On Pilot Symbol Aided Channel Estimation for Time Varying Rayleigh Fading Channels

Xinmin Deng and Alexander M. Haimovich
Department of Electrical and Computer Engineering
New Jersey Institute of Technology
University Height, Newark, NJ 07102 USA
Email: {xd2, haimovic}@njit.edu

***Abstract*— This paper considers the performance of pilot symbol aided channel estimation by investigating the achievable information rate in a fading channel. The channel is modeled as a time varying Rayleigh fading process whose dynamics are characterized by the normalized fading rate. The optimal fractions of bandwidth and power allocated to the pilot symbols are determined and presented in terms of the signal to noise ratio (SNR) and fading rate.**

I. INTRODUCTION

The availability of channel state information (CSI) is crucial to the performance of wireless communication systems. Traditionally, CSI is estimated by training with known pilot symbols inserted in the transmitted sequence. Due to the presence of noise, the receiver is provided with imperfect CSI and therefore the performance depends on the quality of the CSI [1],[3],[5].

In this paper, we investigate the pilot symbol aided channel estimation for a time-varying fading channel. The dynamics of the fading channel are characterized by its normalized fading rate. In particular, we examine a lower bound on the channel capacity with imperfect CSI obtained from the pilot symbols. The capacity lower bound is related to the capacity of a Rayleigh fading channel with perfect CSI, through scale factors representing the loss in bandwidth and power due to the use of pilot symbols. The optimal fractions of bandwidth and power devoted to the pilot symbols are determined by optimizing the capacity lower bound.

Section II describes the system and channel model with pilot symbol aided channel estimation. Section III presents a lower bound on channel capacity and the optimal resources allocation determined by maximizing the capacity lower bound. Numerical results are provided in Section IV and conclusions are given in Section V.

II. SYSTEM MODEL

The channel between the transmitter and receiver is modeled as a flat fading process. The discrete time received signal at time k, r_k, is given by

$$r_k = h_k s_k + n_k, \tag{1}$$

where s_k is the transmitted symbol, h_k and n_k are the samples of the fading and noise processes, respectively. Rayleigh fading is assumed and the fading process is normalized so that h_k is complex Gaussian with zero mean and unit variance. The dynamics of the time varying fading process is characterized by its Doppler spectrum (or equivalently autocorrelation function) with specified normalized fading rate, f_D, defined as the maximum Doppler spread normalized by the symbol rate. The additive noise sample is complex Gaussian with zero mean and variance $N_0/2$ per dimension. The average transmitted signal power is $\mathcal{E}_s = E[|s_k|^2]$. We use ρ to denote the average signal to noise ratio (SNR) per symbol at the receiver, i.e.,

$$\rho = \frac{\mathcal{E}_s}{N_0}. \tag{2}$$

A. Pilot Symbol Aided Channel Estimation

At the transmitter, pilot symbols are periodically inserted into the data sequence to be transmitted. In particular, a pilot symbol is inserted after every block of $T_P - 1$ data symbols. The pilot symbol insertion frequency $1/T_P$ satisfies the Nyquist criterion $1/T_P \geq 2f_D$ so that no aliasing is incurred.

Let α and β denote the fractions of bandwidth and power devoted to pilot symbols, respectively. If we use $\mathcal{E}_P$ and $\mathcal{E}_D$ denote the pilot symbol power and average data symbol power, then we have

$$\alpha = \frac{1}{T_P},$$

$$\mathcal{E}_P = \beta T_P \mathcal{E}_s = \frac{\beta}{\alpha}\mathcal{E}_s, \tag{3}$$

and

$$\mathcal{E}_D = \frac{T_P}{T_P - 1}\left(1-\beta\right)\mathcal{E}_s = \frac{1-\beta}{1-\alpha}\mathcal{E}_s. \tag{4}$$

In the special case where the pilot symbol and data symbols are constrained to have the same power, i.e., $\mathcal{E}_P = \mathcal{E}_D = \mathcal{E}_s$, we have that $\alpha = \beta = 1/T_P$.

At the receiver, channel state information is obtained from received signals, corresponding to known transmitted pilot symbols, by an optimal minimum mean squared error (MMSE) estimator. The CSI is subsequently used by the decoder for data detection.

This work was supported by NSF award ANI-0338788 and by AFOSR grant F49620-03-1-0161.

In the sequel, we shall suppress the time index of the channel parameter for notational compactness. The channel parameter h can be written as

$$h = \hat{h} + \tilde{h}, \tag{5}$$

where $\hat{h}$ denotes the estimate of h and $\tilde{h}$ denotes the estimation error.

Assuming infinite length pilot symbols, the mean squared error of the optimal channel estimator is given by (see, e.g., [1])

$$\sigma_{\tilde{h}}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\left[1 - W(\omega)\right] d\omega$$

where $H(\omega)$ is the power spectral density (PSD) function of the fading process, and

$$W(\omega) = \left[1 + T_P \left(\frac{\mathcal{E}_P}{N_0} H(\omega)\right)^{-1}\right]^{-1}$$

is the transfer function of the optimal Wiener filter used for channel interpolation. For a uniform Doppler spectrum, given by

$$H(\omega) = \left\{ \begin{array}{cc} \frac{1}{2f_D}, & |\omega| < 2\pi f_D \\ 0, & \text{otherwise,} \end{array} \right.$$

the corresponding MMSE of the channel estimate is given by the expression

$$\sigma_{\tilde{h}}^2 = \left(1 + \frac{1}{2f_D T_P} \frac{\mathcal{E}_P}{N_0}\right)^{-1}. \tag{6}$$

B. Effective SNR

We can rewrite the signal model (1) by substituting (5) into it

$$\begin{aligned} r &= s\hat{h} + s\tilde{h} + n \\ &= s\hat{h} + z. \end{aligned}$$

In this equation, we regard $s\hat{h}$ as the signal part and $z = s\tilde{h} + n$ as the additive noise. Thus the effective SNR can be expressed as

$$\begin{aligned} \rho_{\text{eff}} &= \frac{E\left[\left\|s\hat{h}\right\|^2\right]}{E\left[\|z\|^2\right]} \\ &= \frac{\mathcal{E}_D\left(1 - \sigma_{\tilde{h}}^2\right)}{\mathcal{E}_D \sigma_{\tilde{h}}^2 + N_0}. \end{aligned}$$

Upon substitution of (2), (3), (4) and (6), and after some manipulations, (7) becomes

$$\rho_{\text{eff}} = \eta\rho,$$

where the coefficient

$$\eta = \frac{(1-\beta)\beta\rho}{(1-\alpha)(2f_D + \beta\rho) + 2f_D(1-\beta)\rho} \tag{7}$$

represents the loss in SNR due to imperfect CSI.

III. LOWER BOUND ON CAPACITY AND OPTIMAL RESOURCE ALLOCATION

For systems with imperfect CSI, the capacity and capacity achieving signaling is unknown. However, it is possible to derive a useful lower bound on the capacity by assuming Gaussian signaling [1],[5].

The capacity of a system with pilot symbol aided channel estimation is equivalent ot that of a system with perfect CSI but with a loss in bandwidth and power reflecting the use of the pilot signals and the CSI estimation errors. For the Rayleigh fading channel with Gaussian signaling, the ergodic capacity serves as a lower bound to the capacity with imperfect CSI and unconstrained signalling. Thus we have

$$C \geq C_{LB} \triangleq (1-\alpha)\, C_{\text{Rayleigh}}(\eta\rho) \tag{8}$$

where $C_{\text{Rayleigh}}(x)$ is the ergodic capacity per symbol of a Rayleigh fading channel in the presence of perfect CSI, with x being the average SNR, and is defined by [4], [6]

$$\begin{aligned} C_{\text{Rayleigh}}(x) &\triangleq E_\lambda\left[\log(1 + x\lambda)\right] \\ &= -\exp\left(x^{-1}\right) \operatorname{Ei}\left(-x^{-1}\right), \end{aligned}$$

where the expectation is with respect to an exponentially distributed random variable λ with unity mean.

The capacity lower bound (8) is expressed in terms of the capacity with perfect CSI, through the coefficients α and η, which represent the loss in bandwidth and power, respectively, due to the use of pilot symbols. It can be seen, from (7) and (8), that the capacity lower bound depends on the resources allocated to the pilot symbols, through the parameters α and β. In the following subsections, we will find the optimal values of α and β, for fixed values of f_D and ρ, by optimizing the capacity lower bound (8).

A. Optimal Pilot Symbol Insertion Frequency

Following the approach taken in [3], it is possible to show that for fixed values of f_D, ρ, and β, the capacity lower bound (8) is a decreasing function of α. To see this, we examine the derivative of C_{LB} with respect to α,

$$\begin{aligned} \frac{\partial C_{LB}}{\partial \alpha} &= -C_{\text{Rayleigh}}(\eta\rho) + (1-\alpha)\frac{\partial}{\partial\alpha} C_{\text{Rayleigh}}(\eta\rho) \\ &= E_\lambda\left[-\log(1+\eta\rho\lambda) + (1-\alpha)\frac{\rho\lambda}{1+\eta\rho\lambda}\frac{\partial\eta}{\partial\alpha}\right] \\ &= E_\lambda\Big[-\log(1+\eta\rho\lambda) + \\ &\qquad \frac{\eta\rho\lambda}{1+\eta\rho\lambda}\frac{(1-\alpha)(2f_D+\beta\rho)}{(1-\alpha)(2f_D+\beta\rho)+2f_D(1-\beta)\rho}\Big] \\ &< E_\lambda\left[-\log(1+\eta\rho\lambda) + \frac{\eta\rho\lambda}{1+\eta\rho\lambda}\right] \\ &\leq 0, \end{aligned}$$

where the first inequality holds because

$$\frac{(1-\alpha)(2f_D+\beta\rho)}{(1-\alpha)(2f_D+\beta\rho)+2f_D(1-\beta)\rho} < 1$$

for $f_D > 0$, $\rho > 0$, and $\beta < 1$; the second inequality follows from the fact that

$$\log(1+x) \geq \frac{x}{1+x}$$

for all nonnegative values of x.

Therefore the lower bound is maximized when α attains its minimum possible value, which is just the minimum frequency satisfying the Nyquist criterion, i.e.,

$$\alpha_{opt} = 2f_D. \tag{9}$$

Clearly, the optimal pilot insertion frequency does not depend on the SNR ρ. With this choice of pilot symbol insertion frequency, the coefficient η can be written as

$$\eta = \frac{(1-\beta)\beta\rho}{(1-2f_D)(2f_D+\beta\rho)+2f_D(1-\beta)\rho}. \tag{10}$$

B. Optimal Pilot Symbol Power Allocation

Having established that the optimal fraction α of bandwidth used for pilot symbols is given by (9), we turn to find the optimal fraction β of power dedicated to the pilot symbols. That will be the value of β that maximizes η in (10).

Expression (10) can be rewritten

$$\eta = \frac{1}{1-4f_D}\frac{(1-\beta)\beta}{\beta+\gamma}, \tag{11}$$

where

$$\gamma = \frac{2f_D(1-2f_D+\rho)}{\rho(1-4f_D)}. \tag{12}$$

The value of β that maximizes (11) is dependent on the value of f_D. For simplicity, suppose $f_D < 1/4$, which is reasonable for channels of practical interest. Under this assumption, γ is positive, and the optimal fraction β of power allocated to the pilot symbols, in the sense that the coefficient η given in (11) is maximized, can be expressed as

$$\beta_{opt} = g(\gamma), \tag{13}$$

where the function $g(x)$ is defined as

$$g(x) = -x + \sqrt{x(x+1)}. \tag{14}$$

Upon setting β equal to this optimal value and substituting in (11), we obtain

$$\eta_{opt} = \frac{1}{1-4f_D}\left(\sqrt{\gamma+1}-\sqrt{\gamma}\right)^2.$$

It is not hard to verify the following statements:

1) $\gamma > 0$, if $\rho > 0$ and $0 < f_D < 1/4$;
2) γ is an increasing function of f_D;
3) γ is a decreasing function of ρ if $0 < f_D < 1/4$;
4) $g(x)$ is a nonnegative, increasing function of x for $x > 0$, and its value tends to $1/2$ as x tends to infinity;
5) η_{opt} is a decreasing function of γ for $\gamma > 0$.

It follows from the above statements that, β_{opt} is a decreasing function of ρ and an increasing function of f_D, while η_{opt} is an increasing function of ρ and a decreasing function of f_D. In other words, if fading becomes slower or SNR becomes higher, then a smaller fraction of the power should be allocated to pilot symbols, and using pilot symbols becomes more efficient. It can also be shown that $\beta_{opt} > 2f_D$ for all values of ρ and all values of $f_D < 1/4$, implying that the optimal power allocated to pilot symbols is always greater than the average power.

Since for a fixed fading rate, β_{opt} and η_{opt} are functions of the SNR, it is of interest to examine the limiting behavior at high SNR and low SNR. In doing so we will make use of the following result [4], [6]

$$C_{\text{Rayleigh}}(x) \approx \begin{cases} x, & x << 1 \\ \log(1+x) - \mathcal{C}, & x >> 1, \end{cases} \tag{15}$$

where $\mathcal{C} \approx 0.577$ is the Eüler's constant.

1) Low SNR: As SNR ρ tends to zero, from (12) we have that γ tends to infinity, and

$$\lim_{\rho\to 0}\beta_{opt} = \frac{1}{2}. \tag{16}$$

Hence half the power should be devoted to pilot symbols.

For sufficiently small values of ρ, we have

$$\eta_{opt} \approx \frac{\rho}{8f_D(1-2f_D)}.$$

It follows from (15)

$$\begin{aligned} C_{LB} &\approx (1-2f_D)\eta_{opt}\rho \\ &= \frac{1}{8f_D}\rho^2. \end{aligned} \tag{17}$$

This result indicates that it is ineffective to use pilot symbol aided channel estimation when SNR is low, since almost half of the transmitted energy should be devoted to the pilot symbols, and the effective SNR decreases at a quadratic rate with ρ.

2) High SNR: At high SNR, the optimal values of β and η are respectively given by

$$\lim_{\rho\to\infty}\beta_{opt} = \frac{-2f_D + \sqrt{2f_D(1-2f_D)}}{1-4f_D} \tag{18}$$

and

$$\lim_{\rho\to\infty}\eta_{opt} = \frac{1}{1+2\sqrt{2f_D(1-2f_D)}}. \tag{19}$$

Since $1/2 < \lim_{\rho\to\infty}\eta_{opt} < 1$ provided that $0 < f_D < 1/4$, the loss in effective SNR due to the use of channel estimation is less than 3 dB when the SNR is high.

C. Equal Pilot and Data Symbol Power

We have shown that if we are free to choose the fractions α and β, then the optimal solution is given in (9) and (13), and the optimal values of α and β are never equal. More specifically, we have $\beta_{opt} > \alpha_{opt} = 2f_D$ for any SNR ρ and fading rate $f_D < 1/4$, implying that the optimal power of pilot symbols always exceeds the average power.

It is of interest to investigate the special case considered in [1], where the pilot symbol power and average data symbol power are constrained to be equal. To distinguish between the cases with and without the equal pilot and data symbol power constraint, quantities with power constraint will be emphasized by superscripts *. Under the power constraint, the fractions of bandwidth and power allocated the pilot symbols are equal, i.e., $\alpha^* = \beta^*$. It is possible to find the optimal amount of resources dedicated to pilot symbols by maximizing the capacity lower bound

$$C^*_{LB}\left(\alpha^*\right) = \left(1 - \alpha^*\right) C_{\text{Rayleigh}}\left(\eta^* \rho\right), \tag{20}$$

where the maximization is over α^* in the interval $2f_D \leq \alpha^* < 1$, with the coefficient η^* given by

$$\begin{aligned} \eta^* &= \frac{\alpha^* \rho}{2f_D\left(1+\rho\right) + \alpha^* \rho} && (21) \\ &= \left[1 + 2f_D\left(\alpha^*\right)^{-1}\left(1+\rho^{-1}\right)\right]^{-1}. && (22) \end{aligned}$$

No simple analytic expression for the optimal value of α^* appears to be available. It is observed, through numerical computations, that the optimal value of α^* is always greater that $2f_D$ [1].

Although the detailed dependence of the optimal value of α^* on ρ must be determined numerically, simple expressions do exist for sufficiently high or low SNR ρ.

1) Low SNR: At sufficiently low SNR, the optimal value of α^* can be approximated as

$$\alpha^*_{opt} \approx g\left(\delta\right),$$

where

$$\delta = 2f_D\left(1 + \frac{1}{\rho}\right) \tag{23}$$

and the function g is defined in (14). From (12) and (23), we have that $\delta < \gamma$ if $\rho < 1/2$. It follows from the monotonicity of the function g that $\alpha^*_{opt} < \beta_{opt}$. As SNR ρ tends to zero, it follows from (23) that δ tends to infinity, and

$$\lim_{\rho \to 0} \alpha^*_{opt} = \frac{1}{2}. \tag{24}$$

Hence half of the bandwidth and power should be devoted to pilot symbols when SNR is low.

For small values of ρ, we have from (22) and (15)

$$\begin{aligned} C^*_{LB}\left(\alpha^*_{opt}\right) &\approx \left(1 - \alpha^*_{opt}\right) \cdot \\ &\quad \left[1 + 2f_D\left(\alpha^*_{opt}\right)^{-1}\left(1+\rho^{-1}\right)\right]^{-1} \rho \\ &= \left(\sqrt{1+\delta} - \sqrt{\delta}\right)^2 \rho \\ &\approx \frac{1}{4\delta}\rho \\ &\approx \frac{1}{8f_D}\rho^2, \end{aligned}$$

which is the same result as in (17). Hence, for sufficiently low SNR, negligible performance degradation is caused by the equal pilot and data symbol power constraint.

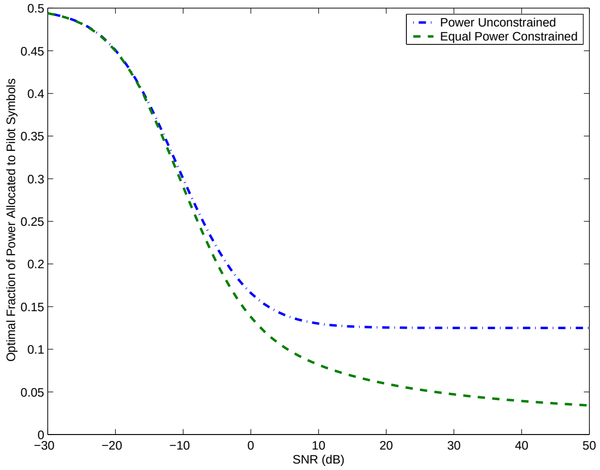

Fig. 1. Optimal fraction of power allocated to pilot symbols ($f_D = 0.01$).

2) High SNR: From (15), for any value of α^* in the interval $2f_D \leq \alpha^* < 1$, we have

$$\begin{aligned} &\lim_{\rho \to \infty} \frac{C^*_{LB}\left(\alpha^*\right)}{C^*_{LB}\left(2f_D\right)} \\ = \; &\frac{1-\alpha^*}{1-2f_D} \\ &\lim_{\rho \to \infty} \frac{\log \rho - \log\left[1 + 2f_D\left(\alpha^*\right)^{-1}\left(1+\rho^{-1}\right)\right] - \mathcal{C}}{\log \rho - \log\left[1 + \left(1+\rho^{-1}\right)\right] - \mathcal{C}} \\ = \; &\frac{1-\alpha^*}{1-2f_D}, \end{aligned}$$

which is decreasing in α^*. It follows that

$$\lim_{\rho \to \infty} \alpha^*_{opt} = 2f_D \tag{25}$$

and

$$\begin{aligned} \lim_{\rho \to \infty} \eta^*_{opt} &= \lim_{\rho \to \infty} \frac{1}{1 + \left(1+\rho^{-1}\right)} \\ &= \frac{1}{2}. && (26) \end{aligned}$$

Therefore, the loss in effective SNR is 3 dB in the high SNR regime.

It can be seen, by comparing (19) and (26), that this loss is always greater than that without the constraint on the power of pilot and data symbols. It is clear from (8),(9), and (25) that for systems with pilot symbol aided channel estimation, the capacity lower bounds have a slope of ($(1 - 2f_D)$ bits/symbol/3 dB) in the high SNR regime. Note that these bounds have the same slope as that of the lower bound developed in [2], where no CSI estimation is attempted.

IV. NUMERICAL RESULTS

The optimal fractions of resources allocated to the pilot symbols are plotted in Fig. 1 as a function of SNR for normalized fading rate $f_D = 0.01$. The optimal fractions

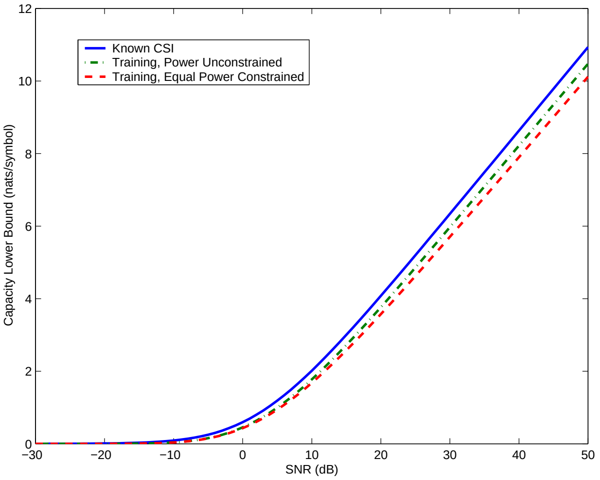

Fig. 2. Lower bounds on capacity ($f_D = 0.01$).

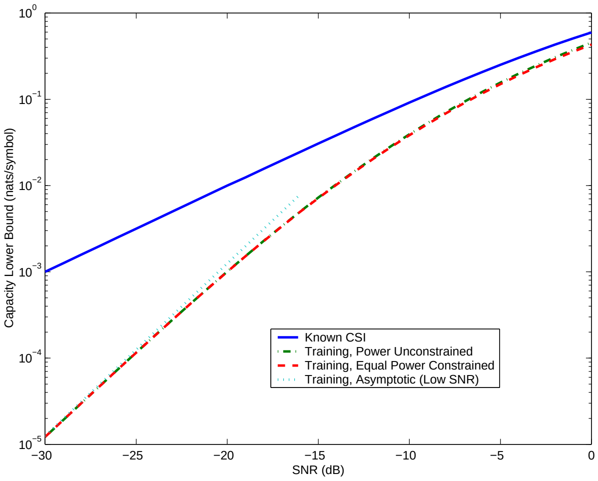

Fig. 3. Lower bounds on capacity in the low SNR regime ($f_D = 0.01$).

β_{opt} of power used for pilot symbols are obtained from the expression (13). The optimal fractions α^*_{opt} of bandwidth and power with the constraint on pilot and data symbols are determined numerically, by optimizing the capacity lower bound (20). It can be seen that when SNR goes to zero, both β_{opt} and α^*_{opt} tend to $1/2$, as given by (16) and (24). On the other hand, when SNR goes to infinity, β_{opt} and α^*_{opt} converge to 0.125 and 0.02, as predicated by (18) and (25), respectively.

Fig. 2 illustrates the capacity lower bound versus the SNR, for normalized fading rate $f_D = 0.01$. As expected from the pervious discussion, while the loss due to the constraint imposed at the powers of the pilot and data symbols is obvious in the high SNR regime, this loss becomes negligible for sufficiently low SNR, as can be seen from Fig. 3 in which the same bounds in the low SNR regime are plotted on a log-log scale.

V. CONCLUSIONS

In this paper, we have investigated pilot symbol aided channel estimation in a time varying Rayleigh fading channel. We have derived the optimal amount of resources allocated to the pilot symbols, by optimizing a lower bound on channel capacity. It has been shown that the optimal fraction of bandwidth devoted to the pilot symbols equals twice the normalized fading rate, and the optimal power of pilot symbols is always higher than the data symbols. If the constraint of equal pilot and data symbol power is imposed, it is observed that performance degradation caused by the constraint is negligible in the low SNR regime, and is dependent of the fading rate in the high SNR regime. In the low SNR regime, it is inefficient to use pilot symbol aided channel estimation, since the lower bound capacity decreases quadratically with SNR, rather than linearly with SNR as the capacity with perfectly known CSI does.

REFERENCES

[1] J. Baltersee, G. Fock, and H. Meyr, “An information theoretic foundation of synchronized detection,” *IEEE Trans. Commun.*, vol. 49, no. 12, pp. 2115-2123, Dec. 2001.
[2] X. Deng and A. M. Haimovich, “Information rates of time varying Rayleigh fading channels,” to appear in *Proc. IEEE Int. Conf. Communications (ICC)*, Paris, France, 2004.
[3] B. Hassibi and B. M. Hochwald, “How much training is needed in multiple-antenna wireless links?” *IEEE Trans. Inform. Theory*, vol. 49, no. 4, pp. 951-963, Apr. 2003.
[4] W. C. Y. Lee, “Estimation of channel capacity in Rayleigh fading environment,” *IEEE Trans. Veh. Tech.*, vol. 39, no. 3, pp. 187-189, Aug. 1990.
[5] M. Médard, “The effect upon channel capacity in wireless communications of perfect and imperfect knowledge of the channel, ” *IEEE Trans. Inform. Theory*, vol. 46, no. 3, pp. 933-946, May 2000.
[6] L. H. Ozarow, S. Shamai, and A. D. Wyner,“Information theoretic considerations for cellular mobio radio,” *IEEE Trans. Veh. Tech.*, vol. 43, no. 2, pp. 359-378, May 1994.