Physics 430, Classical Mechanics **Name ________________________**
Exam 1, 2009 Oct 06

Instructions: No books, notes, or "cheat sheet" allowed. You may use a calculator, but no other electronic devices during the exam. Please turn your cell phone off.

Please note that the NJIT honor code applies to this exam, as it does to all activities related to this course.

Each part of each question is worth 5 points, as noted, for a total of 80 points. You may use additional sheets of paper if you need more room to work. Clearly label the portion of each additional sheet with the problem number. Show all work. Right answers with no work will be marked wrong. **Be careful with notation** (e.g. vectors underlined, unit vectors with hats, etc.).

1. Given the two vectors **a** = (1, 3, –1) and **b** = (0, 0, 5)

   (a) (5 points) Find the dot product and use it to find the angle between the two vectors.

   Start with definition: $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$

   $$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = 1\times 0 + 3\times 0 - 1\times 5 = -5$$

   $$a = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11} \quad \text{and} \quad b = \sqrt{0^2 + 0^2 + 5^2} = 5$$

   $$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{ab} = -\frac{5}{5\sqrt{11}} = -0.3015 \quad \text{so} \quad \theta = 107.5^\circ$$

   (b) (5 points) Find the cross product and use its magnitude to find the (same) angle between the two vectors. If you get a different answer from (a), state why, and say which is the correct angle.

   Start with definition: $|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$

   $$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y)\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\hat{\mathbf{y}} + (a_x b_y - a_y b_z)\hat{\mathbf{z}}$$

   $$= (3\times 5 + 1\times 0)\hat{\mathbf{x}} + (-1\times 0 - 1\times 5)\hat{\mathbf{y}} + (1\times 0 - 3\times 0)\hat{\mathbf{z}} = 15\hat{\mathbf{x}} - 5\hat{\mathbf{y}}$$

   $$|\mathbf{a}\times\mathbf{b}| = \sqrt{15^2 + (-5)^2} = \sqrt{250} = 15.81$$

   $$\sin\theta = \frac{\mathbf{a}\times\mathbf{b}}{ab} = -\frac{15.81}{5\sqrt{11}} = 0.9535 \quad \text{so} \quad \theta = 72.5^\circ$$

   This is different from part (a), because of 180° ambiguity.

   The correct angle is 107.5° as in part (a).

2. A particle's potential energy is $U(\mathbf{r}) = k(x^2 + xy^2 - 2z^2)$, where k is a constant.
(a) (5 points) What is the force on the particle?

$$\mathbf{F} = -\nabla U = -\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)U(\mathbf{r}) = -k\left[(2x+y^2)\hat{\mathbf{x}} + 2xy\hat{\mathbf{y}} - 4z\hat{\mathbf{z}}\right]$$

$$\text{or} \quad \mathbf{F} = -k(2x+y^2, 2xy, -4z)$$

(b) (5 points) Is the force you obtained in part (a) conservative (i.e. is $\nabla \times \mathbf{F} = 0$)? Show your work.

$$\nabla \times \mathbf{F} = -k\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+y^2 & 2xy & -4z \end{vmatrix}$$

$$= -k\left[\left(\frac{\partial}{\partial y}(-4z) - \frac{\partial}{\partial z}2xy\right)\hat{\mathbf{x}} + \left(\frac{\partial}{\partial z}(2x+y^2) - \frac{\partial}{\partial x}(-4z)\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial x}2xy - \frac{\partial}{\partial y}(2x+y^2)\right)\hat{\mathbf{z}}\right]$$

Term by term, all terms are zero, so the force is conservative.

3. The equation of motion for a vertically falling body under linear air resistance (drag force $f(v) = -bv$) is $m\dot{v} = mg - bv$, while for quadratic resistance (drag force $f(v) = -cv^2$) it is $m\dot{v} = mg - cv^2$.
(a) (5 points) Use these equations to derive the expressions for the terminal velocity in the two cases. Explain your reasoning.

The body accelerates until the drag force equals the gravity force.
When that occurs, the body stops accelerating and $\dot{v} = 0$.
The velocity at that time is $v = v_{ter}$. Then:

$mg - bv_{ter} = 0$ so $v_{ter} = \frac{mg}{b}$ (linear case)

$mg - cv_{ter}^2 = 0$ so $v_{ter} = \sqrt{\frac{mg}{c}}$ (quadratic case)

(b) (5 points) Rewrite the quadratic equation $m\dot{v} = mg - cv^2$ in terms of the terminal velocity v_{ter}, and then write it in separated form (terms involving dv on one side, and dt on the other).

$$m\dot{v} = mg - cv^2 \Rightarrow \frac{m}{c}\frac{dv}{dt} = \frac{mg}{c} - v^2 = v_{ter}^2 - v^2.$$

In separated form, this is:

$$\frac{dv}{v_{ter}^2 - v^2} = \frac{c}{m}dt.$$

(c) (5 points) Solve for v, given that $\int \frac{dv}{(v_{ter}^2 - v^2)} = \frac{1}{v_{ter}} \operatorname{arctanh}\left(\frac{v}{v_{ter}}\right)$, for the case of the body starting from rest. What is the time constant τ in terms of other constants in your solution?

Integrating both sides:

$$\int \frac{dv}{v_{ter}^2 - v^2} = \frac{c}{m}t$$

so

$$\frac{1}{v_{ter}} \operatorname{arctanh}\left(\frac{v}{v_{ter}}\right) = \frac{ct}{m} \Rightarrow v = v_{ter} \tanh\left(\frac{cv_{ter}}{m}t\right) = v_{ter} \tanh\left(\frac{t}{\tau}\right).$$

Here, the time constant is $\tau = \frac{m}{cv_{ter}} = \frac{v_{ter}}{g}$.

4. The Apollo 11 mission to the Moon weighed 7.15 million pounds fully fueled. Its payload had to reach a speed of 25,000 mph in order to escape the Earth, and its first-stage exhaust velocity was about 5,700 mph. How many pounds of payload could escape the Earth if the rocket were a single stage? [The actual payload was about 100,000 lbs.] Recall that the relevant equation is $v - v_o = v_{ex} \ln(m_o/m)$. (5 points)

This is a straight plug in. Rearrange the equation to solve for the final mass m, or more correctly, the final weight mg:

$$mg = m_o g\, e^{-v/v_{ex}} = 7.15 \times 10^6 \text{ lb}\, e^{-25000/5700} = 89000 \text{ lb}.$$

5. (a) The element of volume in Cartesian coordinates is $dV = dxdydz$. Give the corresponding expression in spherical polar coordinates r, θ, ϕ. (5 points)

The element of volume in spherical polar coordinates is $dV = r^2 dr \sin\theta d\theta d\phi$.

(b) By direct integration, $V = \int dV$, over the appropriate limits, find the volume of the part of the sphere of radius R shown in Fig. 1. Show all steps, including the limits on each of the three integrals needed. Show that your volume is 1/8th of a complete sphere. (5 points)

$$V = \int dV = \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{\pi/4} d\varphi$$
$$= \frac{R^3}{3}\left(-\cos\theta\Big|_0^\pi\right)\frac{\pi}{4} = \frac{\pi}{6}R^3$$
$$\frac{V}{V_{sphere}} = \frac{\frac{\pi}{6}R^3}{\frac{4}{3}\pi R^3} = \frac{1}{8}$$

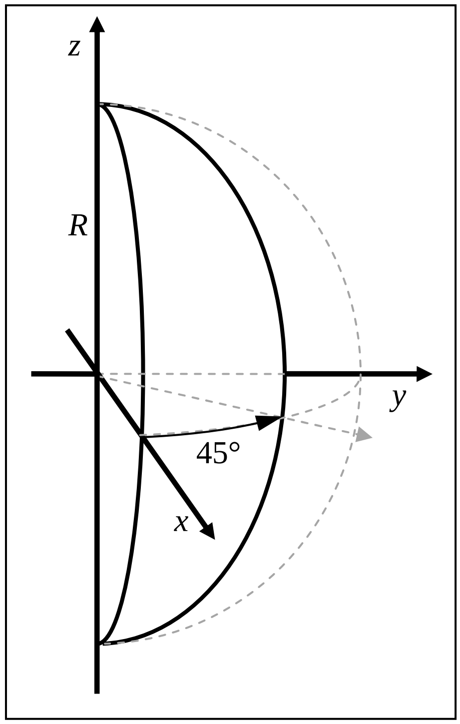


Fig. 1. Slice of a sphere of radius R.

(c) (5 points) Using the definition of center of mass

$$\mathbf{R} = \frac{1}{M}\int \rho \mathbf{r} dV$$

write down the expression for the x-component of the CM vector (call it X), in terms of integrals over dr, $d\theta$ and $d\phi$, assuming the density is constant and the total mass is M. Indicate the appropriate limits for each coordinate. Do not do the integration yet, you'll do it in part (d).

Using $x = r\sin\theta\cos\phi$, then $xdV = r^3 dr \sin^2\theta d\theta \cos\phi d\phi$, so

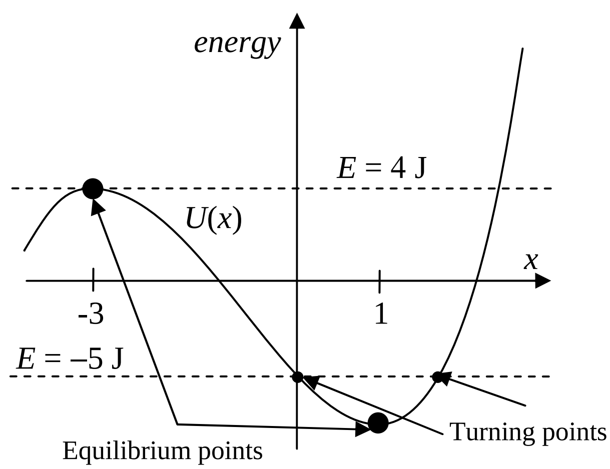
$$X = \frac{\rho}{M}\int_0^R r^3 dr \int_0^\pi \sin^2\theta\, d\theta \int_0^{\pi/4}\cos\phi\, d\phi$$
$$= \frac{6}{\pi R^3}\int_0^R r^3 dr \int_0^\pi \sin^2\theta\, d\theta \int_0^{\pi/4}\cos\phi\, d\phi$$

(d) (5 points) Do the integration to show $X = \frac{3\sqrt{2}}{8}R$. [*Hint:* $\int_0^T \sin^2\theta\, d\theta = \frac{T}{2}$.]

$$X = \frac{6}{\pi R^3}\frac{R^4}{4}\frac{\pi}{2}\int_0^{\pi/4}\cos\phi\, d\phi = \frac{6}{\pi R^3}\frac{R^4}{4}\frac{\pi}{2}\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{8}R.$$

6. The plot below schematically shows the graph of the one-dimensional potential energy, $U(x) = x^3/3 + x^2 - 3x - 5$ J, acting without friction on an object of mass 10 kg, where x is in m.

(a) (5 points) Find all equilibrium points x_n of the object under potential energy $U(x)$, mathematically, and determine which are stable and which are unstable. Plot these equilibrium points on the graph and label their locations on the x axis.



The equilibria occur where the slope of the potential energy curve is zero, i.e.

$$\frac{dU}{dx} = x^2 + 2x - 3 = 0. \quad \text{This has roots } x = +1 \text{ m}, -3 \text{ m}.$$

To find if they are stable or unstable, check the curvature (2nd derivative) at each point:

$$\left.\frac{d^2U}{dx^2}\right|_{x=1} = 2(1) + 2 = 4 > 0 \Rightarrow \text{stable}$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=-3} = 2(-3) + 2 = -4 < 0 \Rightarrow \text{unstable}$$

(b) (5 points) Let the total energy of the object moving in the vicinity of the stable equilibrium point be $E = -5$ J. Show that $x = 0$ is a turning point of the object. Sketch the total energy and both turning points on the graph [do not calculate the second turning point]. What is the force on the object at $x = 0$, in N?

The turning points occur where the potential energy equals the total energy, $E = U(x)$.

$$-5 \text{ J} = \frac{x^3}{3} + x^2 - 3x - 5 \text{ J}. \quad \text{Thus } \frac{x^3}{3} + x^2 - 3x = 0, \text{ which clearly has a root at } x = 0.$$

The force at this location is found from:

$$\mathbf{F} = -\nabla U = -\frac{dU}{dx}\hat{\mathbf{x}} = \left(-x^2 - 2x + 3\right)\hat{\mathbf{x}} = 3\hat{\mathbf{x}} \text{ N at } x = 0.$$

(c) (5 points) For the case of $E = -5$ J, what is the speed (m/s) of the object when it is at the $x = 0$ turning point? What is its speed when it is at the equilibrium point?

At a turning point, the kinetic energy is zero, so the object's speed is zero there.

At the equilibrium point, $x = 1$, we have $E = T + U(x = 1)$:

$$T = E - U(x=1) = -5 - \frac{x^3}{3} - x^2 + 3x + 5 \text{ J} = -\frac{1}{3} - 1 + 3 \text{ J} = \frac{5}{3} \text{ J}.$$

The speed is determined from $T = \frac{1}{2}mv^2$, where $m = 10$ kg, so

$$v = \sqrt{\frac{2T}{m}} = 0.58 \text{ m/s}.$$

(d) (5 points) What is the maximum total energy the object could have and still be bound, i.e. remain always within a finite distance of the stable equilibrium point? Sketch and label this total energy on the graph.

Looking at the graph, the left turning point ceases to exist when the total energy exceeds the $U(x = -3)$ for the unstable turning point, i.e.

$$E = U(x = -3) = -27/3 + 9 + 9 - 5 \text{ J} = 4 \text{ J}.$$