Mathematics 662(101) : Probability Distributions

Final Examination (12/15/2005)

Time : 2hrs 30 minutes
M.S. students answer any four questions.
Ph.D. students answer all five questions.

1. (i) Suppose X is a positive random variable ($P(X > 0) = 1$) and that EX exists.

(i) Use Cauchy-Schwarz inequality, to show that $E(\frac{1}{X}) \geq \frac{1}{EX}$, and hence prove,

$$\text{cov}\ (X, \frac{1}{X}) \leq 0.$$

(ii) Let Y be another random variable, such that $\text{cov}\ (X, Y + \frac{1}{X}) > 0$. Then show that we must have $\text{cov}\ (X, Y) > 0$.

2. $X_i,\ i = 1, 2$ are independent, but not identically distributed, exponential random variables, with densities,

$$f_i(x) = \lambda_i e^{-\lambda_i x} 1_{x>0}, \quad i = 1, 2$$

Since $\lambda_1 \neq \lambda_2$, assume $\lambda_1 > \lambda_2 > 0$ without loss of generality.

(i) Show that the density of $(X_1 + X_2)$ is

$$g(t) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t})\ 1_{\{t>0\}}$$

(ii) Compute the c.d.f. $G(t)$ of the sum $X_1 + X_2$.
(iii) Use (ii) to compute $E(X_1 + X_2)$.
(iv) Show that "*failure intensity*" function $r_G(t)$ of G, defined by

$$r_G(t) = \frac{g(t)}{1 - G(t)}, \quad t > 0,$$

is increasing in t.

3. X_1, X_2 are i.i.d. Cauchy distributed, with a common density,

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$

Using transformation of variables, show that the mean $\bar{X} := \frac{1}{2}(X_1 + X_2)$ has the same Cauchy distribution.

4. (i) Suppose X is exponentially distributed with mean Λ^{-1} $(0 < \Lambda < \infty)$. Suppose moreover that Λ itself is exponentially distributed with mean $\theta^{-1} \in (0, \infty)$. (In other words, conditional on Λ; $X|\Lambda \sim G$ where G is the exponential c.d.f. with mean Λ^{-1}). Show that the unconditional distribution of X has c.d.f.,

$$P(X \leq x) = \frac{x}{x + \theta}, \quad x > 0.$$

What can you say about EX ?

(Think of X as the life of an equipment, and Λ as a parameter which describes in some sense the severity of the environment in which the equipment has to work. The distribution of Λ the describes the varition of the environment)

5. Suppose the joint distribution of the continuous random variables (X, Y) is specified via the following marginal and conditional distributions :

$$X \sim U(0, 1),$$
$$Y|X \sim U(X, X + 1)$$

i.e., X is *Uniformly distributed* on $(0, 1)$, and the conditional distribution of Y given X is *Uniform* on the interval $(x, x + 1)$.

(i) Find the p.d.f. of the marginal diatribution of Y,

(ii) Compute $P(X + Y > 1)$.