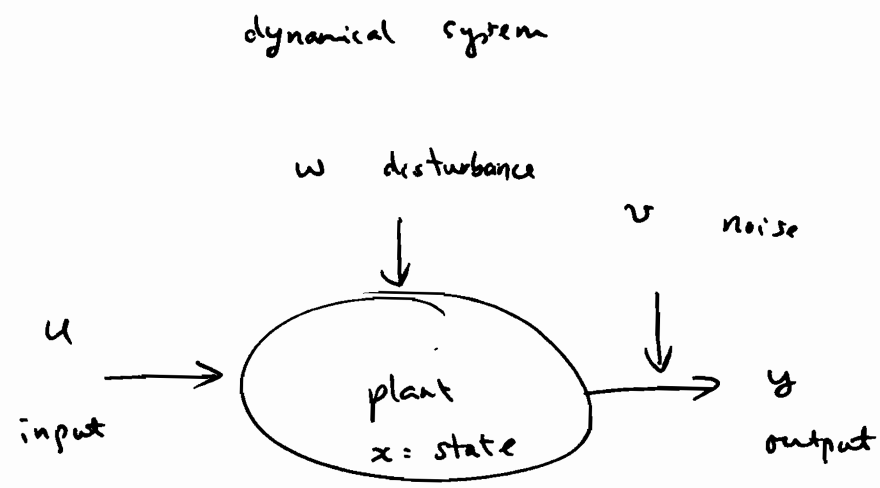
ECE 667 Review



differential equation approach:

x as state variable

$$\dot{x}_1 = f_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots u_m, w_1, \cdots w_n)$$

$\vdots$

$$\dot{x}_n = f_n(x, u, w)$$

$$y_1 = g_1(x_1, x_2, \cdots x_n, u_1, u_2, \cdots u_m, v_1, v_2, \cdots v_r)$$

$\vdots$

$$y_r = g_r(x, u, v)$$

or
$$\left.\begin{aligned} \dot{x} &= f(x, u, w) \\ y &= g(x, u, v) \end{aligned}\right] \text{ in vector description}$$

1) what are $f(\cdot)$, $g(\cdot)$?

2) how to analyse $f(\cdot)$, $g(\cdot)$?

Biological oscillator examples

① Classical circuit models applied to Bio oscillations

- van der pol equation
- duffing equation

② Biologically inspired models for neurons

Hodgkin - Huxley

2 variable model

FitzHugh-Nagumo Mode

Morris Lecar Model

Wilson-Cowan model

multi variable models

Nobel (phenomenological model)
Purkinje Fiber

MNT (physiological model)

Mc Allister - Noble - Tsien

YNI model
Sinoatrial node

Yanagihara - Noma - Irisawa

③ Organ - System level models

- body temperature
- cardiovascular
- endocrinological

(I) linearization

$$\dot{x} = f(x,u) \quad \longrightarrow \quad \dot{x} = Ax + Bu$$
$$y = g(x,u) \qquad\qquad\quad y = Cx + Du$$

note : linearized x is really δx

1. determine equilibrium point x_e

$$\dot{x} = 0 \quad \Rightarrow \quad f(x_e, u_e) = 0$$

2. $$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\substack{x_e \\ u_e}} = \frac{\partial f}{\partial x}$$

$$B = \frac{\partial f}{\partial u}\bigg|_{\substack{x_e \\ u_e}}$$

$$C = \frac{\partial g}{\partial x}\bigg|_{\substack{x_e \\ u_e}} \qquad D = \frac{\partial g}{\partial u}\bigg|_{\substack{x_e \\ u_e}}$$

3. always useful to test stability condition of the linearized system

(II) Solution of State Space System

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} B\, U(s)$$

Calculation of e^{At} & $(sI-A)^{-1}$

(VI) Modal decomposition

$$x(t) = v_1 e^{\lambda_1 t} w_1' x(0) + v_2 e^{\lambda_2 t} w_2' x(0) + \cdots + v_n e^{\lambda_n t} w_n' x(0)$$

→ useful to "see" how internal dynamics are coupled and analyzed

→ 2-D case leads to "phase-plane".

④ Conversion between models

transfer function ⇌ state space

transfer function	state space
– useful to see energy distribution	– useful to see transient characteristic
– easy to handle block dynamics	– easy to handle internal dynamics

⑤ Transfer Function

- Frequency response models : Bode, Nyquist, etc
- Gain / Phase Margin, DC gain, Unity Gain : these are important land marks

⑥ Model conversion using Matlab → state space model block diagram manipulation

⑦ Outline of oscillations

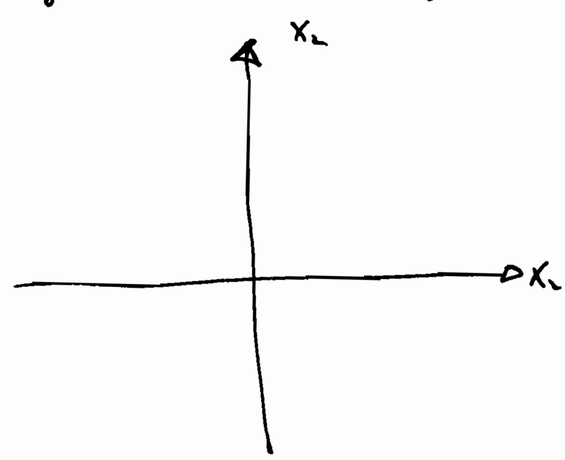
- linear vs nonlinear
- free vs forced
- damped vs undamped

$\dot{x} = f(x, u, t)$

⑧ Free Response, 2^{nd} order system ⇒ phase plane

$\dot{x}_1 = f_1$

$\dot{x}_2 = f_2$



$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$$

isocline : $\frac{dx_2}{dx_1}$ = constant

x_1-nullcline : $f_1 = 0$ ⇒ ↑ or ↓ flow

x_2-nullcline : $f_2 = 0$ ⇒ ⇄ flow pattern

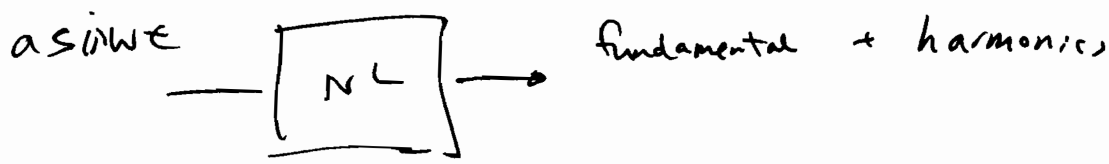
Linearized system with real, non-repeated eigenvalues

$\left.\begin{matrix} v_1 \\ v_2 \end{matrix}\right\}$ eigenvectors

Most nonlinear systems can be locally approximated by linearized models + limit cycles

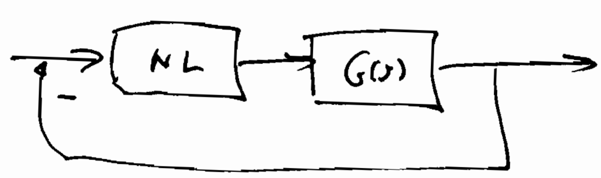
Q How to detect limit cycles?

Describing function



$$\frac{\text{output fundamental}}{\text{input fundamental}} = N(a) \quad \text{describing function}$$

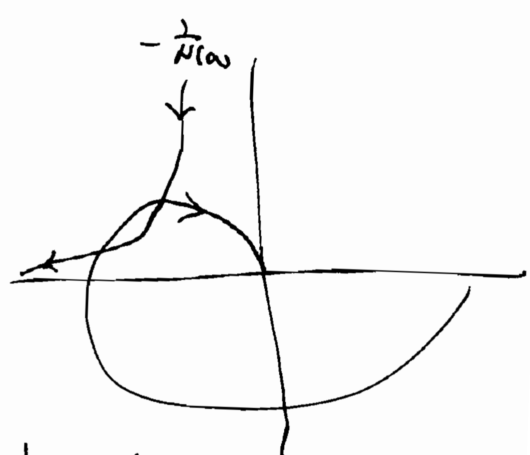
Application (typical but not limited to)



1) generate Nyquist plot for G(s)

2) plot $-\frac{1}{N(a)}$, a, $0 \to \infty$

3) check for intersection
   - frequency
   - amplitude



4) check if oscillations are sustained.

(10) Theoretical results on limit cycles

— Bendixson's Theorem : Sufficient condition to eliminate the existence of limit cycles.

$\nabla f \neq 0$ and does not change sign

in a simply connected region

— Poincaré - Bendixson's Theorem : sufficient condition to determine if a limit cycle exists.

— Index Theorem :

a) Every closed trajectory of $\dot{x} = f(x)$ contains at least 1 equilibrium point in its interior, with net index = 1

b) Index of node, focus = 1
Saddle point = -1

— Bifurcation : system parameter change can profoundly affect the equilibrium points.

(11) More Theoretical Results on limit cycle

approximate analytic methods

- Krylov - Bogolubov (averaging method)

$$\ddot{y} + y = \mu f(y, \dot{y}) \quad , \quad \mu \text{ "small"}$$

$$y(t) \simeq a(t) \sin[t + \phi(t)]$$

$$\dot{a} = \frac{1}{2\pi} \int_0^{2\pi} (\mu \cos\theta)\, f(a\sin\theta, a\cos\theta)\, d\theta$$

$$\dot{\phi} = \frac{1}{2\pi} \int_0^{2\pi} -\frac{\mu}{a} \sin\theta\, f(a\sin\theta, a\cos\theta)\, d\theta$$

- power series (perturbation method)

$$\ddot{y} + \omega_0^2 y + \mu f(y, \dot{y}) = 0$$

$$\left.\begin{aligned} y &= \sum_{i=0}^{\infty} \mu^i y_i \\ \omega^2 &= \sum_{i=0}^{\infty} \mu^i \omega_i^2 \end{aligned}\right\} \text{ Poincaré - Lindstedt method}$$

obtain y_0, y_1, etc.

(12) Circuital equivalence

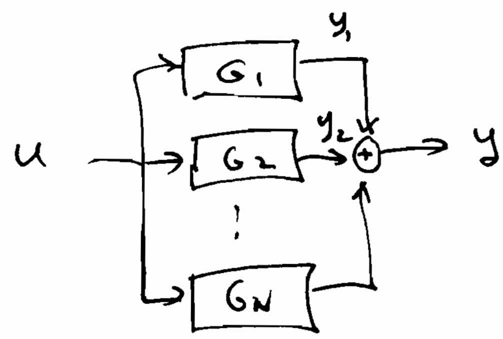
- op-amps
- VCO
- PLL

(13) Oscillation control

- Single pulse perturbation
- Periodic entrainment

} open loop control

- Gain, Phase, Frequency control } closed loop control



$$G_i = \frac{c_{i1} + c_{i2} s}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}$$

$$\omega_d^i = \omega_i \sqrt{1 - \zeta_i^2}$$

ringing frequency

$$y_i = G_i(t) * u(t)$$

baseband components:

$$y_i = y_i^c \cos \omega_d^i t - y_i^s \sin \omega_d^i t = A_i(t) \sin(\omega_d^i t + \theta_i)$$

in-phase quadrature

$$y_i^c = G_i^c * u_i^c - G_i^s * u_i^s$$

$$y_i^s = G_i^s * u_i^c + G_i^c * u_i^s$$

Single input ($u = u_i = u_1$)

& $G_i = 2 G_i^c \cos \omega_d^i t - 2 G_i^s \sin \omega_d^i t$

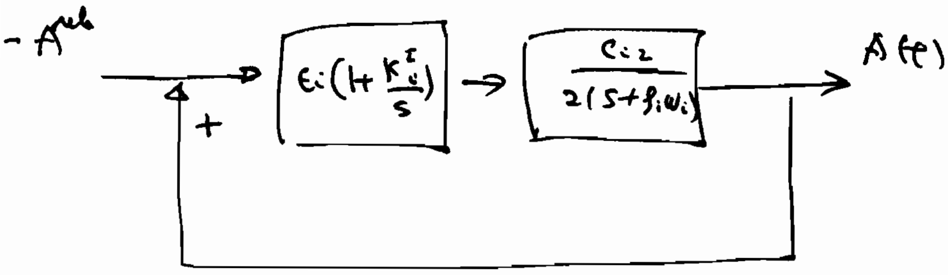
Amplitude control

$$e = (A(t) - A^{ref})$$

$$u^c = \epsilon_i e + \epsilon_i K_i^I \int_0^t e(\tau) d\tau$$

How This works.

Baseband equivalent model



closed loop TF :
$$\frac{\frac{1}{2}\epsilon_i C_{i2}(s+K_i^I)}{1-\frac{\epsilon_i(s+K_i^I)}{s}\,\frac{C_{i2}}{2(s+\zeta_i\omega_i)}}$$

↑ since e is defined as $A - A^{ref}$

denominator polynomial :

$$s^2+\left(\zeta_i\omega_i-\frac{C_{i2}\epsilon_i}{2}\right)s-\frac{C_{i2}\epsilon_i K_i^I}{2}=0$$

So as long as $\epsilon_i C_{i2} < 0$, $K_i^I > 0$, the poles are stable!

Final value theorem : (for A^{ref} constant $= -\frac{A^{ref}}{s}$)

$$A^{\infty}=\lim_{s\to 0} s\,\frac{\frac{1}{2}\epsilon_i C_{i2}(s+K_i^I)}{s^2+\left(\zeta\omega_i-\frac{C_{i2}\epsilon_i}{2}\right)s-\frac{C_{i2}\epsilon_i K_i^I}{2}}\;\frac{-A^{ref}}{s}$$

$= A^{ref}$ so the amplitude converges!