x_1^2 + x_2^2 < \frac{\beta^2}{2}$, radius $= \frac{\beta}{\sqrt{2}}$

Equilibrium pt: $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} \alpha\beta^2 & 1 \\ -1 & \alpha\beta^2 \end{pmatrix} \Rightarrow \begin{array}{l} \lambda^2 - 2\alpha\beta^2\lambda + 1 + \alpha^2\beta^4 = 0 \\ \lambda = \alpha\beta^2 \pm j \end{array}$$

$\therefore$ $\alpha > 0$ unstable focus; $\alpha < 0$ stable focus $\}$ index $= 1$ $\Rightarrow$ limit cycle possible for $x_1^2 + x_2^2 > \frac{\beta^2}{2}$

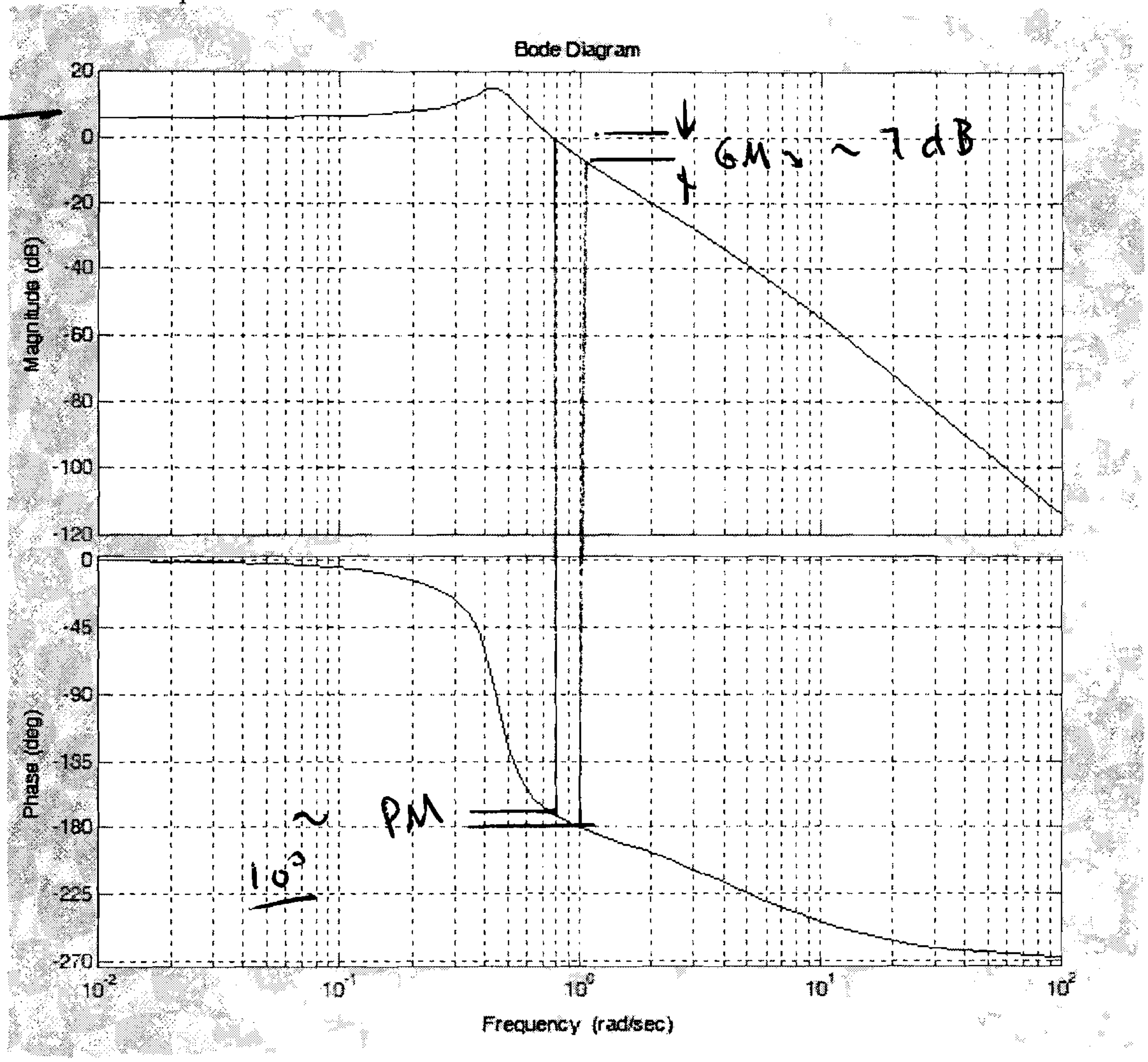
(you may use pplane7 to verify limit cycle at $x_1^2 + x_2^2 = \beta^2$)

4. Bode plot of $G(s) = \dfrac{2}{s^3 + 5s^2 + s + 1}$ is given below, determine

- DC gain,
- gain and phase margins
- state space model

DC gain

~ 6 dB



$$\dddot{y} + 5\ddot{y} + \dot{y} + y = 2 \qquad x_1 = y, \text{ etc}$$

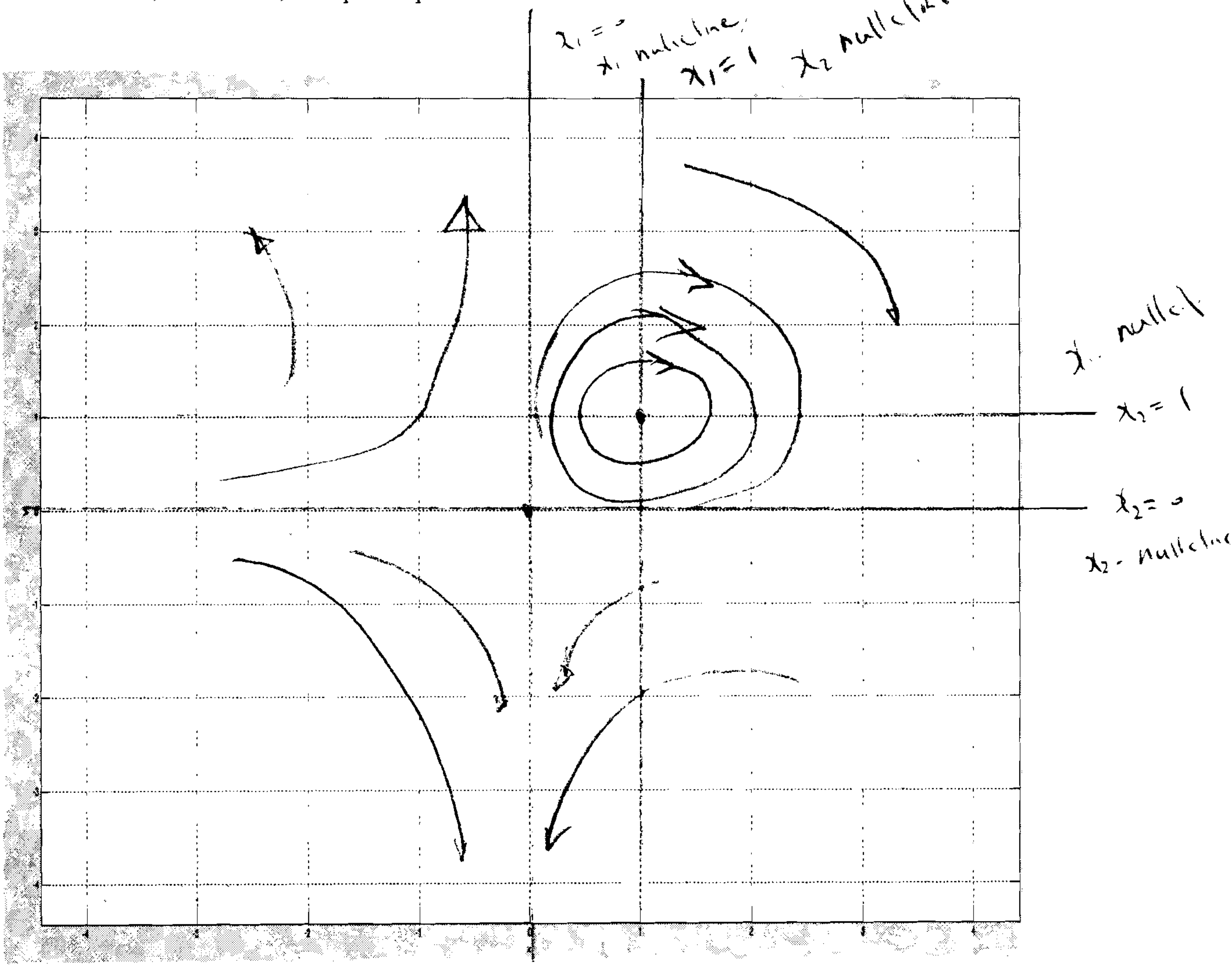
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

5. Given the Volterra predictor-prey model:

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = x_2 - x_1 x_2$$

A print out from pplane7 is shown below. Determine all equilibrium points, stability classification, nullclines, and phase plane construction.



$\dot{x}_1 = 0$ $\qquad x_1(-1+x_2) = 0$

$\dot{x}_2 = 0$ $\qquad x_2(1-x_1) = 0$

$$\Rightarrow \quad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad x_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\frac{dx_2}{dx_1} = \frac{x_2(1-x_1)}{x_1(-1+x_2)} = 0$$

x_2 - nullcline $\qquad x_2(1-x_1) = 0$

x_1 - nullcline $\qquad x_1(-1+x_2) = 0$

$$A = \begin{pmatrix} -1+x_2 & x_1 \\ -x_2 & 1-x_1 \end{pmatrix}$$

$$A\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{saddle pt}$$

$$A_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{center}$$