ECE 667 09F

Mid term Solutions

1)
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^2 + x_1^3 - 2x_2 + u$$

$\dot{x} = 0 \Rightarrow x_2 = 0$ $\quad -x_1^2 + x_1^3 = 0$

$x_1^2(x_1 - 1) = 0$

$$x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -2x_1 + 3x_1^2 & -2 \end{bmatrix}$$

2) $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$

eigenvalue $0, -2$ marginally stable

3) $x_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

eigenvalue : $\underbrace{-1 + \sqrt{2}}_{\text{unstable}}, \; -1 - \sqrt{2} \rightarrow$ saddle pt.

2) $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ eigenvalues. -1, -2

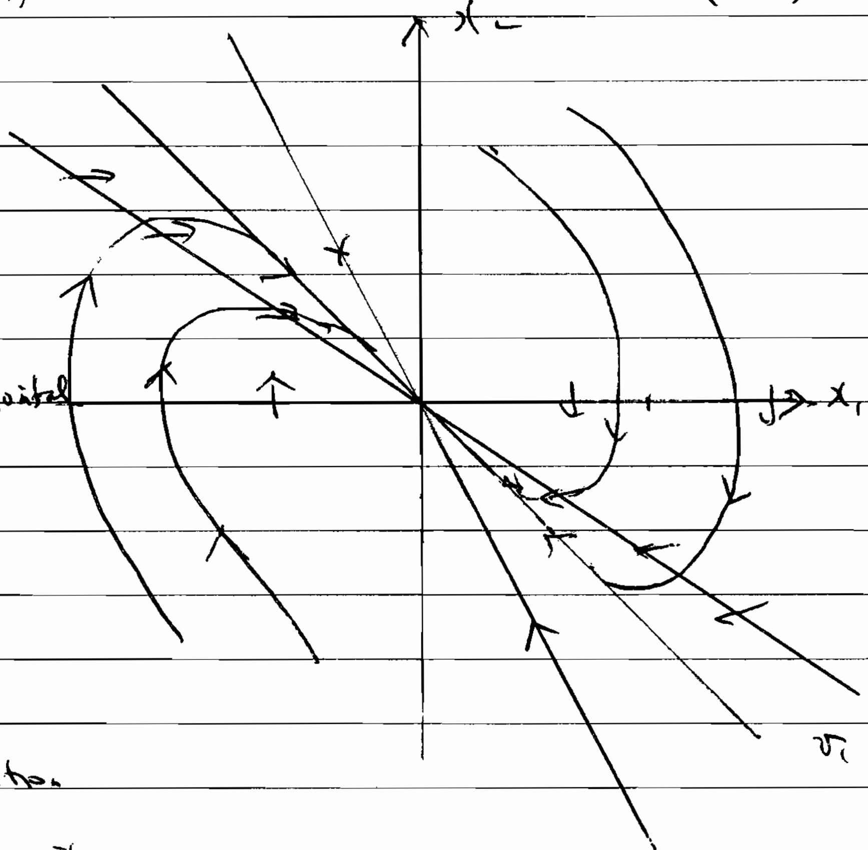
$\lambda_1 = -1$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\lambda_2 = -2$, $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\frac{dx_2}{dx_1} = \frac{-2x_1 - 3x_2}{x_2}$

null clines

$x_2 = 0$ vertical

$-2x_1 - 3x_2 = 0$ horizontal



modal decomposition

$W^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} +2 & +1 \\ -1 & -1 \end{pmatrix}$

$x(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \begin{pmatrix} 2 & 1 \end{pmatrix} x(0) + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} \begin{pmatrix} -1 & -1 \end{pmatrix} x(0)$

$x(0) = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow$ 2^{nd} mode dynamics $= 0$

3) $\nabla f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$

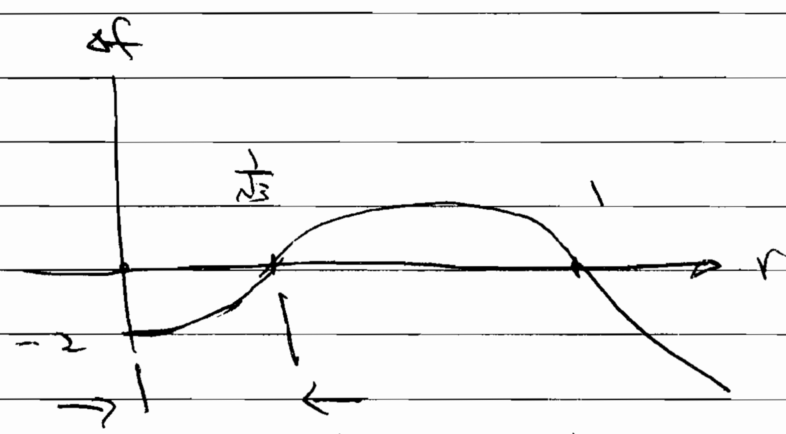
$= -2 + 8(x^2+y^2) - 6(x^2+y^2)^2$

for $x, y \to 0$ $\quad \nabla f \to -2$

to determine sign change, set $\Delta f = 0$

$-2 + 8r^2 - 6r^4 = 0$

$r = 1 \;,\; \sqrt{\frac{1}{3}}$



region without sign change

a) Simply connected region = $\quad r < \frac{1}{\sqrt{3}}$

b) Linearize around the origin

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{\binom{0}{0}} = \begin{pmatrix} -1 & -2Z \\ 2Z & -1 \end{pmatrix}$$

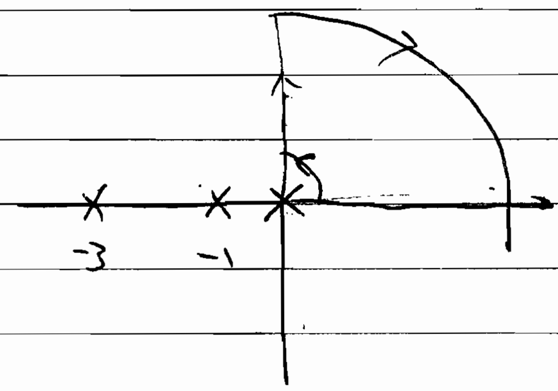
, eigenvalue $-1 \pm j\,2Z$

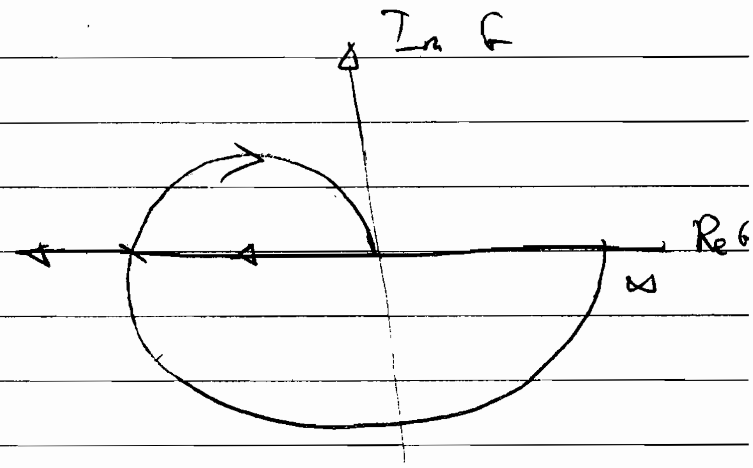
stable focus

index = 1

$\therefore$ possible to have LC

4) Nyquist plot





$$G(j\omega) = \left. \frac{1}{s^3 + 4s^2 + 3s} \right|_{s=j\omega}$$

$$= \frac{1}{-4\omega^2 + j3\omega - j\omega^3} \quad \frac{-4\omega^2 - j(3\omega - \omega^3)}{-4\omega^2 - j(3\omega - \omega^3)}$$

$$= \frac{-4\omega^2 - j(3\omega - \omega^3)}{16\omega^4 + (3\omega - \omega^3)^2}$$

intercept : $\omega = 0$, $\omega = \pm\sqrt{3}$

at $\omega = \sqrt{3}$ $\qquad \mathrm{Re}\, G = \dfrac{-4 \cdot 3}{16 \cdot (3^2)} = -\dfrac{1}{12}$

$$-\frac{1}{N(a)} = -3a^2$$ sustained oscillation

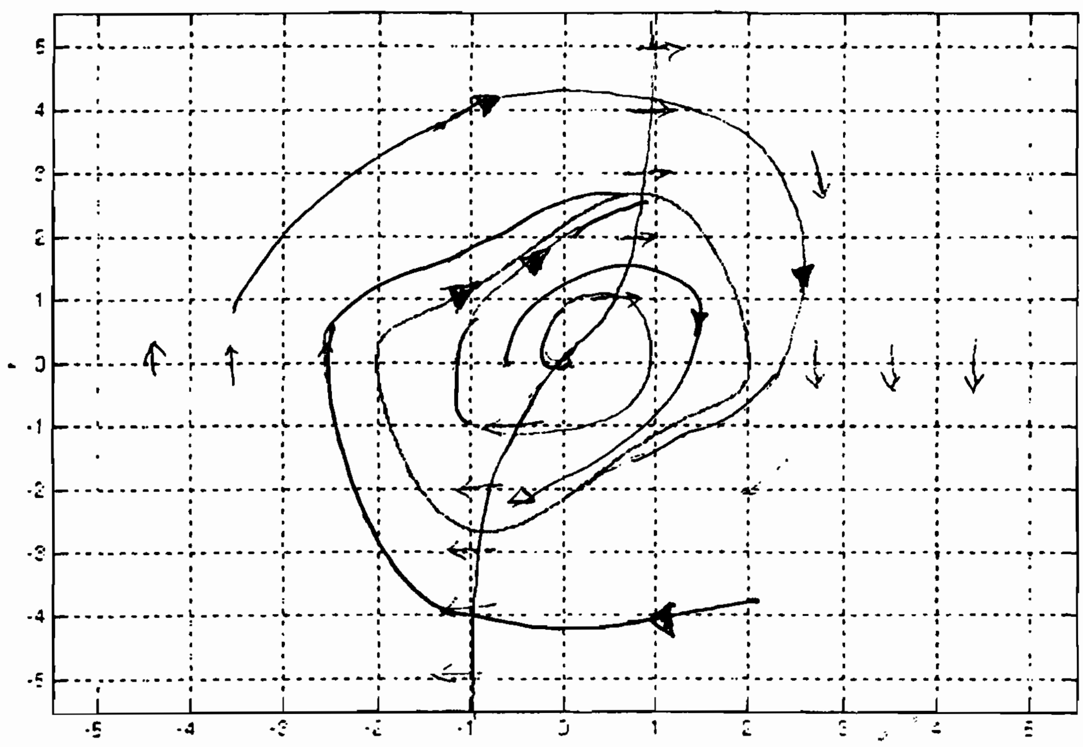
$$-3a^2 = -\frac{1}{12}$$

or $a = \dfrac{1}{6}$, at $\sqrt{3}$ rad/s

5. Given the Van de Pol's equation,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \mu(1-(x_1)^2)x_2$$

For $\mu = 1$, generate at least two phase trajectory from inside the limit cycle and two trajectory from outside of the limit cycle. Justify your answers. Indicate all equilibrium points, nullclines, etc. A partial phase plane with a limit cycle is shown below:



equilibrium pt : $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

$$\frac{dx_2}{dx_1} = \frac{-x_1 + (1-x_1^2)x_2}{x_2}$$

eigenvalue $\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ unstable focus

x_1 - nullclines : $-\infty$ $x_1 > 0$, $+\infty$ $x_1 < 0$

x_2 - nullclines $-x_1 + (1-x_1^2)x_2 = 0$

$x_2 = \frac{x_1}{1-x_1^2}$ $\to \infty$ as $x_1 \to 1$, $\to -\infty$ as $x_1 \to -1$

$$\frac{dx_2}{dx_1} = m = \frac{-x_1 + (1-x_1)^2 x_2}{x_2}$$

or $x_2 = \dfrac{x_1}{1-x_1^2-m}$ series of hyperbolas