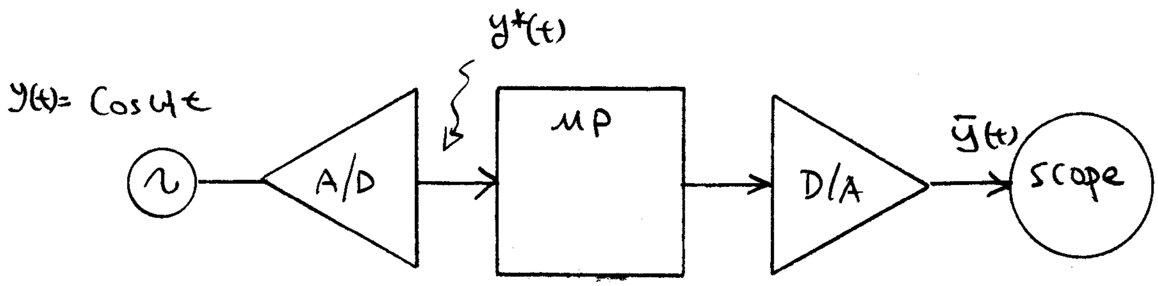
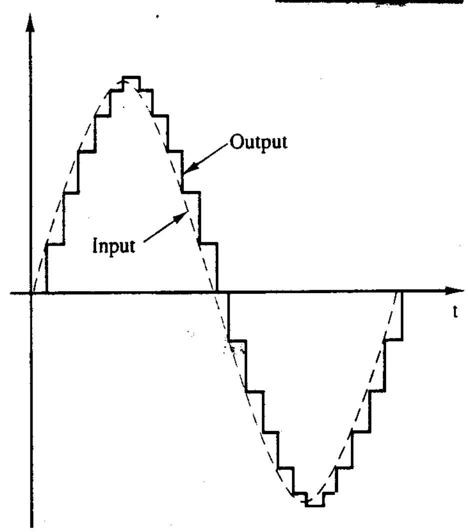
Sampling and Reconstruction

— the gateway between the world and the computer control system

Consider the following experiment:

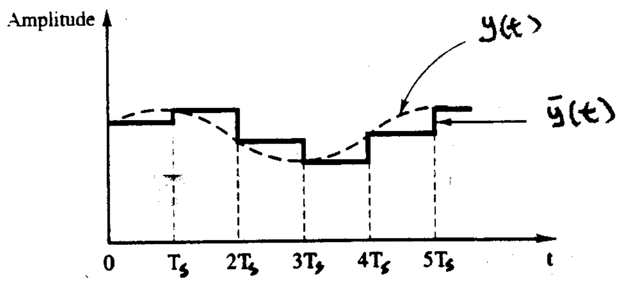




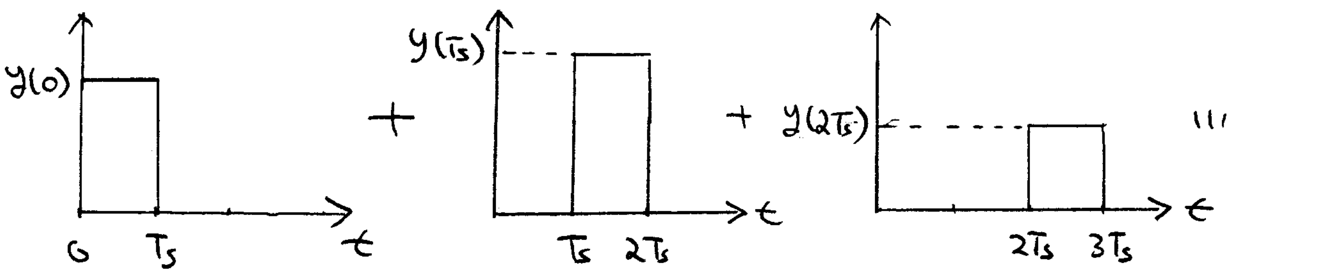
- What caused the output to have a staircase appearance?

- What are the properties of $y^*(t)$ and $\bar{y}(t)$ in relation to $y(t)$?

To examine the sample & hold operation in more detail :



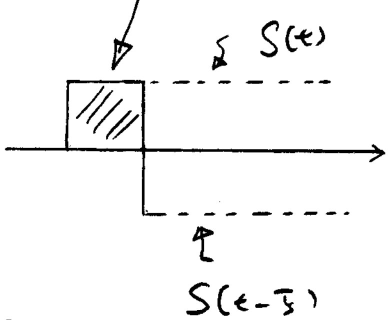
$\bar{y}(t)$ can be consider as a sequence of pulses:



$y(0)\left[S(t) - S(t-T_s)\right]$

$y(T_s)\left[S(t-T_s) - S(t-2T_s)\right]$

$y(2T_s)\left[S(t-2T_s) - S(t-3T_s)\right]$



unit pulse

Thus we can write

$$\bar{y}(t) = \sum_{n=0}^{\infty} y(nT_s)\left[\underbrace{S(t-nT_s) - S(t-(n+1)T_s)}_{\substack{\text{unit pulse at} \\ nT_s \le t < (n+1)T_s}}\right]$$

Apply Laplace transform to $\bar{y}(t)$

$$\bar{Y}(s) = \sum_{n=0}^{\infty} y(nT_s)\,\mathcal{L}\left[S(t-nT_s) - S(t-(n+1)T_s)\right]$$

Fact: $\mathcal{L}\left[S(t-T)\right] = \dfrac{e^{-sT}}{s}$

Therefore

$$\bar{Y}(s) = \sum_{n=0}^{\infty} y(nT_s)\,\frac{e^{-snT_s} - e^{-s(n+1)T_s}}{s}$$

$$= \sum_{n=0}^{\infty} y(nT_s)\,e^{-snT_s}\;\frac{1-e^{-sT_s}}{s}$$

$$= Y^*(s)\;G_{ho}(s)$$

$Y^*(s)$ = star transform of Y

$G_{ho}(s)$ = transfer function of zero order hold

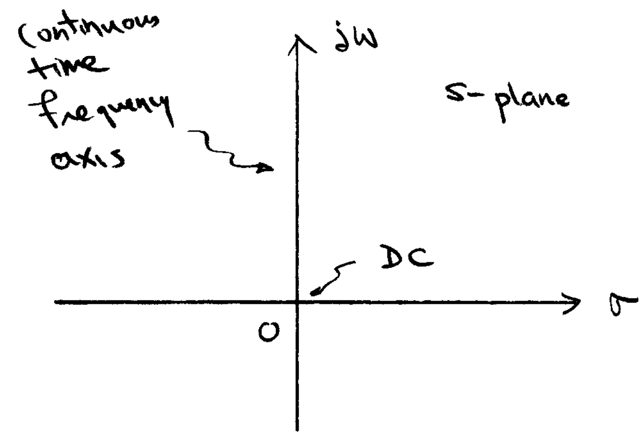
Note:

1. Y^* and G_{ho} must appear together in real world
2. Y^* accounts for the sampling process
3. G_{ho} accounts for the reconstruction

(A) Properties of $Y^*(s)$

Best to examine the frequency response, i.e.

$s = j\omega$



$Y^*(j\omega)$ or just $Y^*(\omega)$

Properties of $Y^*(\omega)$

1. $Y^*(\omega) = Y^*\left(\omega + \frac{2\pi}{T_s}\right)$ periodic with $\frac{2\pi}{T_s}$

T_s = sampling period

$f_s = \frac{1}{T_s}$ = sampling frequency (Hz)

$\omega_s = \frac{2\pi}{T_s}$ = sampling frequency (rad/sec)

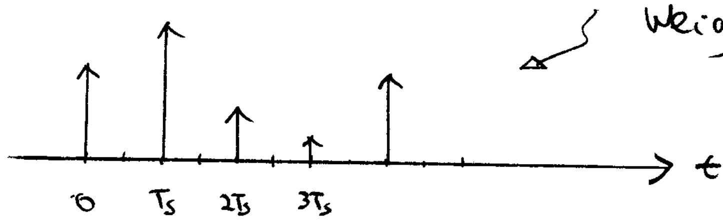
(2) $y^*(t) = \mathcal{L}^{-1}[Y^*(s)]$

$$= \sum_{n=0}^{\infty} y(nT_s)\, \mathcal{L}^{-1}[e^{-snT_s}]$$

Fact $\mathcal{L}^{-1}[e^{-sT}] = \delta(t-T)$

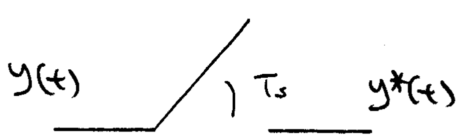
↑ impulse function

$\therefore$ $$y^*(t) = \sum_{n=0}^{\infty} y(nT_s)\, \delta(t - nT_s)$$

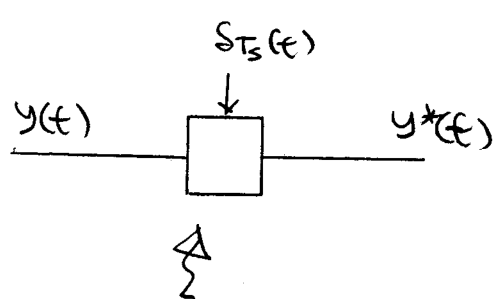


weighted impulse train

$\therefore$ sampling process is AKA impulse sampling



or



$$\delta_{T_s}(t) = \sum_{n=0}^{\infty} \delta(t - nT_s)$$

impulse modulator

③ Poisson's formula,

$$Y^*(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} Y\left(\omega + \frac{2\pi n}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} Y(\omega + n\omega_s)$$

Proof:

a) $$\frac{1}{2\pi} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} Y^*(\omega) e^{j\omega n T_s} d\omega = \frac{1}{2\pi} \sum_m y(mT_s) \underbrace{\int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} e^{j\omega m T_s} e^{-j\omega n T_s} d\omega}_{\substack{= 2\pi/T_s,\ m = n \\ = 0,\ m \neq n}}$$

$$= \frac{y(nT_s)}{T_s}$$

b) $$y(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega n T_s} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} Y(\omega) e^{j\omega n T_s} d\omega + \int_{\frac{\pi}{T_s}}^{\frac{2\pi}{T_s}} Y(\omega) e^{j\omega n T_s} d\omega + \int_{-\frac{2\pi}{T_s}}^{-\frac{\pi}{T_s}} Y(\omega) e^{j\omega n T_s} d\omega + \cdots \right]$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{(2k-1)\frac{\pi}{T_s}}^{(2k+1)\frac{\pi}{T_s}} Y(\omega) e^{j\omega n T_s} d\omega$$

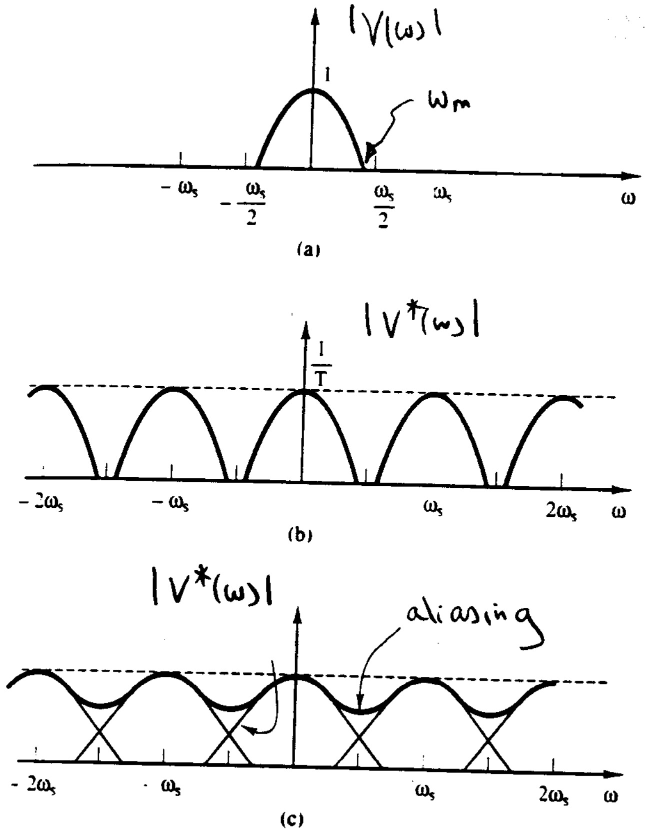
On letting $\omega' = \omega - \frac{2\pi k}{T_s}$

$$y(nT_s) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} Y\left(\omega' + \frac{2\pi k}{T_s}\right) e^{j n T_s \left(\omega' + \frac{2\pi k}{T_s}\right)} d\omega'$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{T_s}}^{\frac{\pi}{T_s}} \sum_{n=-\infty}^{\infty} Y\left(\omega + \frac{2\pi n}{T_s}\right) e^{j n T_s \omega} d\omega$$

Comparison of a) & b) establishes the result.

Implications of Property 3, (Poisson's Formula)



(I) $\omega_m < \frac{\omega_s}{2}$

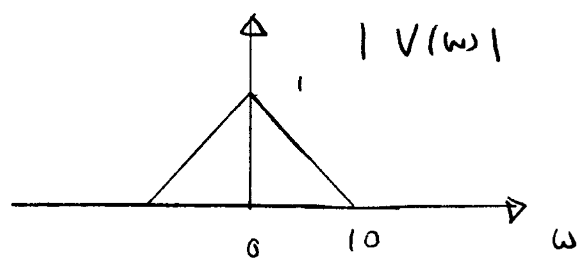
(II) $\omega_m > \frac{\omega_s}{2}$

(I) $V(\omega)$ component can be recovered

(II) $V(\omega)$ cannot be recovered due to aliasing

Class Ex

For $|V(\omega)|$ given by



Sketch $V^*(\omega)$ when

a) $T_s = \frac{2\pi}{10}$ sec

b) $T_s = \frac{2\pi}{20}$ sec

c) $T_s = \frac{2\pi}{50}$ sec

e.g.

$$v(t) = \sin 2t$$

$$v(nT_s) = \sin 2nT_s$$

what is $V^*(\omega)$?

$$V^*(\omega) = \sum_{n=0}^{\infty} \sin(2nT_s)\, e^{-j\omega nT_s}$$

But $$\sin(2nT_s) = \frac{e^{j2nT_s} - e^{-j2nT_s}}{2j}$$

Thus $$V^*(\omega) = \sum_{n=0}^{\infty} \frac{e^{jnT_s(2-\omega)} - e^{-jnT_s(2+\omega)}}{2j}$$

$$\boxed{\text{Handy formula:} \quad 1 + x + x^2 + \cdots = \frac{1}{1-x} \qquad |x| < 1}$$

$$\therefore V^*(\omega) = \frac{1}{2j}\left\{ \frac{1}{1 - e^{jT_s(2-\omega)}} - \frac{1}{1 - e^{-jT_s(2+\omega)}} \right\}$$

$$= \frac{1}{2j}\left\{ \frac{1 - e^{-jT_s(2+\omega)} - 1 + e^{jT_s(2-\omega)}}{\left(1 - e^{jT_s(2-\omega)}\right)\left(1 - e^{-jT_s(2+\omega)}\right)} \right\}$$

$$= \frac{e^{-j\omega T_s}\left(e^{j2T_s} - e^{-j2T_s}\right)/2j}{1 - e^{-j2T_s}e^{-j\omega T_s} - e^{j2T_s}e^{-j\omega T_s} + e^{-j2\omega T_s}}$$

$$= \frac{e^{-j\omega T_s}\sin(2T_s)}{1 - 2\cos(2T_s)e^{-j\omega T_s} + e^{-j2\omega T_s}}$$

Similarly, $v(t) = \cos 2t$

$$V^*(\omega) = \frac{1 - e^{-j\omega T_s}\cos(2T_s)}{1 - 2\cos(2T_s)e^{-j\omega T_s} + e^{-j2\omega T_s}}$$

__Ex__ : Using matlab, plot

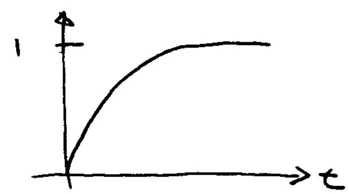
$V^*(\omega)$ for $\sin 2t$
$\cos 2t$

using $T_s = 0.1$ sec
$T_s = 0.01$ sec

Class Ex

e.g. $v(t) = (1 - e^{-t})\, S(t)$

step function



$$v(nT_s) = 1 - e^{-nT_s} \qquad n = 0, 1, 2, 3, \ldots$$

$$V^*(\omega) = \sum_{n=0}^{\infty} v(nT_s) e^{-jn\omega T_s}$$

$$= \sum_{n=0}^{\infty} (1 - e^{-nT_s})\, e^{-jn\omega T_s}$$

$$= \sum_{n=0}^{\infty} e^{-jn\omega T_s} - \sum_{n=0}^{\infty} e^{-(1+j\omega) nT_s}$$

$$= \frac{1}{1 - e^{-j\omega T_s}} - \frac{1}{1 - e^{-T_s(1+j\omega)}}$$

$$= \frac{(1 - e^{-T_s})\, e^{j\omega T_s}}{(e^{j\omega T_s} - 1)(e^{j\omega T_s} - e^{-T_s})}$$

Shannon Sampling & Reconstruction Theorems

(I) A continuous time signal $v(t)$ with bandwidth ω_m so that

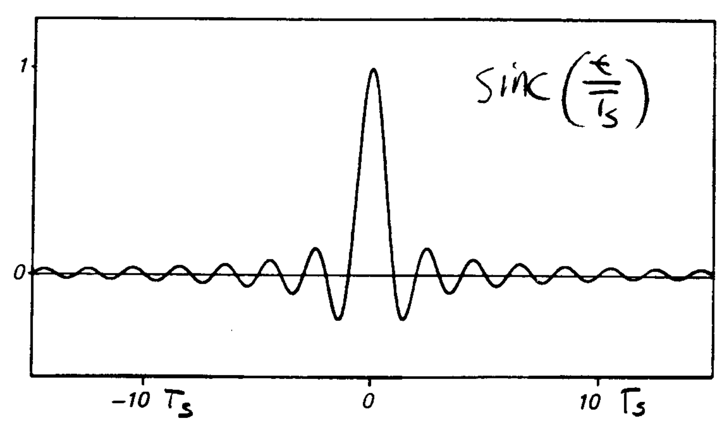
$$|V(\omega)| = 0 \qquad |\omega| > \omega_m$$

is given uniquely by its values at $t = nT_s$ ($v(nT_s)$) if

$$\omega_s > 2\omega_m$$

(II) $v(t)$ can be reconstructed from $v(nT_s)$ by

$$v(t) = \sum_{n=-\infty}^{\infty} v(nT_s)\ \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$



non-causal

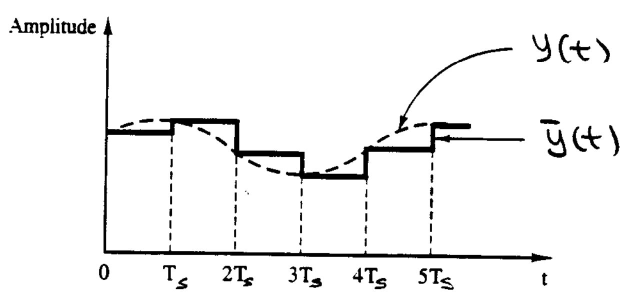
Zero Order Hold & Properties

$$y(t) = y(nT_s) + \dot{y}(nT_s)(t - nT_s) + \ddot{y}(nT_s)\cdot\frac{(t-nT_s)^2}{2!} + \ldots$$

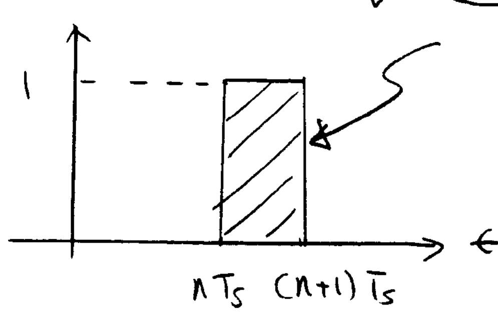
$$nT_s \le t < (n+1)T_s$$

zero order hold :

$$\bar{y}(t) = y(nT_s) \quad , \quad nT_s \le t < (n+1)T_s$$



$$\bar{y}(t) = \sum_{n=0}^{\infty} y(nT_s)\left[s(t - nT_s) - s(t - (n+1)T_s) \right]$$



$$\bar{Y}(s) = \mathcal{L}\left[\bar{y}(t)\right] = \int_0^\infty \bar{y}(t)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} y(nT_s)\ \frac{1 - e^{-sT_s}}{s}\ e^{-snT_s}$$

$$= G_{ho}(s)\ Y^*(s)$$

$$G_{ho}(s) = \frac{1 - e^{-sT_s}}{s}$$

<u>Frequency Response of $G_{ho}(s)$</u>

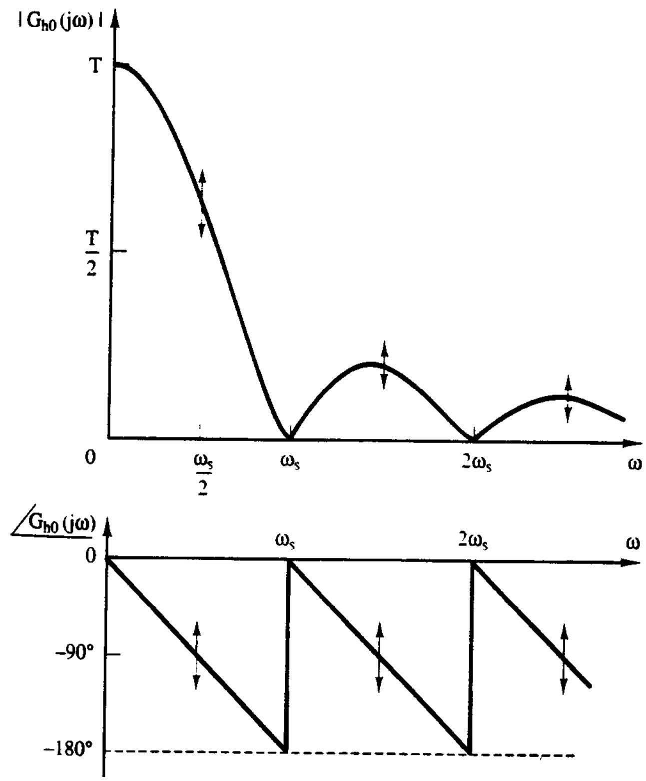
$$G_{ho}(\omega) = \frac{1 - e^{-j\omega T_s}}{j\omega}$$

$$= \frac{2}{\omega}\, e^{-\frac{j\omega T_s}{2}}\ \sin\frac{\omega T_s}{2}$$

$$|G_{ho}(\omega)| = T_s \left|\operatorname{sinc}\left(\frac{\omega}{\omega_s}\right)\right|$$

$$\measuredangle G_{ho}(\omega) = -\frac{\omega\pi}{\omega_s} + \measuredangle \operatorname{sinc}\left(\frac{\omega}{\omega_s}\right)$$

$\measuredangle$ either 0 or 180°



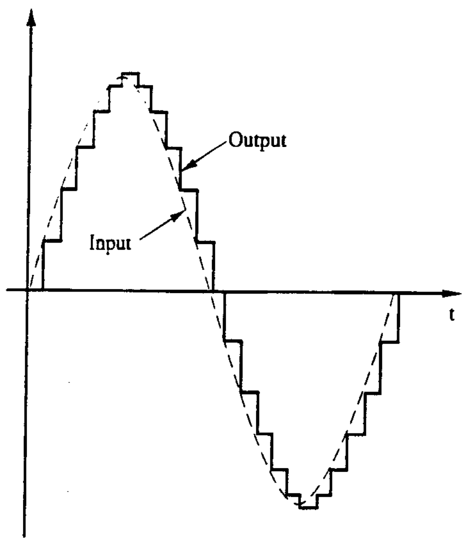
Error of Zero Order Hold :

$$e_{ho} = \max_{n} | y((n+1)T_s) - y(nT_s) |$$

$$\leq T_s \max_{t} [y'(t)]$$

Example

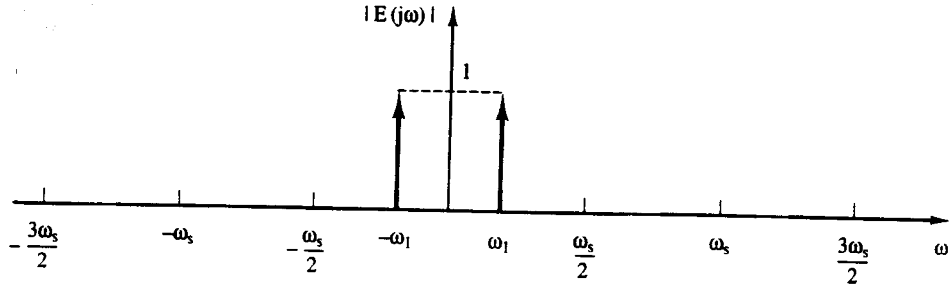
$y(t) = \cos \omega_1 t \longrightarrow$ zero order hold $\longrightarrow \cos \omega_1 n T_s = \bar{y}(t)$



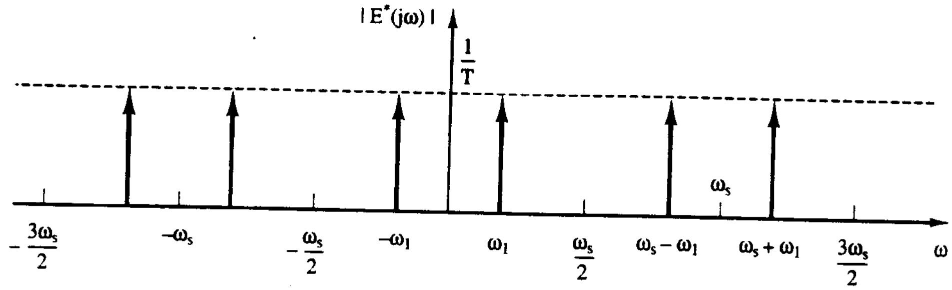
$$\bar{Y}(\omega) = \mathcal{F}\left[\bar{y}(t)\right] = Y^*(\omega)\; G_{ho}(\omega)$$

$$Y^*(\omega) = \sum_{n=0}^{\infty} y(nT_s)\, e^{-j\omega n T_s}$$

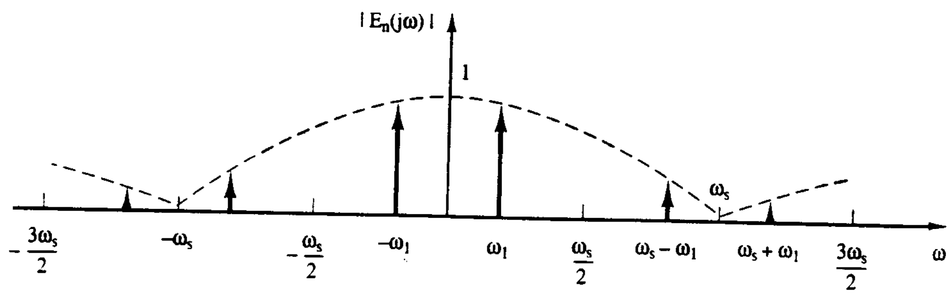
$$G_{ho}(\omega) = \frac{1 - e^{-j\omega T_s}}{j\omega}$$



(a)



(b)



(c)

Ex : Given $y(t) = \cos 5t$

and $T_s = 0.1$ s

Calculate the frequencies of all harmonic components within a ± 30 Hz range.

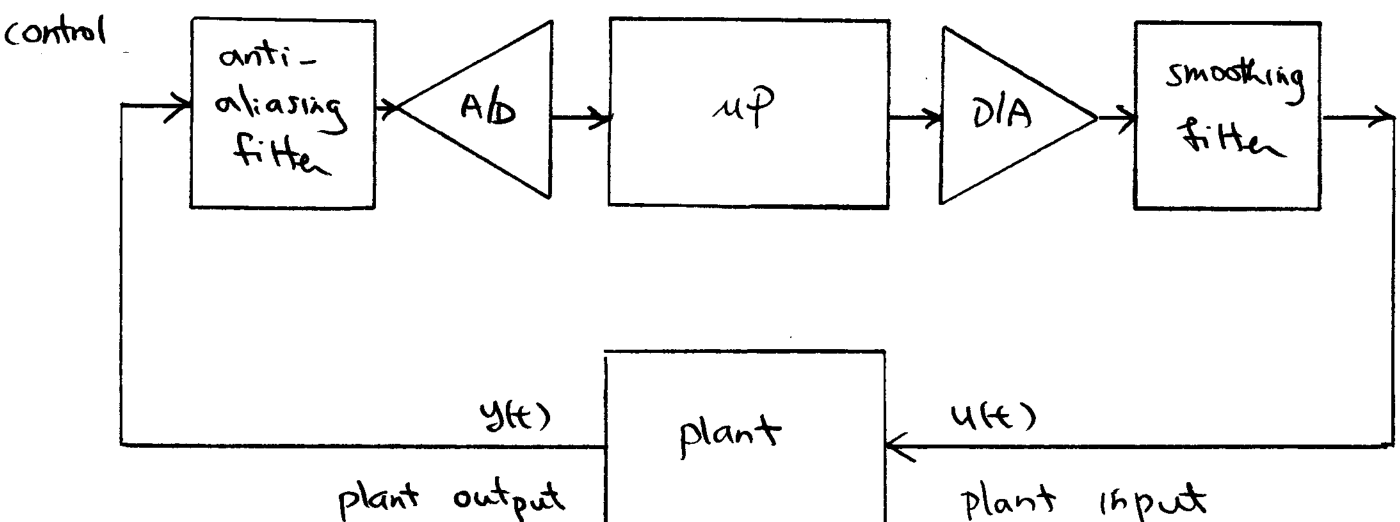
Summary of proper sampling & reconstruction

① Apply low pass filter to the input to restrict signal bandwidth

→ anti-aliasing filter

② Apply low pass filter to the D/A output to smooth out the "staircase"

→ smoothing filter



— Further Exploration —

ECE664 FALL, 00
Real-Time Control Systems

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Tentative Schedule

Week	Date	Topic
1		Introduction to real-time control systems
2		Architecture of DSP systems: the TMS320C25
3		Programming the C25
4		A/D, D/A, PLA, and other peripherals
5		Properties of sampled-data systems
6		Properties of sampled-data systems (cont'd)
7		Review of Z-transform
8		Mid-term Examination
9		Digital controller design I: Parameter optimized controllers
10		Digital controller design II: State controllers
11		Digital controller design III: Feedforward and Cancellation controllers
12		Command shaping and applications
13		Practical issues: sampling rate, dead-time, scaling, reset windup, etc.
14		Experiment/ project presentation
15		Final Exam

Grading Scheme: 30% Midterm, 30 % Final, 40% Project

Text: Computer Control Systems: Design and Theory, 3rd Ed., Astrom & Wittenmark, Prentice-Hall.

Required Software:
MATLAB Student Version, Mathworks, Inc.

References

1. Applied Optimal Control & Estimation: digital design & implementation, by F.L. Lewis, Prentice-Hall, 1992.

2. Feedback Control Systems, Van de Vegte, Prentice-Hall.