Exact Average Symbol Error Probability of Optimum Combining With Arbitrary Interference Power

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***Abstract*—A new expression is derived for the exact average symbol error probability for optimum combining with M-ary phase-shift keying. The expression adds to the significant body of work in this field by handling interferers with arbitrary power levels. The expression involves a single integral with finite limits and finite integrand. A closed-form expression is also derived for the average symbol error probability for binary phase-shift keying.**

***Index Terms*—Error probability, fading channels, interference suppression, optimum combining (OC).**

I. INTRODUCTION

FOR communications systems with receive diversity, optimum combining (OC) is a well-known technique to combat fading and suppress cochannel interference. Performance analysis of OC is challenging due to the multiple random processes involved in the signal model. Analysis for the case of a single interference source with binary phase-shift keying (BPSK) modulation can be found in [1], [2]. The performance of systems with more than one interferer has been studied extensively through the use of Monte Carlo simulations [1], upper bounds and approximation [3], exact expressions with integral forms [4], and closed-form expressions [5]–[7]. In previous work, the power levels of the interferers were assumed uniform.

In this letter, we consider the general case where the interferers may have unequal power levels. We derive an expression for the exact symbol error probability (SEP) for M-ary phase-shift keying (M-PSK) modulation. The final expression involves only a single integral with finite limits and a finite integrand. We also derive a close-form expression for the error probability of BPSK modulation. As far as we know, the new expressions for both M-PSK and BPSK are the first that can be used to evaluate the exact SEP of systems with interferers of arbitrary power levels. They are easy and fast to evaluate and their validity has been demonstrated by simulations.

The paper is organized as follows: following the system model in Section II, the SEP expressions for M-PSK and BPSK are developed in Section III. Numerical results are shown in Section IV. Finally, conclusions are drawn in Section V.

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II. SYSTEM MODEL

Consider a wireless communications system with N independent receive branches and $L+1$ users. Of the users, one is the desired user and it transmits M-PSK signals with power P_s; the other are considered interferers with arbitrary powers $P_n (n = 1, 2, \ldots, L)$. The white Gaussian noise variance is σ^2. The channel gains of the desired user and interferers are assumed to be independently and identically distributed (i.i.d.), zero-mean, complex Gaussian random variables (flat Rayleigh fading), with variance 1/2 per dimension. It is also assumed that conditioned on the channel of the interferers, the summation of interference and noise has a multivariate complex Gaussian distribution with zero mean.

Define the signal-to-noise ratio (SNR) $\gamma = P_s/\sigma^2$, and the signal to interference ratio (SIR) for the n-th interferer $\beta_n = P_s/P_n$. The reliability function $R(\tau)$ is defined as the probability that the output signal-to-interference-plus-noise ratio (SINR) of OC is greater than a threshold τ. It can be written [8]

$$R(\tau) = \left[\sum_{m=1}^{N-L} \frac{1}{(m-1)!}\left(\frac{\tau}{\gamma}\right)^{m-1} + \sum_{m=\max(N-L+1,1)}^{N} \frac{1}{(m-1)!} \times \frac{\sum_{i=0}^{N-m} C_i \tau^i}{\prod_{n=1}^{L}(\tau+\beta_n)}\left(\frac{\tau}{\gamma}\right)^{m-1}\right] e^{-\tau/\gamma} \quad (1)$$

where C_i is the coefficient of τ^i in the product $\prod_{n=1}^{L}(\tau+\beta_n)$. By mathematical induction it can be proved that

$$C_i = \sum_{n_1+n_2+\cdots+n_L=L-i,\ n_i\in\{0,1\}} \beta_1^{n_1}\beta_2^{n_2}\cdots\beta_L^{n_L}. \quad (2)$$

III. SEP ANALYSIS

In this section, we carry out the theoretical analysis of the average SEP for OC with M-PSK and BPSK modulation.

A. SEP for M-PSK

For OC with M-PSK modulation and Gaussian distributed interference plus noise, the SEP conditioned on the output SINR τ can be written as [9]

$$P_{s,M\text{-PSK}}(E|\tau) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left[-\tau\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\theta}\right] d\theta \quad (3)$$

where M is the cardinality of the modulation. The ensemble average SEP, $P_{s,M\text{-PSK}}(E)$ is obtained by averaging $P_{s,M\text{-PSK}}(E|\tau)$ over the PDF $p_\tau(\tau)$,

$$P_{s,M\text{-PSK}}(E) = \int_0^\infty P_{s,M\text{-PSK}}(E|\tau)\, p_\tau(\tau) d\tau. \quad (4)$$

Substituting the PDF $p_\tau(\tau) = -dR(\tau)/d\tau$ in (4) and using the method of integration by parts, after some straightforward manipulations, we have

$$P_{s,M\text{-PSK}}(E) = \frac{M-1}{M} - \frac{\sin\left(\frac{\pi}{M}\right)}{2\sqrt{\pi}} \int_0^\infty \tau^{-(1/2)} R(\tau) \times e^{-\sin^2(\pi/M)\tau} \times \operatorname{erfc}\left[-\sqrt{\tau}\cos\left(\frac{\pi}{M}\right)\right] d\tau \quad (5)$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function. Changing the variable $\tau = \tan^2\varphi$, we obtain the final expression for the average SEP

$$P_{s,M\text{-PSK}}(E) = \frac{M-1}{M} - \frac{\sin\left(\frac{\pi}{M}\right)}{\sqrt{\pi}} \int_0^{(\pi/2)} R\left(\tan^2\varphi\right) \times \exp\left[-\sin^2\left(\frac{\pi}{M}\right)\tan^2\varphi\right] \times \operatorname{erfc}\left[-\tan\varphi\cos\left(\frac{\pi}{M}\right)\right] \sec^2\varphi d\varphi. \quad (6)$$

Notice that even though the argument of R could be infinite, the range of R and the integrand are finite. The SEP can be evaluated fast and accurately.

B. SEP for BPSK Modulation

A closed-form expression of the SEP can be derived for the special case of BPSK modulation. The following is the outline of the derivation.

In this case, $M = 2$. Substituting (1) in (5), we have

$$P_{s,\text{BPSK}}(E) = \frac{1}{2} - \Upsilon_1 - \Upsilon_2 \quad (7)$$

where

$$\Upsilon_1 = \frac{1}{2\sqrt{\pi}} \sum_{m=1}^{N-L} \frac{1}{(m-1)!\gamma^{m-1}} \int_0^\infty \tau^{m-(3/2)} e^{-\alpha\tau} d\tau \quad (8)$$

$$\Upsilon_2 = \frac{1}{2\sqrt{\pi}} \sum_{m=\max(N-L+1,1)}^{N} \sum_{i=0}^{N-m} \frac{C_i}{(m-1)!\gamma^{m-1}} \times \int_0^\infty \frac{\tau^{m+i-(3/2)}}{\prod_{n=1}^{L}(\tau+\beta_n)} e^{-\alpha\tau} d\tau \quad (9)$$

and $\alpha = 1 + 1/\gamma$. By using [10, eq. 3.381.4 and 8.339], Υ_1 can be expressed in the following closed form:

$$\Upsilon_1 = \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}} \sum_{m=0}^{N-L-1} \binom{2m}{m} \frac{1}{[4(\gamma+1)]^m}. \quad (10)$$

Next Υ_2 will be expressed in closed form.

The L SIR terms $\beta_n (n = 1, 2, \ldots, L)$ are separated into O sets, with the SIR's in the same set equal to each other. The SIR's from different sets are distinct. More specifically, assume the first set contains n_1 SIR β_n's with each equaling β'_1; the second set contains n_2 SIR β_n's with each equaling $\beta'_2, \ldots,$ the O-th set contains n_O β_n's with each equaling β'_O. Then $n_1 + n_2 + \cdots + n_O = L$ and $\beta'_1, \beta'_2, \ldots, \beta'_O$ are distinct. With this assumption we have

$$\frac{1}{\prod_{n=1}^{L}(\tau+\beta_n)} = \sum_{k=1}^{O} \sum_{l=1}^{n_k} \frac{A_{k,l}}{(\tau+\beta'_k)^l} \quad (11)$$

where

$$A_{k,l} = \frac{1}{(n_k-l)!} \frac{d^{(n_k-l)}}{d\gamma_t^{(n_k-l)}} \frac{(\tau+\beta'_k)^{n_k}}{\prod_{i=1}^{O}(\tau+\beta'_k)^{n_i}} \Bigg|_{\tau=-\beta'_k}. \quad (12)$$

Substitute (11) into (9), then

$$\Upsilon_2 = \frac{1}{2\sqrt{\pi}} \sum_{m=\max(N-L+1,1)}^{N} \sum_{i=0}^{N-m} \frac{C_i}{(m-1)!\gamma^{m-1}} \times \sum_{k=1}^{O} \sum_{l=1}^{n_k} A_{k,l} B_{m,i,k,l} \quad (13)$$

where

$$B_{m,i,k,l} = \int_0^\infty \frac{\tau^{m+i-(3/2)}}{(\tau+\beta'_k)^l} e^{-\alpha\tau} d\tau = \sum_{p=\max(l-m-i+1,1)}^{l} D_{m,i,k,l,p} G_{k,p} + \sum_{q=0}^{m+i-1-l} F_{m,i,k,l,q} H_q. \quad (14)$$

In (14), $D_{m,i,k,l,p}$, $G_{k,p}$, $F_{m,i,k,l,q}$, and H_q are defined as

$$D_{m,i,k,l,p} = \frac{1}{(l-p)!} \frac{(m+i-1)!}{(m+i-1-l+p)!} (-\beta'_k)^{m+i-1-l+p} \quad (15)$$

$$G_{k,p} = \int_0^\infty \frac{\tau^{-(1/2)} e^{-\alpha\tau}}{(\tau+\beta'_k)^p} d\tau \quad (16)$$

$$F_{m,i,k,l,q} = -\frac{1}{q!} \sum_{p=\max(l-m-i+1,1)}^{l} (-1)^{m+i-1-l+p+q} \times (\beta'_k)^{m+i-1-l-q} \frac{1}{(l-p)!} \times \frac{(m+i-1)!}{(m+i-1-l+p)!} \frac{(p+q-1)!}{(p-1)!} \quad (17)$$

$$H_q = \int_0^\infty \gamma_t^{q-(1/2)} e^{-\alpha\tau} d\tau = \sqrt{\frac{\pi}{\alpha}} \frac{(2q)!}{(4\alpha)^q q!}. \quad (18)$$

In deriving (18), [10, eqs. 3.381.4 and 8.339] are used. From (16), it can be shown that

$$G_{k,0} = \sqrt{\frac{\pi}{\alpha}} \quad (19)$$

$$G_{k,1} = \frac{\pi}{\sqrt{\beta'_k}} e^{\alpha\beta'_k} \operatorname{erfc}\left(\sqrt{\alpha\beta'_k}\right) \quad (20)$$

$$G_{k,p} = \frac{2p - 2\alpha\beta'_k - 3}{2(p-1)\beta'_k} G_{k,p-1} + \frac{\alpha}{(p-1)\beta'_k} G_{k,p-2} \quad (21)$$

for $p \geq 2$. The above equation shows that the value of $G_{k,p}$ for $p \geq 2$ can be evaluated recursively from the values of $G_{k,p-1}$ and $G_{k,p-2}$. Since (21) is a second order difference equation with initial values $G_{k,0}$ and $G_{k,1}$, it can be solved by the method detailed in [11]. The solution is omitted here for brevity.

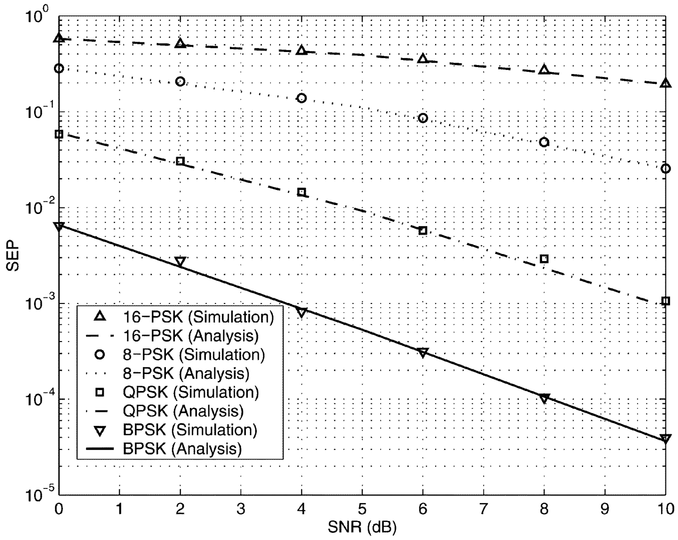

Fig. 1. SEP versus SNR for $N = 8$ branches and $L = 6$ interferers.

By combining (7) and the related expressions (10), (13), (14), and (18), the closed-form expression for the error probability of BPSK can be obtained

$$\begin{aligned} P_{s,\mathrm{BPSK}}(E) &= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{\gamma+1}} \sum_{m=0}^{N-L-1} \binom{2m}{m} \left[\frac{1}{4(\gamma+1)}\right]^m \\ &\quad - \frac{1}{2\sqrt{\pi}} \sum_{m=\max(N-L+1,1)}^{N} \sum_{i=0}^{N-m} \frac{C_i}{(m-1)!} \frac{1}{\gamma^{m-1}} \\ &\quad \times \sum_{k=1}^{O} \sum_{l=1}^{n_k} A_{k,l} \left[\sum_{p=\max(l-m-i+1,1)}^{l} D_{m,i,k,l,p} G_{k,p} \right. \\ &\quad \left. + \sum_{q=0}^{m+i-1-l} F_{m,i,k,l,q} \sqrt{\frac{\pi}{\alpha}} \frac{(2q)!}{(4\alpha)^q\, q!} \right]. \end{aligned} \tag{22}$$

For all the cases we tried, (22) yields the same numerical results as (6). It can be easily proved that for the special case of one interferer, (22) simplifies to (16) in [2].

IV. Numerical Results

Figures shown present both analysis results and simulation results. The analysis results were calculated using (6).

Fig. 1 shows the SEP versus SNR for $N = 8$ receive branches and $L = 6$ interferers. The SIRs for the six interferers were 10, 10, 2, 2, 0, and 0 dB, respectively. Fig. 2 shows the SEP versus the number of receive branches N for 32 interferers at fixed $\mathrm{SNR} = 10$ dB. The SIRs for 16 interferers were 0 dB, while the SIRs for the other 16 interferers were 2 dB.

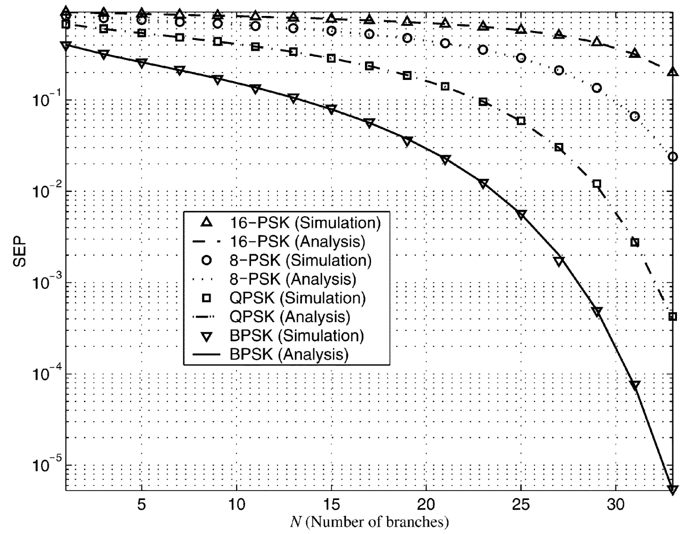

Fig. 2. SEP versus the number of branches for $L = 32$ interferers and $\mathrm{SNR} = 10$ dB.

The analysis results match the simulation results in both figures. That demonstrates we can use the analytical expression to evaluate the SEP.

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