EXIT Charts for Turbo Trellis Coded Modulation

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***Abstract*— In this paper we extend the previously proposed extrinsic information transfer charts (EXIT) method to the analysis of the convergence of turbo codes to turbo trellis coded modulation (TTCM) schemes. The effectiveness of the proposed method is demonstrated through examples. The proposed method provides a convenient way to systematically compare between schemes and thus can be used as a tool in the design of TTCM.**

***Index Terms*— Turbo trellis coded modulation, convergence analysis, EXIT chart.**

I. Introduction

Convergence analysis of iterative decoding algorithms for turbo codes has received much attention recently due to its useful application to predicting code performance, its ability to provide insights into the encoder structure, and its usefulness in helping with the code design. Several methods have been proposed to analyze the convergence of the iterative decoders for binary turbo codes [1], [2], [3]. In particular, the extrinsic information transfer (EXIT) method [1] has created a lot of interest. The EXIT method has been extended to the analysis of nonbinary turbo codes in [4].

Turbo trellis coded modulation (TTCM) [5], [6], conjoins signal mapping techniques, such as Ungerboeck's signal space partition, with turbo coding, to achieve significant coding gains without increasing bandwidth. However the need for signal mapping makes the encoder structure more complex to design and analyze than binary turbo codes. Hence the convergence analysis is a very important tool for the design and comparison between TTCM schemes.

In TTCM, the systematic bits and parity bits are usually mapped to and transmitted as a single symbol. At the receiver, a symbol decoder is used instead of a binary decoder [5], [6]. Consequently, the systematic information and extrinsic information are not naturally separated as in binary turbo codes. The separation is not necessary for carrying out the decoding. It is however, needed for generating the EXIT chart. Due to the mixing of the systematic and extrinsic information, the approach for generating the EXIT chart for binary decoders, or even for nonbinary decoders [4], can not be directly applied to TTCM.

In this paper, we develop the EXIT chart for TTCM by showing a way to explicitly separate the systematic information and extrinsic information. To be specific, we develop the analysis based on the TTCM codes introduced in [5]. The application of the proposed method to other structures is fairly straightforward.

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The rest of the paper is organized as follows. In section II we briefly introduce the structure of TTCM and develop the EXIT chart for TTCM. A specific example is used to illustrate and demonstrate the proposed method in section III. Conclusions are drawn in section IV.

II. TTCM Structure and EXIT analysis

A. Encoder Structure

The structure of the TTCM encoder [5] is presented in Fig. 1. Assuming 2^{m+1}-ary constellation, at each time interval k, a symbol is transmitted representing m information bits $\mathbf{b}[k] = \{b[k,i]\}_{i=0}^{m-1}$. Of the m bits, possibly only $\widetilde{m}$ bits $\{b[k,i]\}_{i=0}^{\widetilde{m}-1}$, are encoded, whereas the other $(m-\widetilde{m})$ uncoded bits are input directly to the symbol mapper. The turbo encoder consists of two rate $\widetilde{m}/(\widetilde{m}+1)$ component recursive systematic convolutional (RSC) encoders, which are connected by a symbol ($\widetilde{m}$ bits) interleaver. These $\widetilde{m}$ encoded bits, are labeled for later use as the $2^{\widetilde{m}}$-ary level symbol $d[k] \triangleq \sum_{i=0}^{\widetilde{m}-1} b[k,i]2^i$. The parity bits $c_1[k]$ and $c_2[k]$ from the two component encoders (the parity bits from the encoder with interleaved input are deinterleaved to their original order) are alternatively punctured, and the surviving parity bit $c[k]$ is mapped together with the m data bits to M-QAM or MPSK symbols $s[k]$, and transmitted.

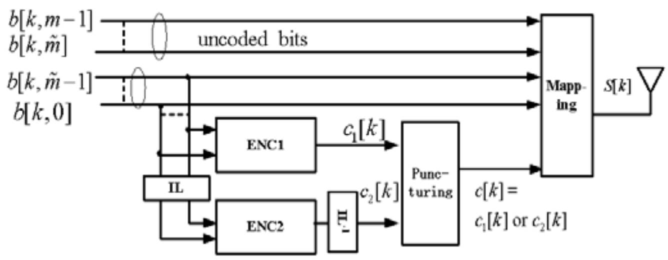

Fig. 1. Structure of TTCM encoder.

B. Decoder Structure

The structure of the TTCM decoder is depicted in Fig. 2. The decoder consists of two symbol maximum *a posteriori* (MAP) decoders, which exchange information about the symbols $\{d[k]\}$ formed by the *encoded* data bits. Of course, no information on uncoded bits is exchanged. Hence in TTCM, the systematic information represents the $\widetilde{m}$ coded bits $\{b[k,i]\}_{i=0}^{\widetilde{m}-1}$. Let the TTCM codeword consist of N symbols $(s[0], s[2], \ldots, s[N-1])$. The channel observation is given by

$$y[k] = s[k] + n[k], \tag{1}$$

where $n[k]$ is zero mean white Gaussian noise with variance σ^2. The signal to noise ratio (SNR) is then defined as $E[|s[k]|^2]/\sigma^2$.

Due to fact that the symbol $s[k]$ embodies both the systematic and parity bits, and consequent to the alternative puncturing of the parity bits, it is shown in [5] that half of the information exchanged between component decoders contains systematic information and half does not. Unlike the binary turbo code case, the *a posteriori* probabilities (APP) $\mathbf{L}_p[k] \triangleq \{\log P(d[k] = u|\{y[k]\}_{k=0}^{N-1})\}_{u=0}^{2^{\widetilde{m}}-1}$ computed by either component decoder can only be decomposed into two parts. The first part is the *a priori* input $\mathbf{L}_a[k] \triangleq \{\log P(d[k] = u)\}_{u=0}^{2^{\widetilde{m}}-1}$, passed from the other decoder. The second part is different for two cases: (i) at time interval k, $s[k]$ contains the parity bit from the *corresponding* component encoder; (ii) at time interval k, $s[k]$ contains parity bit from the *other* component encoder. For case (ii), the second part only contains the newly generated extrinsic information output, denoted as $\mathbf{L}_e^{out}[k]$. For case (i) however, the second part is a mixture of the extrinsic information and systematic information, denoted as $\mathbf{L}_{es}^{out}[k]$. That is,

$$\mathbf{L}_p[k] = \begin{cases} \mathbf{L}_a[k] + \mathbf{L}_{es}^{out}[k], & k \in \text{case (i)} \\ \mathbf{L}_a[k] + \mathbf{L}_e^{out}[k], & k \in \text{case (ii)} \end{cases} . \quad (2)$$

Accordingly, the *a priori* input to each component decoder $\mathbf{L}_a[k]$ is, for case (i), only extrinsic information, denoted as $\mathbf{L}_e^{in}[k]$; and is, for case (ii), the mixture of both extrinsic information and systematic information, denoted as $\mathbf{L}_{es}^{in}[k]$. That is

$$\mathbf{L}_a[k] = \begin{cases} \mathbf{L}_e^{in}[k], & k \in \text{case (i)} \\ \mathbf{L}_{es}^{in}[k], & k \in \text{case (ii)} \end{cases} . \quad (3)$$

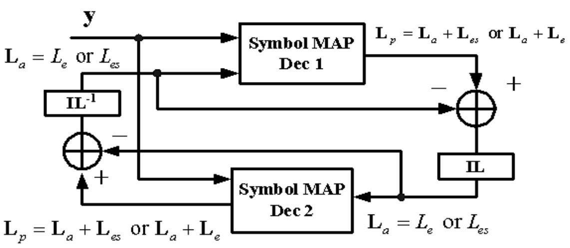

Fig. 2. Structure of TTCM decoder.

C. EXIT charts for TTCM

The EXIT chart is a tool for studying the convergence of turbo decoders without simulating the whole decoding process. The chart is generated by a single component decoder by mimicing the *a priori* input (as if generated by the other decoder) and measuring the decoder output. EXIT chart for binary codes was introduced in [1] and, was extended to nonbinary decoders in [4]. These methods are based on the assumption that the interleaver is long enough and the extrinsic information for each bit/symbol are mutually independent.

To generate the EXIT chart for TTCM, we follow [4] to create the *a priori* $\mathbf{L}_e^{in}[k]$ for $k \in$case (i) in (3), which satisfies $I(\mathbf{L}_e^{in}[k]; d[k]) = I_e^{in}$. Basically, this method assumes the log-likelihood ratios $\lambda[k,i] = \log \frac{P(b[k,i]=1)}{P(b[k,i]=0)}$, $i = 0,..,\widetilde{m}-1$, are i.i.d Gaussian random variables. The mean value and variance of these random variables are decided by the mutual information $I(\lambda[k,i]; b[k,i]) = I_e^{in}/\widetilde{m}$, according to [1] (9)-(18). After $\lambda[k,i]$ is generated, $\mathbf{L}_e^{in}[k]$ can be computed using $\{\lambda[k,i]\}_{i=0}^{\widetilde{m}-1}$.

For $k \in$ case (ii), the *a priori* is a mixture of both extrinsic and systematic information $\mathbf{L}_{es}^{in}[k]$ as shown in (3). Before generating the mixture, we need to find out what the systematic information would be, even it does not appear explicitly during the decoding process in TTCM. In binary turbo coding, the systematic information is the information associated with the channel observation of the systematic bit. This should also be the case in TTCM, although each sample of the channel observation contains not only systematic bits (symbol) but also uncoded bits and parity bits. That is, the systematic information about symbol $d[k]$ must be $\{P(d[k] = u|y[k])\}_{u=0}^{2^{\widetilde{m}}-1}$. Assuming the transmitted data symbols are equiprobable, $P(d[k] = u|y[k])$ is computed as

$$\begin{aligned} P(d[k] &= u|y[k]) = \sum_{c[k]=0}^{1} \sum_{b[k,\widetilde{m}]=0}^{1} \cdots \sum_{b[k,m-1]=0}^{1} \quad (4) \\ P(y[k]|s[k] &= \text{map}(d[k] = u, c[k], b[k,\widetilde{m}], ..., b[k,m-1])). \end{aligned}$$

The log-likelihood values in vector form

$$\mathbf{L}_s[k] = \{\log P(d[k] = u|y[k]), u = 0, ..., 2^{\widetilde{m}} - 1\} \quad (5)$$

We have now explicitly formed the systematic information $\mathbf{L}_s[k]$, for $k \in$ case (ii). To generate the mixture, we also generate for each $k \in$ case (ii) an extrinsic part $\mathbf{L}_e^{in}[k]$ with $I(\mathbf{L}_e^{in}[k]; d[k]) = I_e^{in}$, in the same way as for case (i). The mixture $\mathbf{L}_a[k] = \mathbf{L}_{es}^{in}[k]$ is generated as the combination of extrinsic and systematic information, i.e., $\mathbf{L}_a[k] = \mathbf{L}_e^{in}[k] + \mathbf{L}_s[k]$. To conclude, the *a priori* input to the TTCM component decoder is generated as

$$\mathbf{L}_a[k] = \begin{cases} \mathbf{L}_e^{in}[k], & k \in \text{ case (i)} \\ \mathbf{L}_e^{in}[k] + \mathbf{L}_s[k], & k \in \text{ case (ii)} \end{cases} \quad (6)$$

We turn now to the measurement of the extrinsic information at the output of the TTCM component decoder. According to (2), for $k \in$ case (i) the extrinsic information is mixed with systematic information. However, we can form the systematic information using (4) and (5) and subtract it. That is, the extrinsic information $\mathbf{L}_e^{out}[k]$ for data symbol $d[k]$ is obtained as:

$$\mathbf{L}_e^{out}[k] = \begin{cases} \mathbf{L}_p[k] - \mathbf{L}_a[k] - \mathbf{L}_s[k], & k \in \text{case(i)} \\ \mathbf{L}_p[k] - \mathbf{L}_a[k], & k \in \text{case (ii)} \end{cases} \quad (7)$$

Due to the independence assumption, $\{\mathbf{L}_e^{out}[k]\}_{k=0}^{N-1}$ are mutually independent and follow the same distribution $p_e(\mathbf{L}_e^{out}|d)$. The multi-dimensional joint distribution $p_e(\mathbf{L}_e^{out}|d)$ is obtained by measuring the histogram of $\mathbf{L}_e^{out}$ using its samples $\{\mathbf{L}_e^{out}[k]\}_{k=0}^{N-1}$. Then the mutual information between the extrinsic output and the transmitted symbol d is computed as

$$I_e^{out} = I(\mathbf{L}_e^{out}; d) \quad (8)$$
$$= \frac{1}{2^{\widetilde{m}}} \sum_{u=0}^{2^{\widetilde{m}}-1} \int p_e^{out}(\boldsymbol{\xi}|d=u) \log \frac{2^{\widetilde{m}} \cdot p_e^{out}(\boldsymbol{\xi}|d=u)}{\sum_{u=0}^{2^{\widetilde{m}}-1} p_e^{out}(\boldsymbol{\xi}|d=u)} d\boldsymbol{\xi}.$$

The EXIT chart for TTCM is then generated by measuring the output mutual information I_e^{out} as a function of a family of inputs with specified mutual information I_e^{in} and channel observation of specified SNR, $I_e^{out} = f(I_e^{in}, \text{SNR})$.

III. Examples

In this section, we apply the proposed EXIT chart method to analyze and compare two TTCM schemes. The first code is from [5], Table I, first row. This is an 8-state, rate 1/2 code with 8-PSK modulation, and $m = \widetilde{m} = 2$. The second code has the same structure but the parameter $H^0(D)$ in the first row of table I [5] is changed from $(11)_8$ to $(13)_8$. This changes the feedback polynomial of the component encoder from nonprimitive to primitive. Hence, we compare a primitive code and a nonprimitive code. Both codes have bandwidth efficiency 2 b/s/Hz.

The EXIT charts for these two codes are presented in Fig. 3, for channel SNR 5.8 dB, 6.2 dB and 7.0 dB, respectively. For schemes with identical component encoders/decoders, we only need to plot the EXIT chart for one of the decoders. In the figure, the abscissa is $I_e^{in} = I(\mathbf{L}_e^{in}; d)$ in bits, and the ordinate is $I_e^{out} = I(\mathbf{L}_e^{out}; d)$ also in bits.

It is observed from the figure that for both codes, the curves of SNR=5.8 dB intersect with the y = x line, which means that at some point during the iteration the decoder can no longer increase the extrinsic information compared with the input. In this case the decoding will fail no matter how many iterations are run. Both curves for SNR = 6.2 dB are just a little above the y = x line, which means for that any SNR $\geqslant$ 6.2 dB, the extrinsic information output at the component decoder will always be better than the input, and the decoder will converge to the correct codeword with high probability. That is, the convergence threshold of these two codes is about 6.2 dB. The simulated bit error rate (BER) performance curves are shown in Fig. 4 after 4 and 8 iterations, respectively. The interleaver size is 4096. It is observed that the BER curves (8 iter) actually turn sharply down at about 6.2 dB, which verifies the convergence threshold predicted by the EXIT chart.

In Fig. 3, it is also observed that the curves of the primitive code are always higher than the nonprimitive code, especially at higher values of input mutual information. This implies a faster convergence speed. This agrees with the observation in binary turbo codes that codes with primitive feedback polynomial have faster convergence speed than nonprimitive ones [3]. This observation is also verified by the BER curves in Fig. 4, where with the same number iterations, the performance of the primitive code is better than that of the nonprimitive code.

IV. Conclusions

We have a proposed the method to generate the EXIT chart for TTCM decoders. The accuracy and usefulness of this method has been verified through examples. Since the complexity of generating EXIT charts is significantly lower than a full simulation, the proposed method supplies a convenient way to compare between various TTCM schemes as well as to verify and compare between new designs.

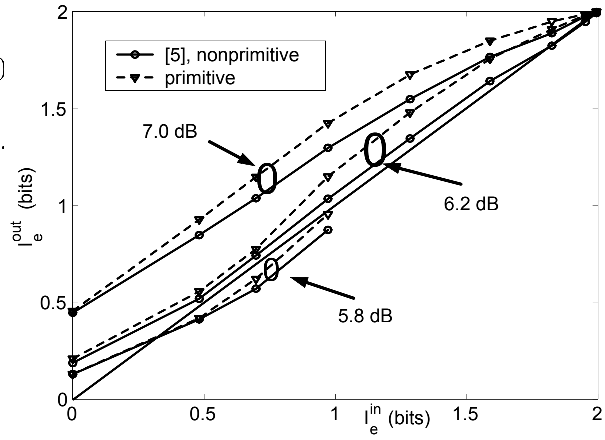

Fig. 3. EXIT charts comparison of two codes.

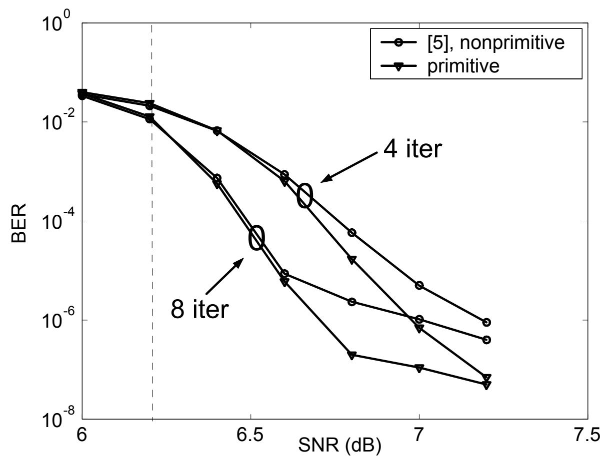

Fig. 4. BER comparison of two codes.

References

[1] S. T. Brink, "Convergence behavior of iterative decoded parallel concatenated codes," *IEEE Transactions on Communications*, vol. 49, pp. 1727–1737, Oct. 2001.

[2] H. E. Gamal and A. R. Hammons, "Analyzing the turbo decoder using Gaussian approximation," *IEEE Transactions on Information Theory*, vol. 47, pp. 671–686, Feb. 2001.

[3] D. Divsalar, S. Donlinar, and F. Pollara, "Iterative turbo deocder analysis based on density evolution," *IEEE Journal on Selected Areas in Communications*, vol. 19, pp. 891–907, May 2001.

[4] A. Grant, "Convergence of non-binary iterative decoding," *GLOBECOM '01*, vol. 2, pp. 1058–1062, Nov 2001.

[5] P. Robertson and T. Worz, "Bandwidth-efficient turbo trellis-coded modulation using punctured component codes," *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 206–218, Feb 1998.

[6] C. Fragouli and R. Wesel, "Turbo-encoder design for symbol-interleaved parallel concatenated trellis-coded modulation," *IEEE Transactions on Communications*, vol. 49, pp. 425 –435, March 2001.