Biol635/Math635/Biol432/Math430
Fall 2024
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Assignment 5

Answer the following questions.

- Justify your answers.
- Explain your results.
- Provide the necessary calculations in a clear way.
- Provide the codes you used (if any).

Consider the following (slightly) modified version of the *logistic equation with a threshold*

$$\frac{dV}{dt} = F(V) \tag{1}$$

where t represents time, V is the dependent variable (e.g., voltage) and the function $F(V)$ is given by

$$F(V) = -r\,V\,\left(1 - \frac{V}{T}\right)\left(1 - \frac{V}{K}\right) + I. \tag{2}$$

The parameters r, K, T and I are constants, $r, K, T > 0$ and $T < K$. The parameter r represents the rate constant (inverse of the time constant for the unbiased case when $I = 0$). The units depend on the problem at hand. If the model is intended for the dynamics of a neuron, then the units of t are milliseconds (ms) and the units of V are millivolts (mV).

This equation has many applications. In particular, it can be used to describe the transition of a system from a resting state (V_{rest}, lower equilibrium V_{eq}) to and activated state (V_{act}, higher equilibrium V_{eq}), when they exist, through a threshold (V_{th}, intermediate equilibrium V_{eq}). V_{act} is also referred to as the saturation value V_{sat}.

In the context of neuronal systems, eq. (1) can be used as a toy / phenomenological (very simple) model of action potential generation (in the absence of termination) where the occurrence of an action potential is determined by the transition from V_{rest} to V_{act}. In this case, the parameter I is interpreted as the current (I_{app}) the neuron receives (e.g., an external DC current that is controlled by the experimenter/modeler). The dynamics governing the termination of an action potential are not described by this model, and requires additional mechanisms and additional model complexity (e.g., a system of two or more differential equations).

The existence and values of the equilibria (V_{rest}, V_{th} and V_{sat}) depend on the model parameters (r, T, K and I). For $I = 0$, eq (1) has three equilibria (V_{eq}): $V_{rest} = 0$, $V_{th} = T$ and $V_{sat} = K$. Because of this, T and K are referred as the threshold and carrying capacity, respectively, in the literature. For other values of I, the activity attributes (V_{rest}, V_{th} and V_{sat} are not longer explicitly given by a single parameter (T, K) and the attributes may even cease to exist for certain ranges of values of I (all other parameters fixed).

1. Write a code to solve numerically the ODE (1)-(2) (or adapt the template code provided in the course website). The simulation output for each set of parameter values must be
   (a) A graph of the solution $V(t)$.
   (b) The equilibrium value(s) $V_{eq} = \lim_{t\to\infty} V(t)$
   (c) A graph of F as a function of V.
   - Each simulation should be run long enough (large enough value of t) so that $V(t)$ reaches values close enough to V_{eq}, but not too long so the changes in $V(t)$ are clearly shown and can be appreciated.
   - Plot the two graphs as two panels in the same Figure.
   - Use correct labels for the axes.
   - Use large enough fonts (suggested: “fontsize" = 24)
2. Consider the following parameter values: $r = 1$, $T = 0.25$, $K = 1$. Perform simulations as described above for $V(0) = 0.01$ and three values of I: $I = 0$, $I = 0.05$, $I = 0.1$.
3. Consider the following parameter values and initial condition: $r = 1$, $T = 0.4$, $K = 1$, $I = [0, 0.2]$ with intervals $\Delta I = 0.02$ (11 values), and $X(0) = 0.25$
   (a) Simulate the model as described above in **ascending** order of the values of I_{app}. Plot the graph of V_{eq} as a function of I.
   (b) Simulate the model as described above in **descending** order of the values of I_{app}. Plot the graph of V_{eq} as a function of I_{app}.
4. Consider the following parameter values and initial condition: $r = 1$, $T = 0.4$, $K = 1$, $I_{app} = [0, 0.2]$ with intervals $\Delta I = 0.02$ (11 values)
   (a) Simulate the model as described above in **ascending** order of the values of I. For $I = 0$ use $V(0) = 0$. For $I_{app} > 0$ set $V(0)$ equal to V_{eq} in the simulation for the previous value of I_{app}.
   (b) Simulate the model as described above in **descending** order of the values of I_{app}. For $I_{app} = 0.2$ set $V(0)$ equal to the value of V_{eq} computed in the previous simulation for $I_{app} = 0.2$. For $I_{app} < 0.2$ set $V(0)$ equal to V_{eq} in the previous simulation.
   (c) Plot a single graph with all the values of V_{eq} as a function of I_{app}.
5. (Graduate level) Provide examples from the literature on bistability in neuronal systems