Math 244-Final Exam

Name:__________________

Student #: _________________

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Instructor: Dhar

Must show all work to receive full credit!!!

I pledge that I have not violated the NJIT code of honor__________________________

1. How many 5-digit numbers can be formed form the integers 1, 2, …, 9 if no digit can appear more than twice? (For instance, 41434 is not allowed.) (12 points) (Self-test problems and exercises #10, page 20)

five repeats; last remaining #

$$9^5 - \left[9 + \binom{5}{4}\binom{9}{1}\binom{8}{1} + \binom{5}{3}\binom{9}{1}(8)(8) \right]$$

slots for 4 repeats; digit for 4 repeat

$$9^5 - [9 + 360 + (90)(64)]$$

$$= 9^5 - 6129 = 52920$$

slots; slots; digits; remaining

OR $\underbrace{9(8)(7)(6)(5)}_{\text{all distinct}} + \underbrace{\binom{9}{1}\binom{5}{2}(8)(7)(6)}_{\text{exactly 2 repeated numbers}} + \underbrace{\binom{5}{4}\binom{9}{2}\frac{4!}{2!2!}7}_{\text{2 distinct pairs repeats}}$

digit for pair-repeat

$15120 + 7560 + 30240 = 52920.$

2. A total of 30 percent of American makes smoke cigarettes, 6 percent smoke cigars and 3 percent smoke both cigars and cigarettes. What percentage of males smokes neither cigars nor cigarettes? What percentage smokes cigars but not cigarettes? (12 points) (Homework problem 11, page 51)

C: Cigarettes

I: Cigars

(i) $P(C^c \cap I^c)$

$= 1 - P(C \cup I) = 1 - [P(C) + P(I) - P(C \cap I)]$

$= 1 - [.3 + .06 - .03] = 1 - .33 = .67$

ii $P(I \cap C^c) = P(I) - P(I \cap C) = .06 - .03$

$= 0.03$

3. Let X_1 be Gamma with ($\lambda = 7, \alpha = 4$) and X_2 be Gamma with ($\lambda = 7, \alpha = 5$). If X_1 and X_2 are independent derive the distribution of Y= $X_1 + X_2$. (12 points) (Done in class for general case, also on page 254-255)

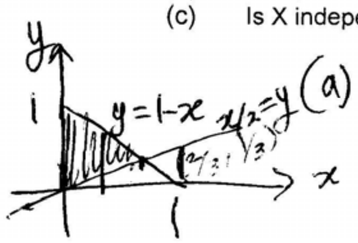
$$M_{X_1}(t) = \left(\frac{7}{7-t}\right)^4, \; t<7 \qquad M_{X_2}(t) = \left(\frac{7}{7-t}\right)^5, \; t<7$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = \left(\frac{7}{7-t}\right)^4 \left(\frac{7}{7-t}\right)^5 = \left(\frac{7}{7-t}\right)^9, \; t<7$$

$$\text{''} E e^{t(X_1+X_2)}$$

Hence Y is Gamma $(\lambda = 7, \alpha = 9)$.

4. Let $f(x, y) = 24xy \; 0 \le x \le 1, 0 \le y \le 1, 0 \le x + y \le 1$ and let it equal zero otherwise.
   (a) Compute the marginal density of X.
   (b) Compute the probability P(X<2Y)
   (c) Is X independent of Y? Why or why not? (18 points)



(a)

$$f_X(x) = \int_0^{1-x} 24xy \, dy, \quad 0 < x < 1$$

$$= 24x \left. \frac{y^2}{2} \right|_0^{1-x}, \quad 0 < x < 1$$

$$= \begin{cases} 12x(1-x)^2, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Beta($\alpha = 2, \beta = 3$).

$x = 2y$
$3y = 1$
$y = \frac{1}{3}$
$x = 2/3$

$\alpha = 2$
$\beta = 3$

$\Gamma(\alpha+\beta) = \Gamma(5) = (6)(4) = 24$

$\Gamma(\alpha) \cdot \Gamma(\beta) = 1!\,2! = 2$

$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} = 12$

(b)

$$\int_0^{2/3} \int_{x/2}^{1-x} 24xy \, dy \, dx$$

$$\int_0^{2/3} 24x \left(\left. \frac{y^2}{2} \right|_{x/2}^{1-x} \right) dx$$

$$= \int_0^{2/3} 12x \left[(1-x)^2 - \frac{x^2}{4} \right] dx = 12 \int_0^{2/3} x(1-x)^2 dx - 3 \int_0^{2/3} x^3 dx$$

$$12 \int_0^{2/3} (x - 2x^2 + x^3) dx - \frac{3}{4} x^4 \Big|_0^{2/3} = 8/3 - \frac{64}{27} + \frac{16}{27} - \frac{4}{27} = \frac{20}{27}$$

(c) No! $f(x,y) \neq f_X(x) \cdot f_Y(y)$ because of the restriction $0 \le x+y \le 1$

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5. If X_1 and X_2 are independent exponential random variables, each having parameter λ find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$. (12 points) (page 290, #6.58)

Step1 Solve for x_1, x_2

$x_1 = \ln y_2$

$x_2 = y_1 - \ln y_2$

Step 2

$$J = \begin{vmatrix} 0 & \frac{1}{y_2} \\ 1 & -\frac{1}{y_2} \end{vmatrix} = -\frac{1}{y_2}$$

Check

$$\int_1^\infty \frac{\lambda^2}{y_2} \int_{\ln y_2}^\infty e^{-\lambda y_1} dy_1\, dy_2 = \int_1^\infty \frac{\lambda}{y_2} e^{-\lambda \ln y_2} dy_2 = \int_0^\infty \lambda e^{-\lambda u} du = 1$$

$\ln y_2 = u$, $\frac{dy_2}{y_2} = du$

Step3 Substitute × $|J|$

$f(x_1, x_2) = \lambda^2 e^{-\lambda(x_1 + x_2)}, \quad x_1 > 0, x_2 \geq 0$

$$f(y_1, y_2) = \begin{cases} \lambda^2 \frac{e^{-\lambda y_1}}{y_2}, & y_2 > 1, \ \ln(y_2) < y_1 \\ 0 & \text{elsewhere} \end{cases}$$

6. Let X and Y have joint density given by

$$f(x,y) = \begin{cases} 2e^{-2x}/x, \ 0 \le x < \infty, 0 \le y \le x, \\ 0, \quad \textit{elsewhere.} \end{cases}$$

Compute E(XY), E(X) and E(Y) (10 points) (7.4, page 373 and 7.38, page 375)

(i) $E(XY) = \int_0^\infty \int_0^x \cancel{x} y \frac{2e^{-2x}}{\cancel{x}} dy\, dx = \int_0^\infty \frac{y^2}{2}\Big|_0^x 2e^{-2x} dx = E\frac{W^2}{2}$

$W \sim Exp(\lambda)$, $\lambda = 2$

(i) $EW^2 = Var(W) + (EW)^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{2}{\lambda^2} = \frac{1}{2}$

(i) $\frac{EW^2}{2} = \boxed{\frac{1}{4}}$

$E(X) = \int_0^\infty \int_0^x \cancel{x} \frac{2e^{-2x}}{\cancel{x}} dy\, dx$

$= \int_0^\infty x\, 2e^{-2x} dx = EW = \frac{1}{\lambda} = \boxed{\frac{1}{2}}$

$EY = \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x} dy\, dx = \int_0^\infty \frac{x}{2} 2e^{-2x} dx = \frac{EW}{2} = \frac{1}{2\lambda} = \boxed{\frac{1}{4}}$

7. Suppose that a die is rolled twice. Find the probability mass function associated with the random variable X: minimum value to appear in the two rolls. Compute the variance of X. (14 points) (4.7 and 4.8 (b) page 173)

Y_1 = Dice 2 values

X_1 = Dice 1 values

$\min(X_1, Y_1) = X_1 \wedge Y_1$ values	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

$X = M = X_1 \wedge Y_1$	$p(m)$	$m\,p(m)$	$m^2 p(m)$
1	11/36	11/36	11/36
2	9/36	18/36	36/36
3	7/36	21/36	63/36
4	5/36	20/36	80/36
5	3/36	15/36	75/36
6	1/36	6/36	36/36
	36/36	91/36	301/36

$$EX = EM = \frac{91}{36} = 2.52\dot{7}$$

$$Var(X) = \frac{301}{36} - \left(\frac{91}{36}\right)^2 = 1.971450617$$

$$= 8.36\dot{1} - 6.389660494$$

8. A parallel system functions whenever at least one of its components works. Consider a parallel system of n components, and suppose that each component works independently with probability 1/3. Find the conditional probability that component 1 works given that the system is functioning. (10points)

P(Component 1 works / System is functioning)

$$= \frac{P(\text{Component 1 works} \cap \text{System is functioning})}{P(\text{system is functioning})}$$

$$= \frac{P(\text{component 1 works})}{P(\text{system is functioning})}$$

$$= \frac{1/3}{1 - P(\text{system is not functioning})}$$

$$= \frac{1/3}{1 - P(\text{none of the components work})}$$

$$= \frac{1/3}{1 - (2/3)^n}$$

END