ECE667 A9 Solutions

$$\ddot{y} + y = \mu\left(\dot{y} - \frac{\dot{y}^3}{3}\right)$$

a) -KB Method-

$$f(a\sin\theta, a\cos\theta) = a\cos\theta - \frac{a^3}{3}\cos^3\theta = \left(a - \frac{a^3}{4}\right)\cos\theta - \frac{a^3}{12}\cos 3\theta$$

$$\dot{a} = \frac{1}{2\pi}\int_0^{2\pi} \mu\cos\theta\left\{\left(a-\frac{a^3}{4}\right)\cos\theta - \frac{a^3}{12}\cos 3\theta\right\}d\theta$$

$$= \frac{\mu}{2\pi}\left(a - \frac{a^3}{4}\right)\int_0^{2\pi}\cos^2\theta\, d\theta - \frac{\mu a^3}{24\pi}\int_0^{2\pi} \cancelto{0}{\cos\theta\cos 3\theta\, d\theta}$$

$$= \frac{\mu}{2}\left(a - \frac{a^3}{4}\right)$$

$$\dot{\phi} = \frac{1}{2\pi}\int_0^{2\pi} -\frac{\mu\sin\theta}{a}\left\{\left(a-\frac{a^3}{4}\right)\cos\theta - \frac{a^3}{12}\cos 3\theta\right\}d\theta$$

$$= 0$$

$$\Rightarrow \quad \phi = \phi_0$$

Steady state : $\dot{a} = 0 \Rightarrow a = 2$

$$\frac{da}{dt} = \frac{\mu}{2}\left(a - \frac{a^3}{4}\right) \quad \text{or} \quad \frac{da}{a\left(1-\frac{a^2}{4}\right)} = \frac{\mu}{2}dt$$

so that

$$\frac{1}{a} + \frac{-\frac{1}{2}}{a-2} + \frac{-\frac{1}{2}}{a+2}\, da = \frac{\mu}{2}t + c$$

or
$$\frac{a}{\sqrt{a^2-4}} = \alpha_0 e^{\frac{\mu}{2}t}$$

ie
$$a = 2\sqrt{\frac{1}{1 - \frac{1}{\alpha_0^2}e^{-\mu t}}}$$ — see notes —

$$y(t) = 2\sqrt{\frac{1}{1-\frac{1}{\alpha_0^2}e^{-\mu t}}}\,\sin(t + \phi_0)$$

3) Power series method

$$\ddot{y} + y = \mu\left(\dot{y} - \frac{\dot{y}^3}{3}\right)$$

$$\ddot{y} + (\omega^2 - \mu\omega_1^2)y = \mu\left(\dot{y} - \frac{\dot{y}^3}{3}\right)$$

$$(\ddot{y}_0 + \ddot{y}_1\mu) + (\omega^2 - \mu\omega_1^2)(y_0 + \mu y_1) = \mu\left[(\dot{y}_0 + \mu\dot{y}_1) - \frac{(\dot{y}_0 + \dot{y}_1\mu)^3}{3}\right]$$

μ^0 term : $\quad \ddot{y}_0 + \omega^2 y = 0 \quad \Rightarrow \quad y_0 = a\cos\omega t$

μ^1 term : $\quad \ddot{y}_1 - \omega_1^2 y_0 + \omega^2 y_1 = \left(\dot{y}_0 - \frac{\dot{y}_0^3}{3}\right)$

or $\quad \ddot{y}_1 + \omega^2 y_1 = \omega_1^2 a\cos\omega t - a\omega\sin\omega t + \frac{a^3\omega^3}{3}\sin^3\omega t$

$$= a\omega_1^2\cos\omega t + \left(\frac{a^3\omega^3}{4} - a\omega\right)\sin\omega t - \frac{a^3\omega^3}{12}\sin 3\omega t$$

no secular terms $\Rightarrow \quad \omega_1 = 0 \quad \Rightarrow \quad \omega \doteq 1$

$a = 2$

$$\ddot{y}_1 + y_1 = -\frac{2}{3}\sin 3\omega t$$

$$y_1 = \frac{\sin 3t}{12} - \frac{\sin t}{4}$$

$\therefore$

$$y = 2\cos t + \mu\left[\frac{\sin 3t}{12} - \frac{\sin t}{4}\right]$$