Introduction to Computational Neuroscience

Biol 698
Math 635
Math 430

Bibliography:

"Mathematical Foundations of Neuroscience", by G. B. Ermentrout & D. H. Terman - Springer (2010), 1st edition. ISBN 978-0-387-87707-5

* "Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting", by Eugene M. Izhikevich. The MIT Press, 2007. ISBN 0-262-09043-8

Figures were taken from some of these books

Overview

- The Hodgkin-Huxley model (review)
- Dynamical systems: Basic concepts
- Reduced one-dimensional models
- Reduced two-dimensional models
- Morris-Lecar and FitzHugh-Nagumo models
- Quadratic integrate-and-fire models
- Bifurcations: basic concepts

Hodgkin-Huxley model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
\dot{n} &= \alpha_n(V)(1-n) - \beta_n(V) n \\
\dot{m} &= \alpha_m(V)(1-m) - \beta_m(V) m \\
\dot{h} &= \alpha_h(V)(1-h) - \beta_h(V) h \,,
\end{aligned}
$$

$$\alpha_n(V) = 0.01 \frac{10 - V}{\exp(\frac{10-V}{10}) - 1}$$

$$\alpha_m(V) = 0.1 \frac{25 - V}{\exp(\frac{25-V}{10}) - 1}$$

$$\alpha_h(V) = 0.07 \exp\left(\frac{-V}{20}\right)$$

$$\beta_n(V) = 0.125 \exp\left(\frac{-V}{80}\right)$$

$$\beta_m(V) = 4 \exp\left(\frac{-V}{18}\right)$$

$$\beta_h(V) = \frac{1}{\exp(\frac{30-V}{10}) + 1}$$

Hodgkin-Huxley model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = (n_\infty(V) - n)/\tau_n(V) \,,$$

$$\dot{m} = (m_\infty(V) - m)/\tau_m(V) \,,$$

$$\dot{h} = (h_\infty(V) - h)/\tau_h(V) \,,$$

$$n_\infty = \alpha_n/(\alpha_n + \beta_n) \,, \qquad \tau_n = 1/(\alpha_n + \beta_n) \,,$$

$$m_\infty = \alpha_m/(\alpha_m + \beta_m) \,, \qquad \tau_m = 1/(\alpha_m + \beta_m) \,,$$

$$h_\infty = \alpha_h/(\alpha_h + \beta_h) \,, \qquad \tau_h = 1/(\alpha_h + \beta_h)$$

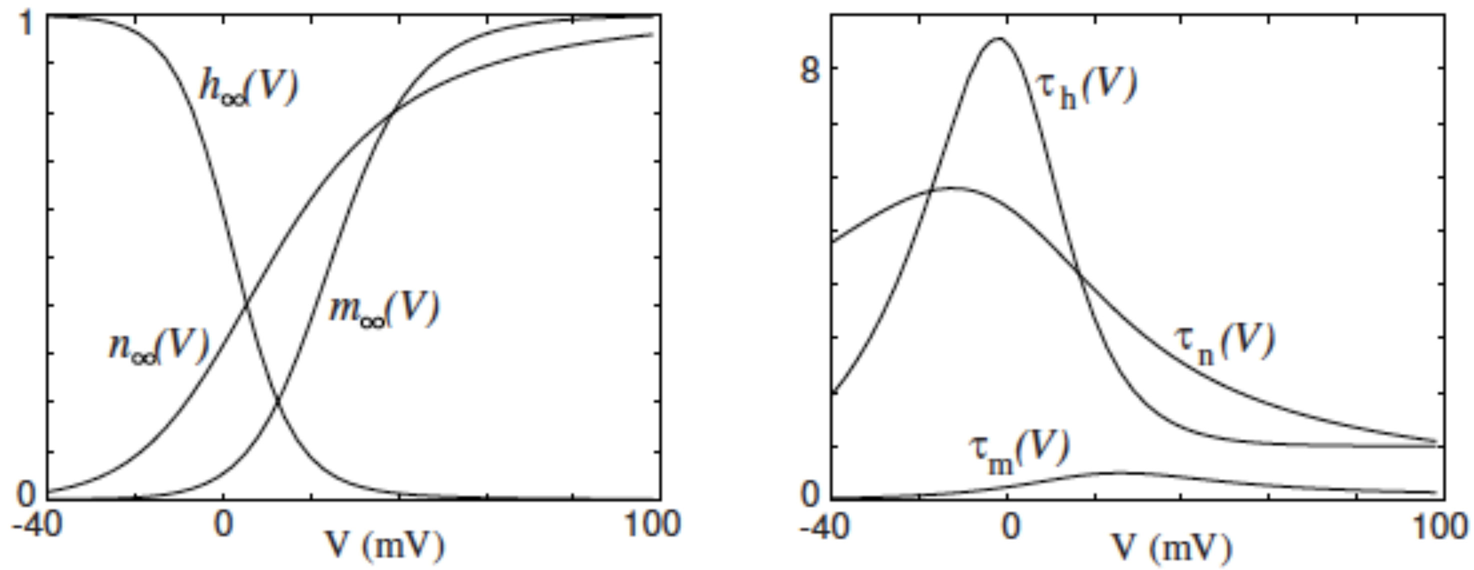


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Hodgkin-Huxley model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
\dot{n} &= \alpha_n(V)(1-n) - \beta_n(V) n \\
\dot{m} &= \alpha_m(V)(1-m) - \beta_m(V) m \\
\dot{h} &= \alpha_h(V)(1-h) - \beta_h(V) h \,,
\end{aligned}
$$

$E_{\mathrm{K}} = -12 \text{ mV}$

$E_{\mathrm{Na}} = 120 \text{ mV}$

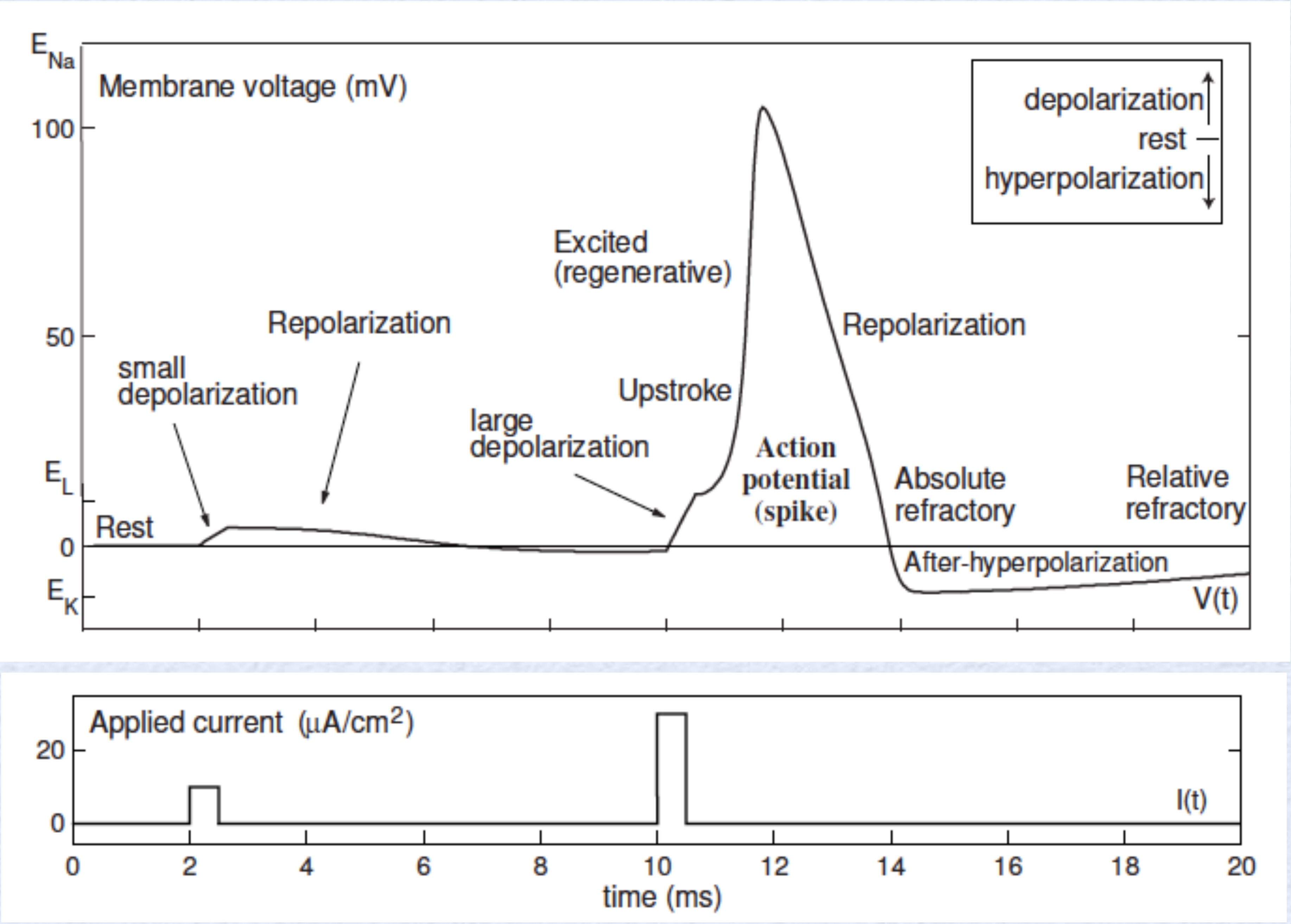
$E_{\mathrm{L}} = 10.6 \text{ mV}$

$\bar{g}_{\mathrm{K}} = 36 \text{ mS/cm}^2$

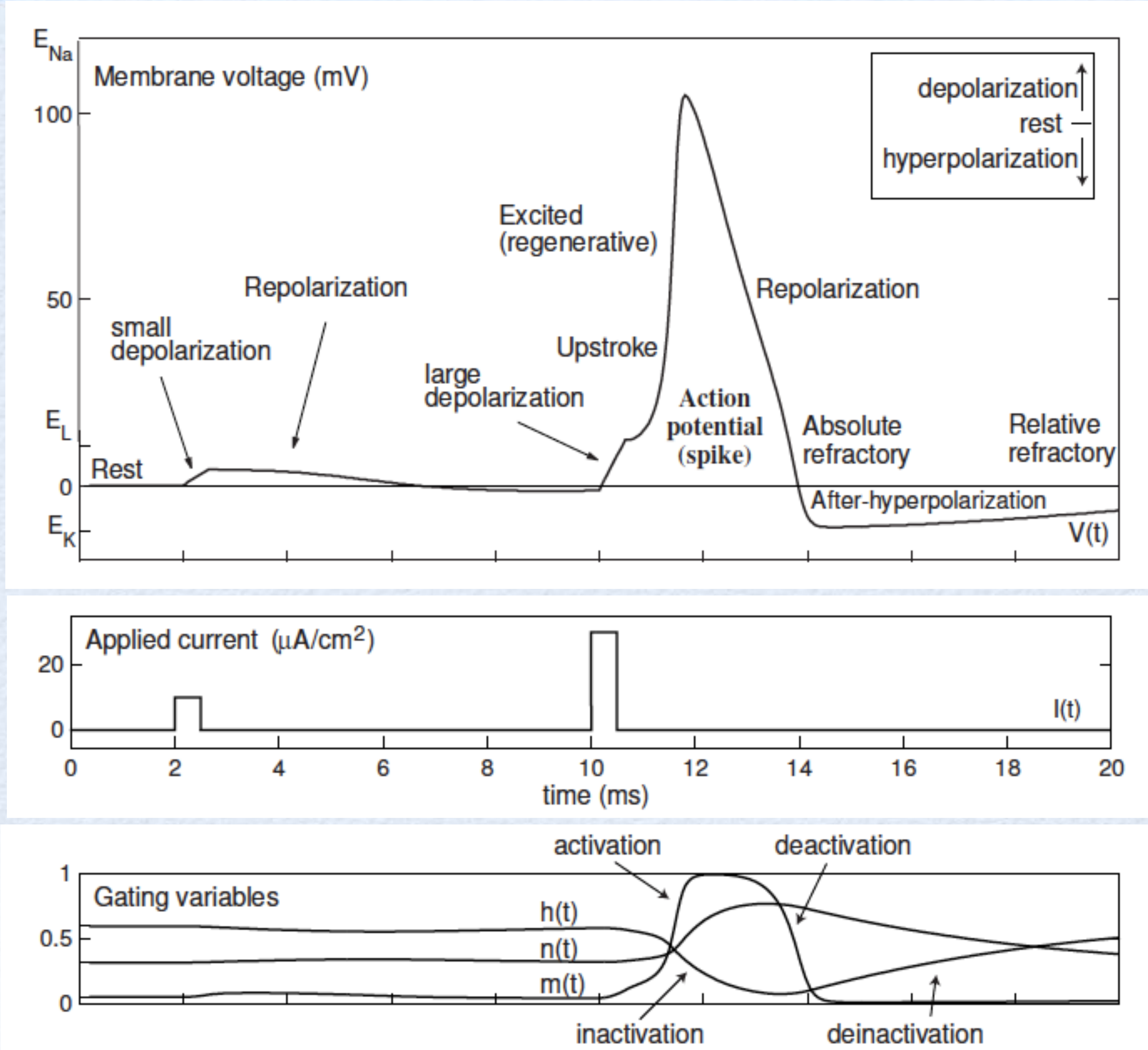
$\bar{g}_{\mathrm{Na}} = 120 \text{ mS/cm}^2$

$g_{\mathrm{L}} = 0.3 \text{ mS/cm}^2$

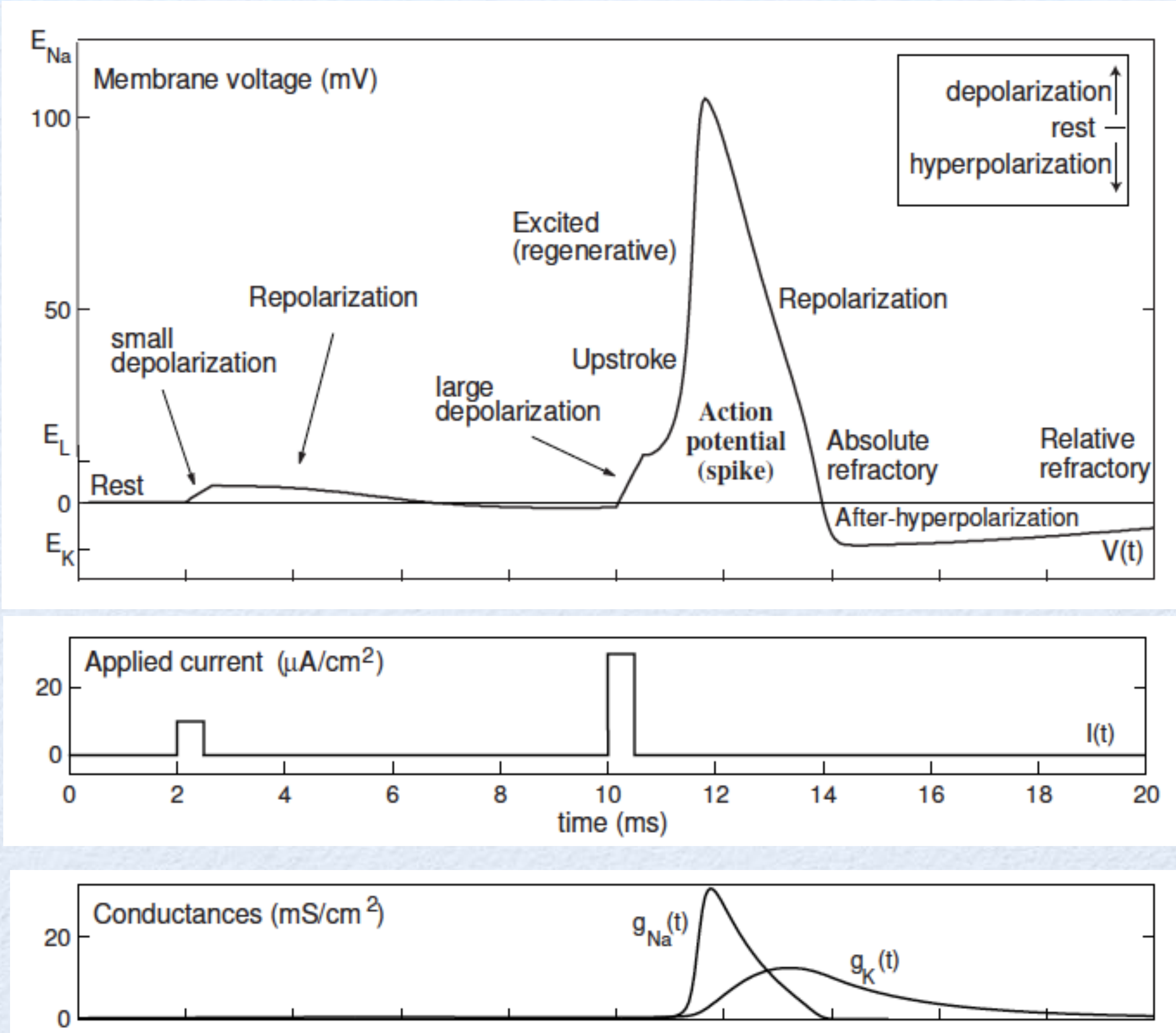
Hodgkin-Huxley model



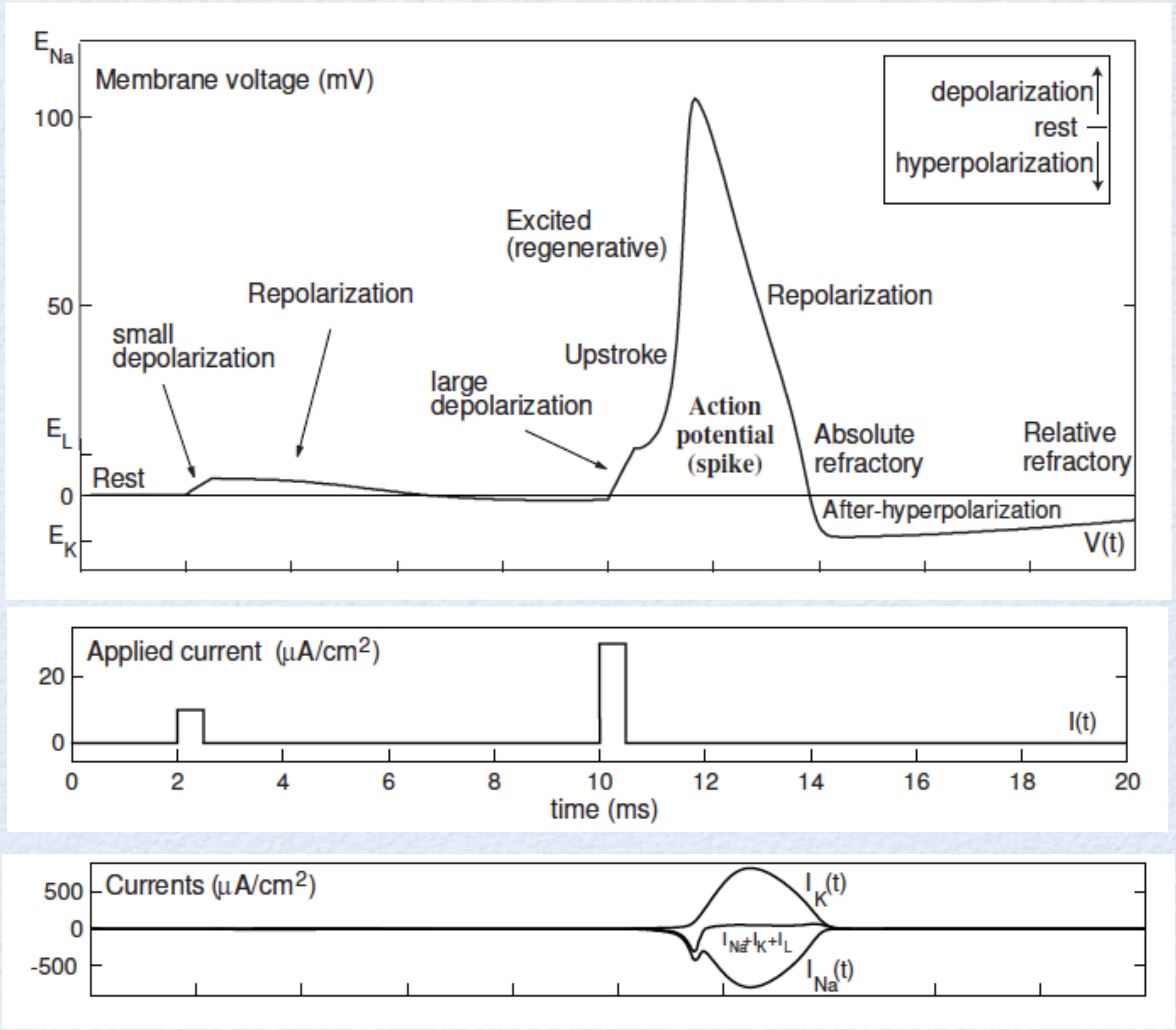
Hodgkin-Huxley model



Hodgkin-Huxley model



Hodgkin-Huxley model



Hodgkin-Huxley model

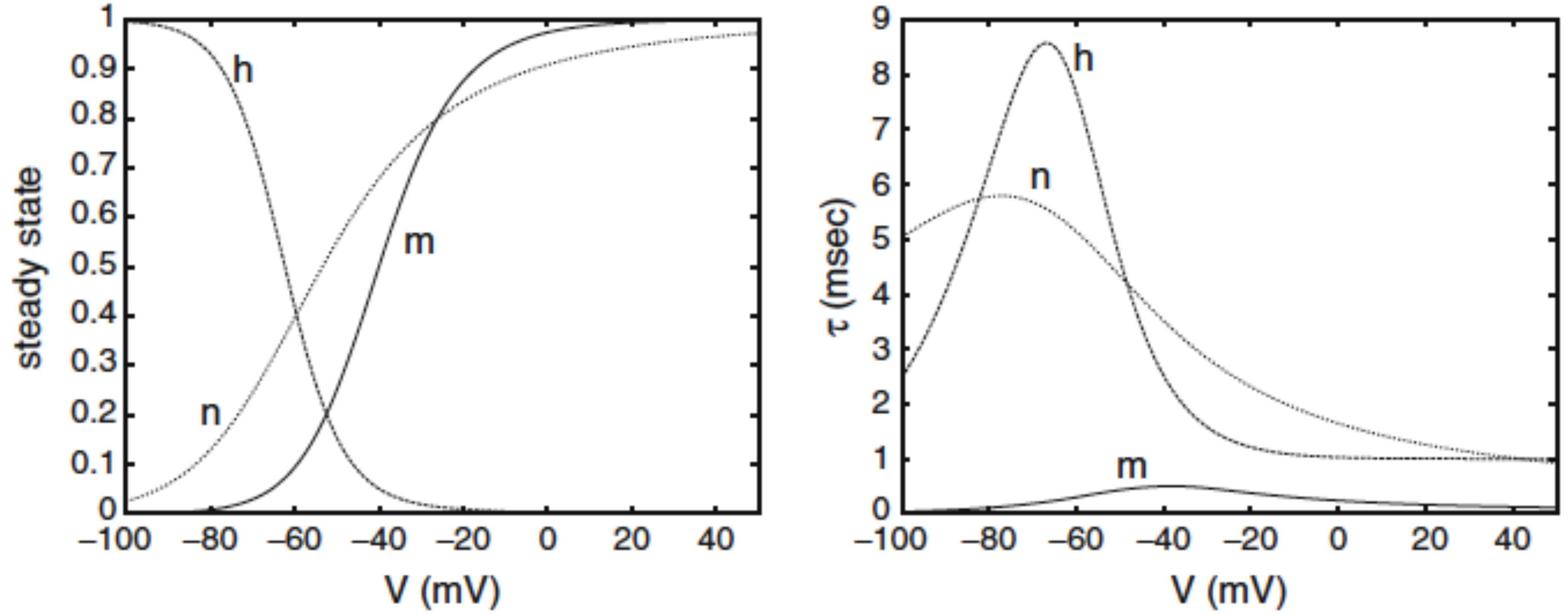


Fig. 1.11 Hodgkin–Huxley functions. *Left* the steady-state opening of the gates and *right* the time constants

Hodgkin-Huxley model

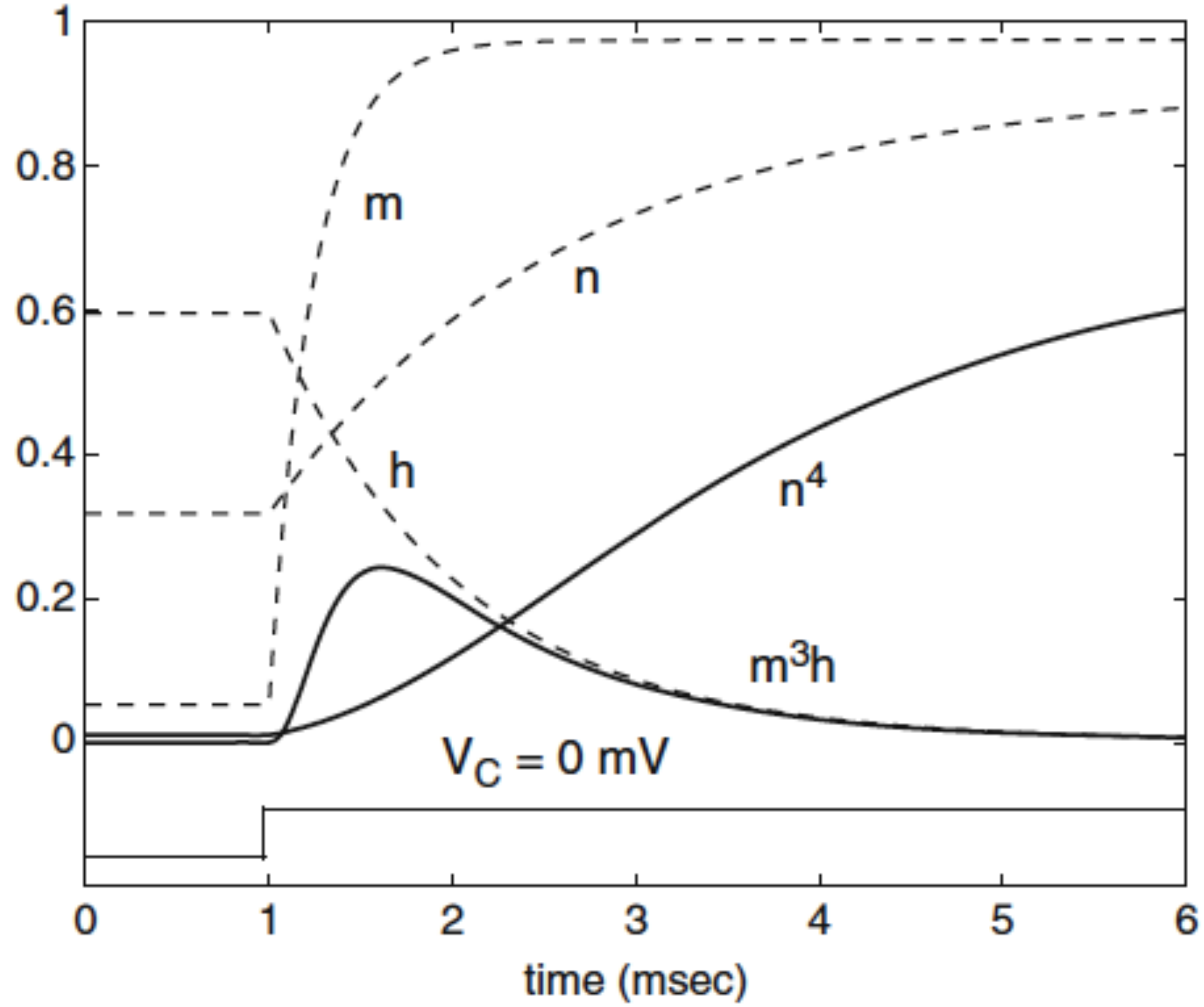


Fig. 1.12 Response of the activation and inactivation variables *m*, *h*, and *n* to a step in voltage

Hodgkin-Huxley model

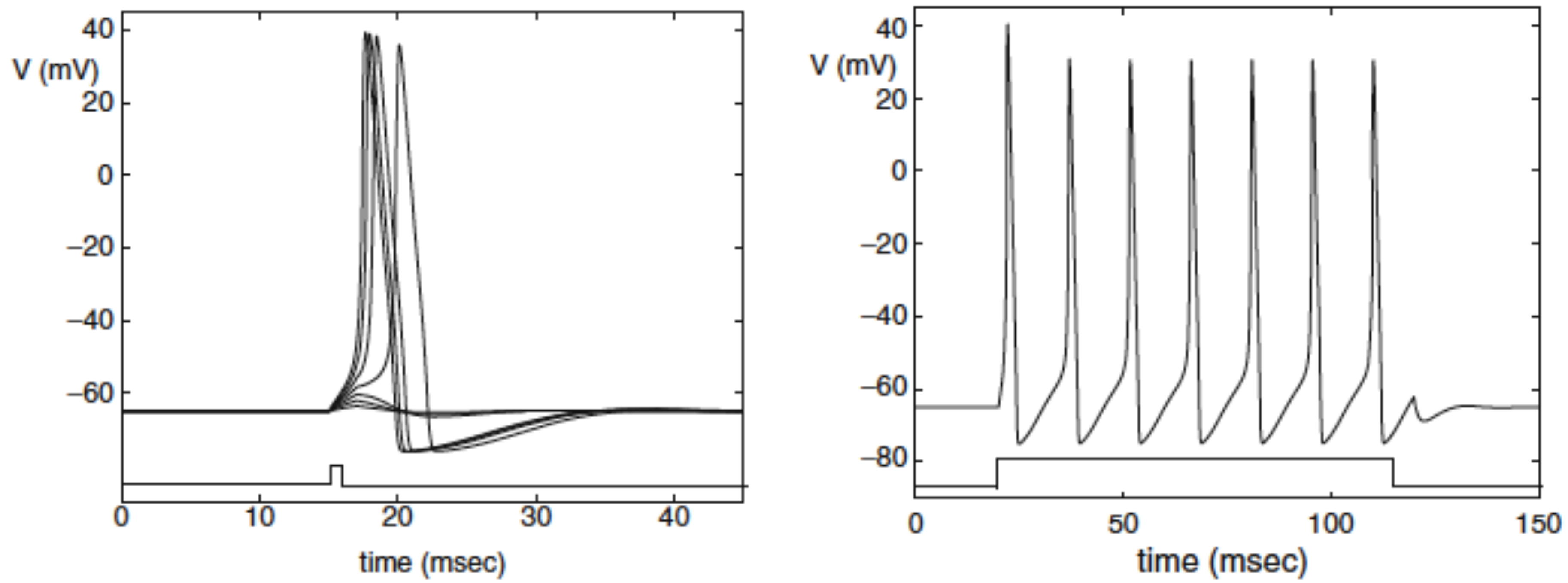


Fig. 1.13 Responses of the Hodgkin–Huxley model to applied currents. *Left* transient responses showing "all-or-none" behavior and *right* sustained periodic response

Hodgkin-Huxley model

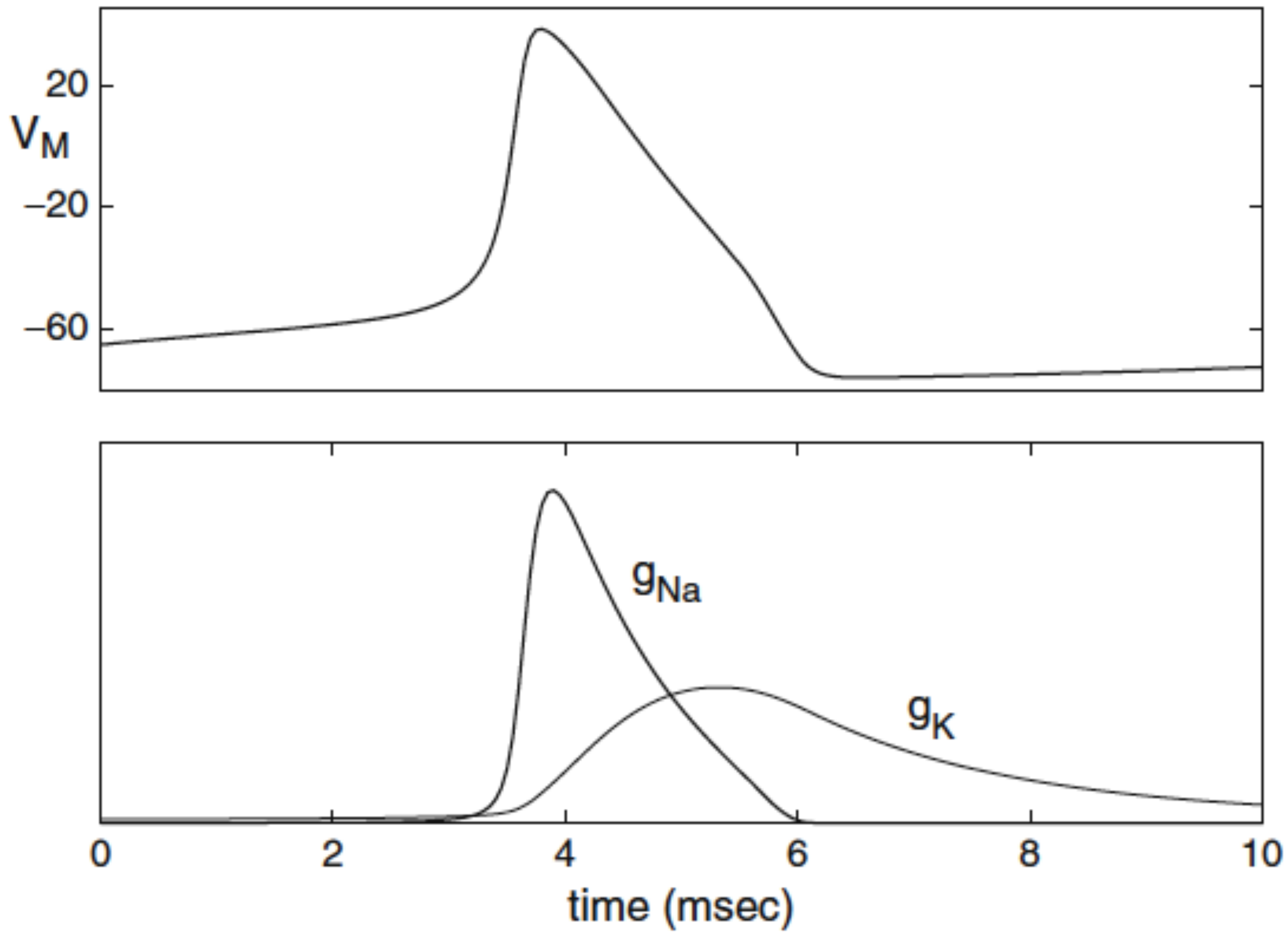
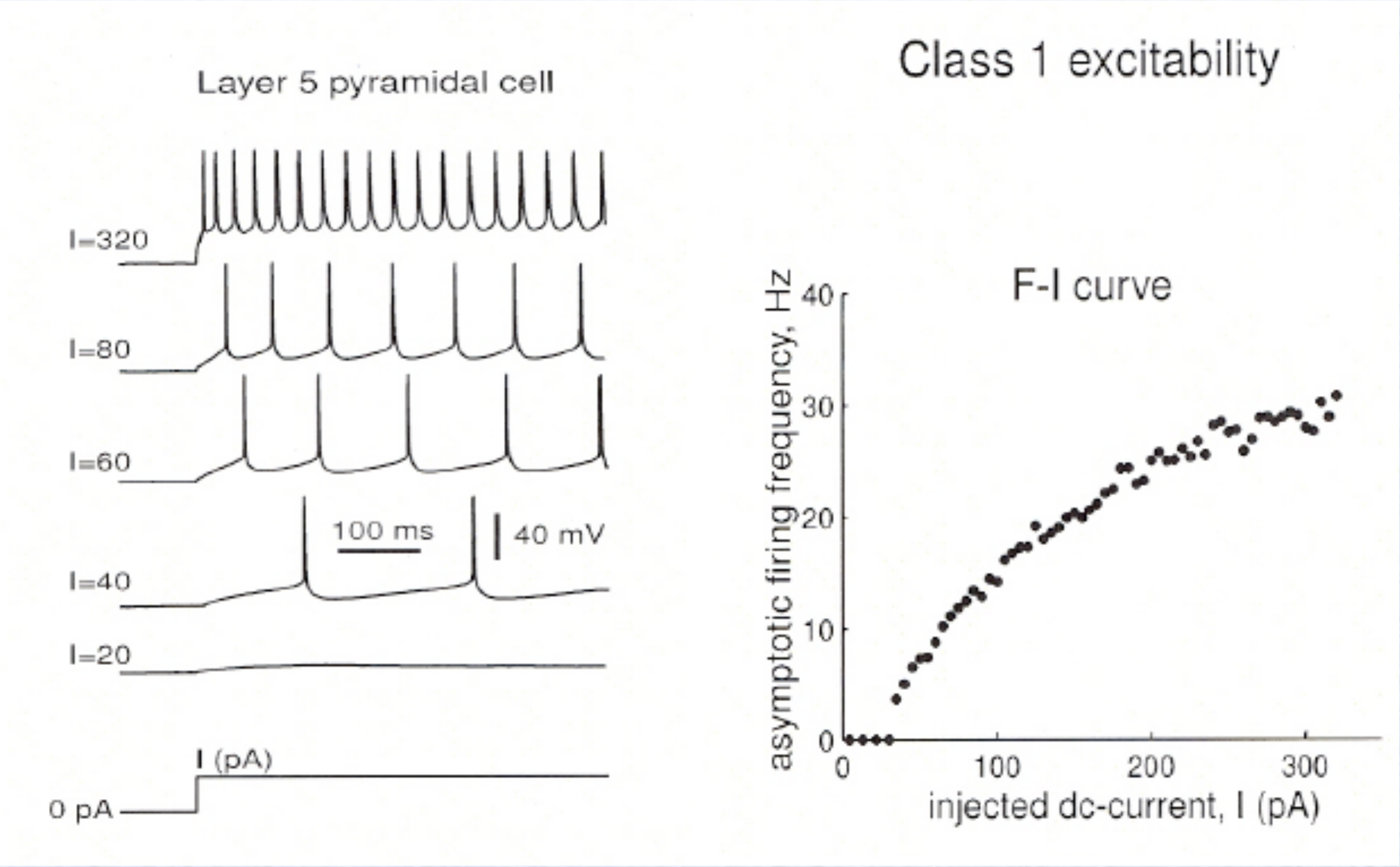
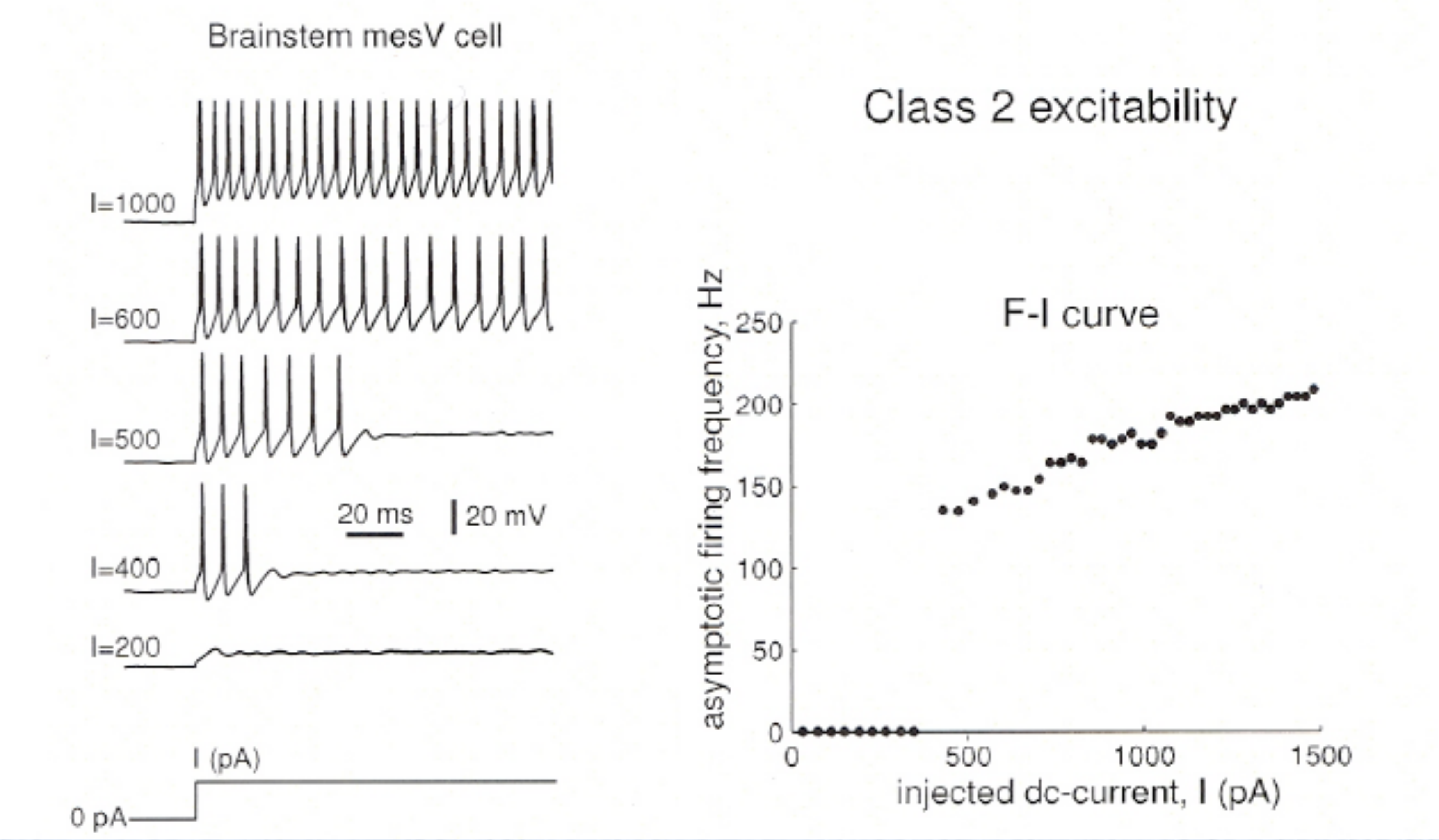


Fig. 1.14 Solution of the Hodgkin–Huxley equations showing an action potential. Also shown are the Na^+ and K^+ conductances

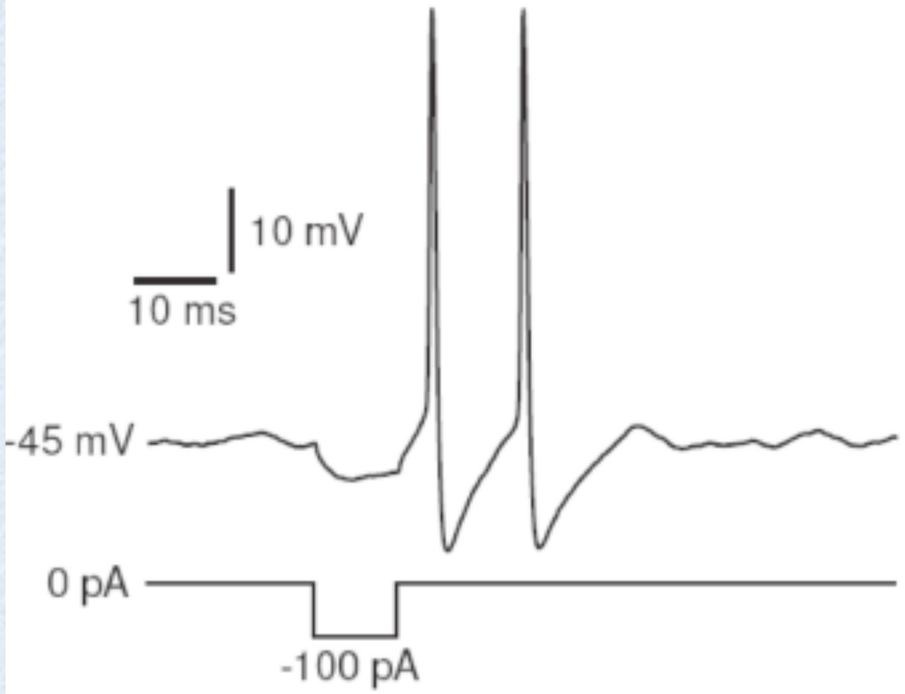
Dynamics of single neurons



Dynamics of single neurons



Dynamics of single neurons



In vitro recording of **rebound spikes** of a rat's brainstem mesencephalic V neuron in response to a brief hyperpolarizing pulse of current.

(Izhikevich)

Dynamics systems

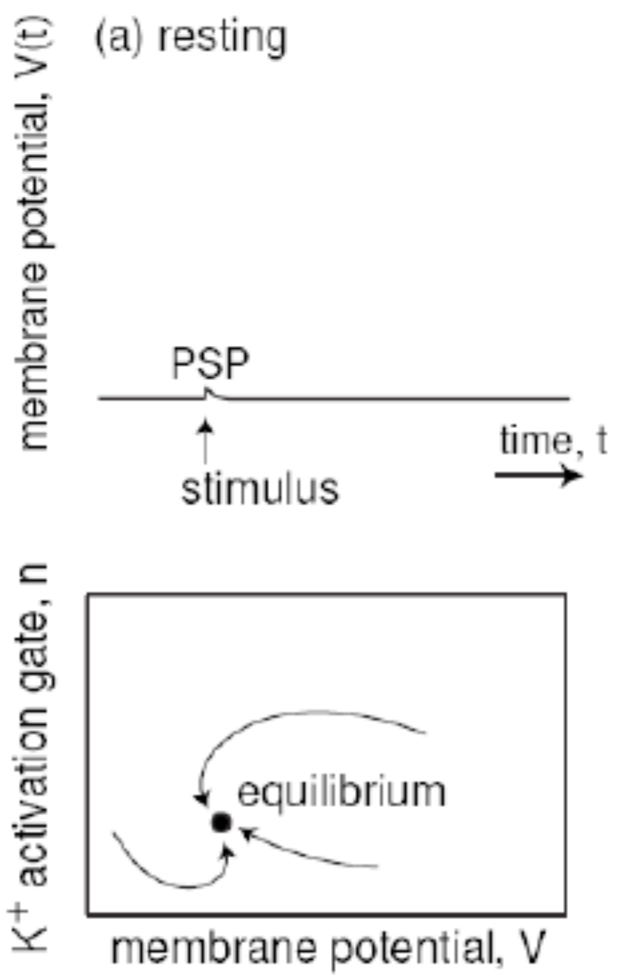
Dynamical System:

- set of variables that describes its state
- law that describes the evolution of the state variables with time

Dynamical systems approach to neuroscience: One can tell things about a system without knowing all the details that govern the system evolution

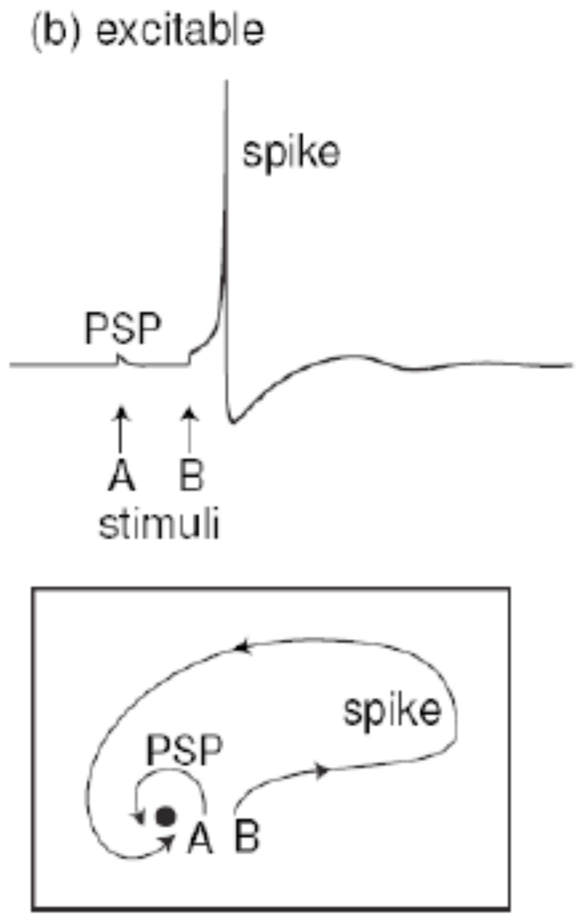
Dynamics systems

- Stable equilibrium point



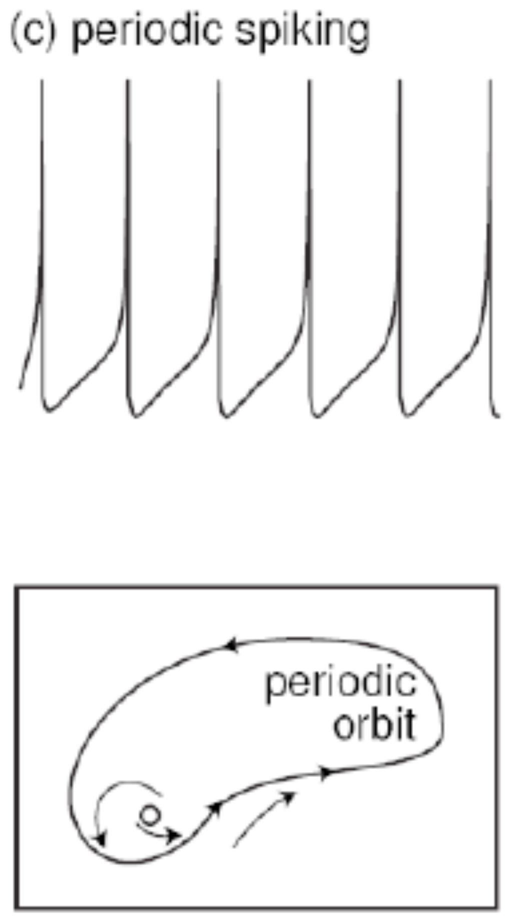
Dynamics systems

- Stable equilibrium point



Dynamics systems

- Stable limit cycle



Dynamics systems

Important concepts:

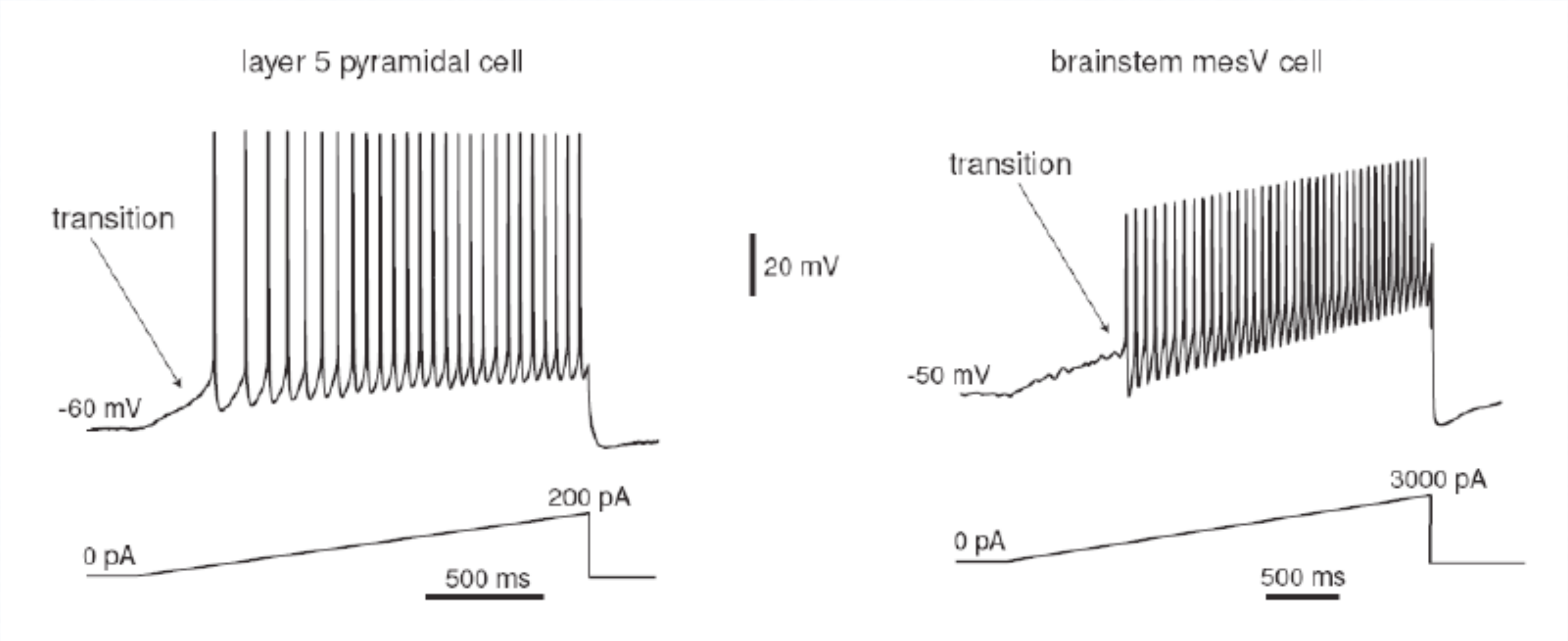
- Phase portrait (phase-plane / phase-space)
- Equilibrium (fixed-) point
- Limit cycle
- Stability
- Threshold
- Attractor
- Attraction domain (basin of attraction)
- Separatrix
- Perturbation
- Bifurcation

Bifurcations

- Qualitative change of the phase portrait of a dynamical system as the magnitude of a control parameter changes
- Example of a control parameter: Applied (injected) current
- Example of a bifurcation: transition from a fixed-point to a limit cycle

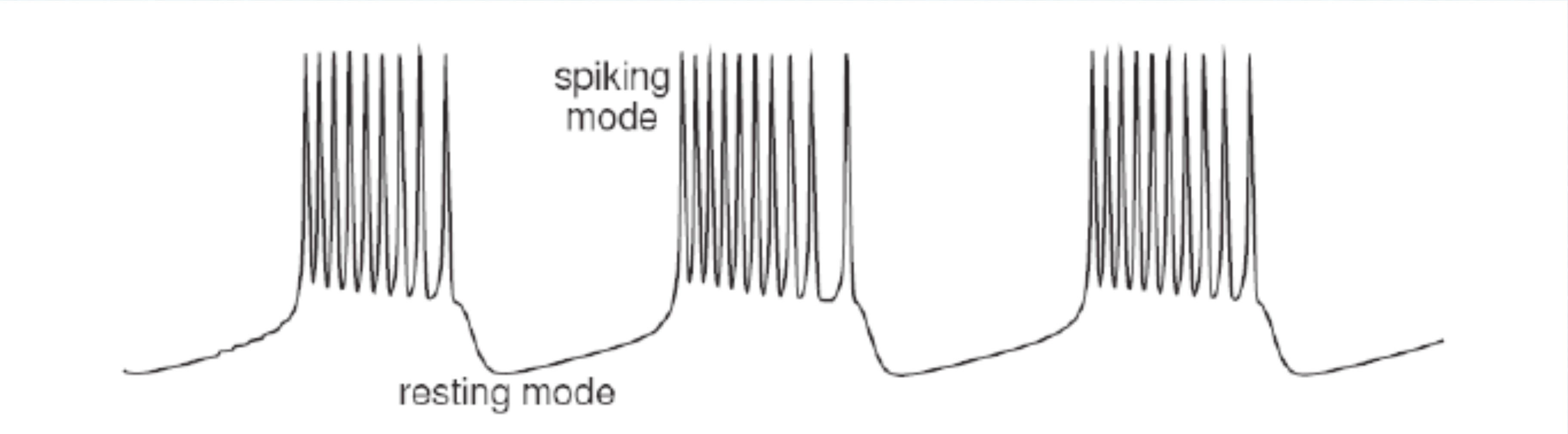
Bifurcations

- As the magnitude of the injected current slowly increases, the neurons bifurcated from resting (equilibrium) to tonic spiking (limit cycle) modes



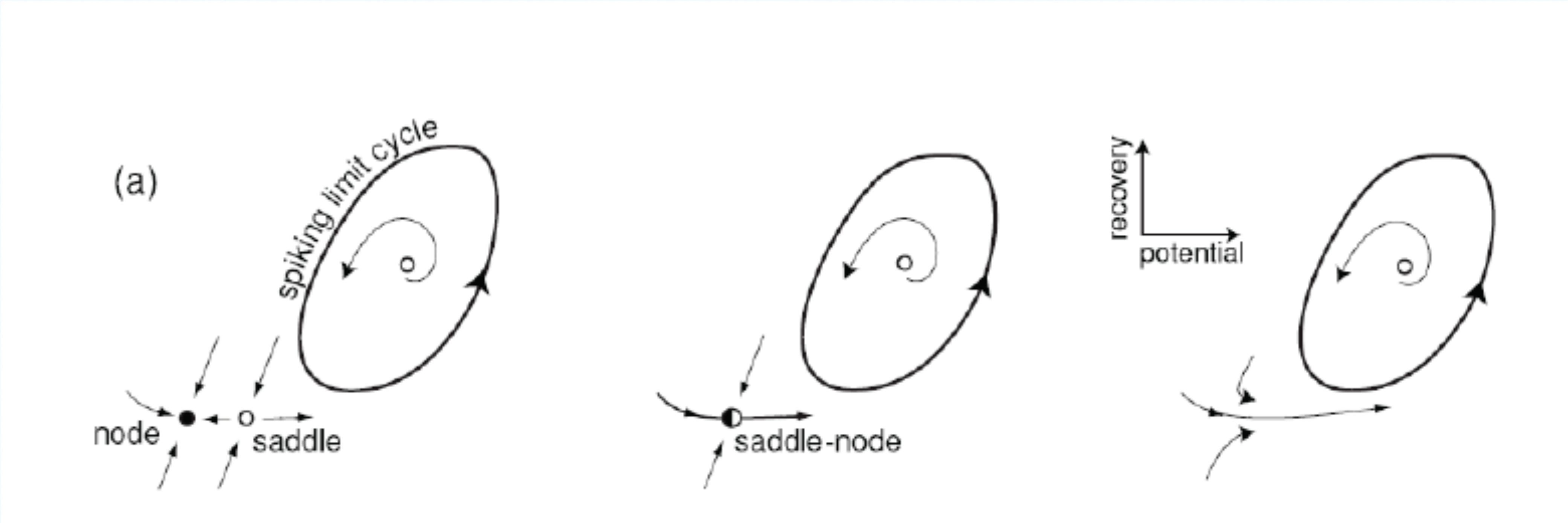
Bifurcations

- Transitions between rhythmic and spiking modes result in bursting behavior



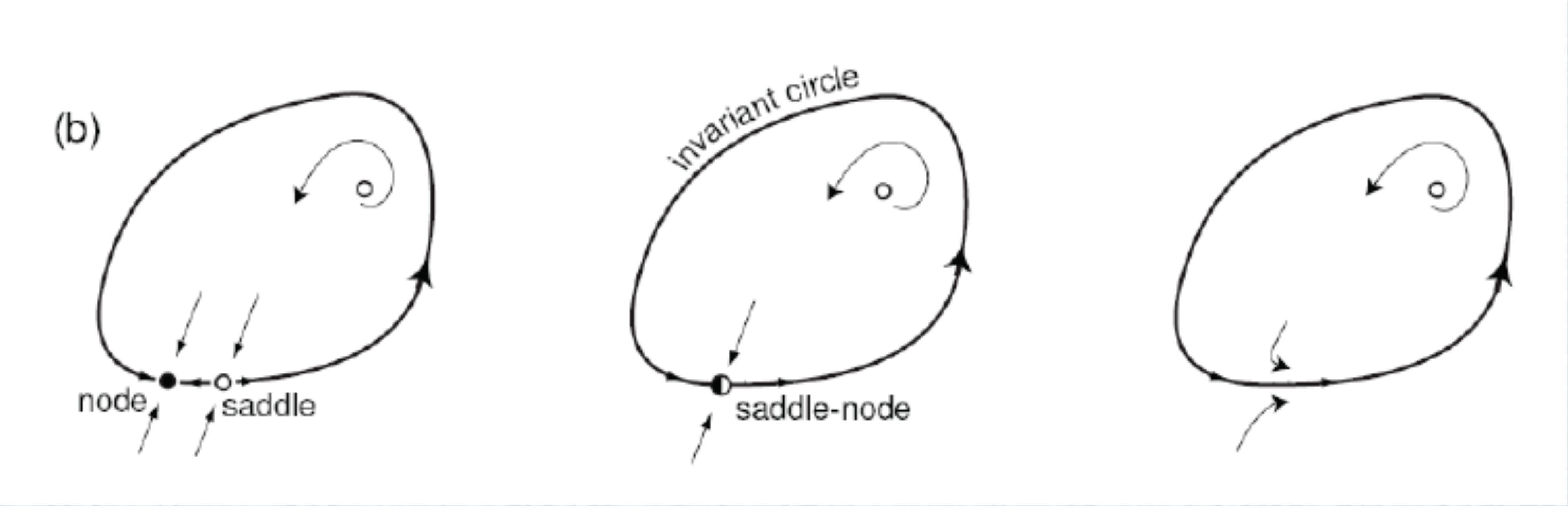
Bifurcations

- Saddle-node bifurcation



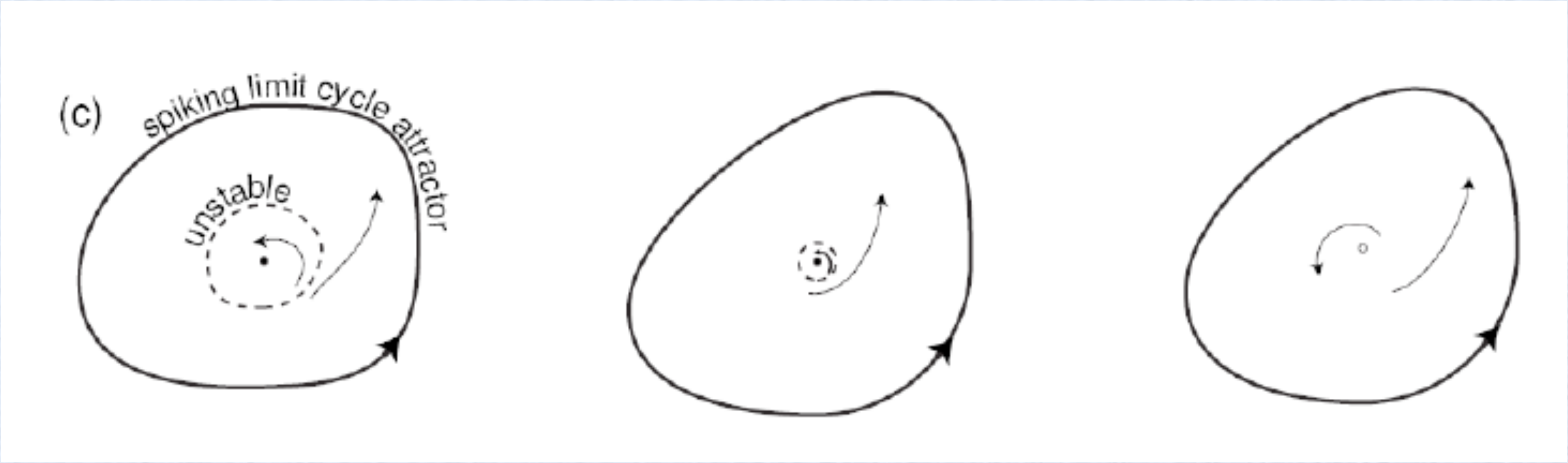
Bifurcations

- Saddle-node on invariant cycle (SNIC) bifurcation



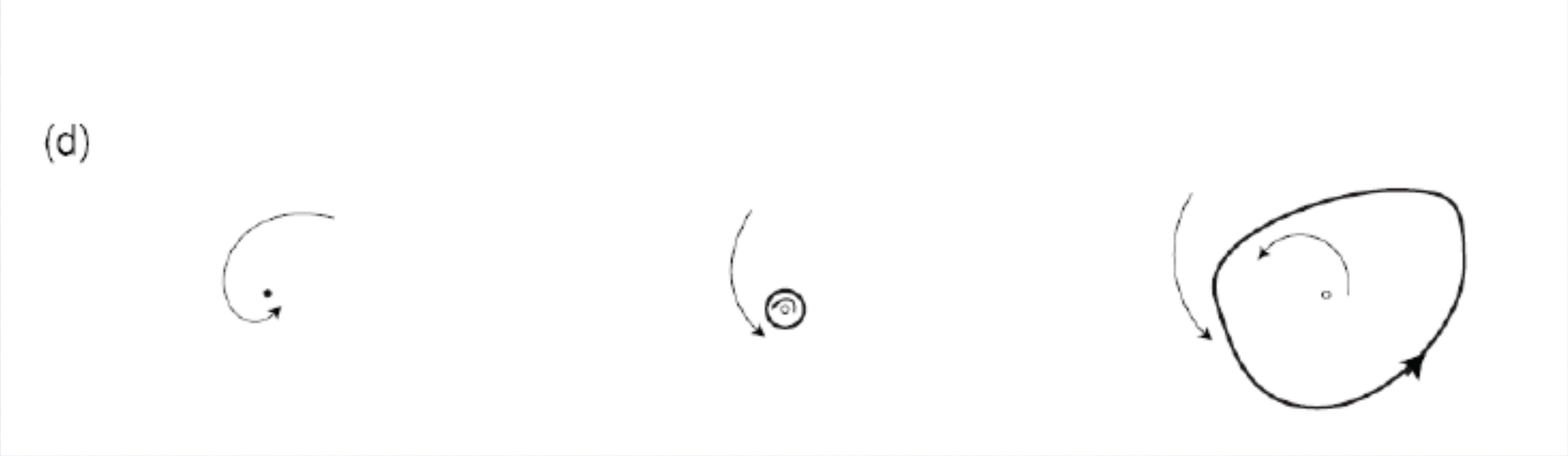
Bifurcations

- Subcritical Hopf (Andronov-Hopf) bifurcation



Bifurcations

- Supercritical Hopf (Andronov-Hopf) bifurcation



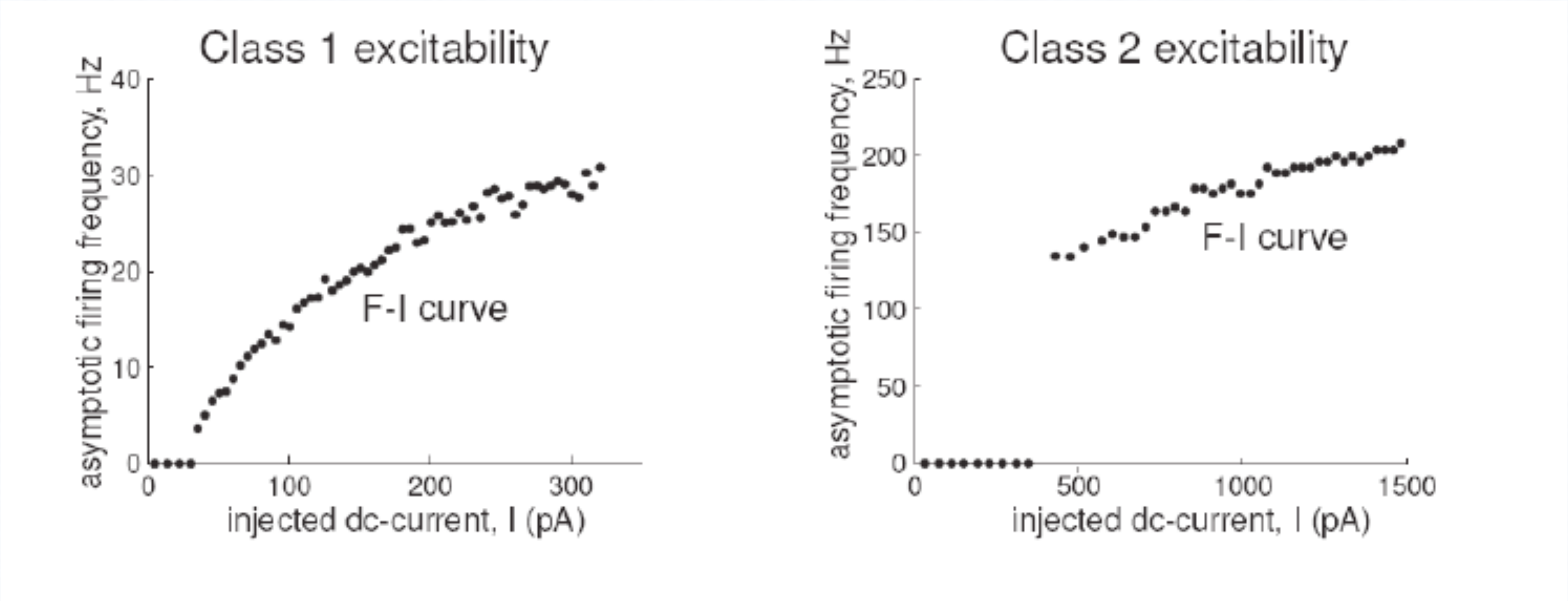
Bifurcations

- Classification of neurons according to the bifurcation of the resting state

subthreshold oscillations	co-existence of resting and spiking states: YES (bistable)	NO (monostable)
NO (integrator)	saddle-node	saddle-node on invariant circle
YES (resonator)	subcritical Andronov-Hopf	supercritical Andronov-Hopf

Bifurcations

- Frequency-current (F-I) curves of cortical pyramidal neurons and mes V neurons



One-dimensional neural model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L} \\
\dot{n} &= (n_\infty(V) - n)/\tau_n(V)\,, \\
\dot{m} &= (m_\infty(V) - m)/\tau_m(V)\,, \\
\dot{h} &= (h_\infty(V) - h)/\tau_h(V)\,,
\end{aligned}
$$

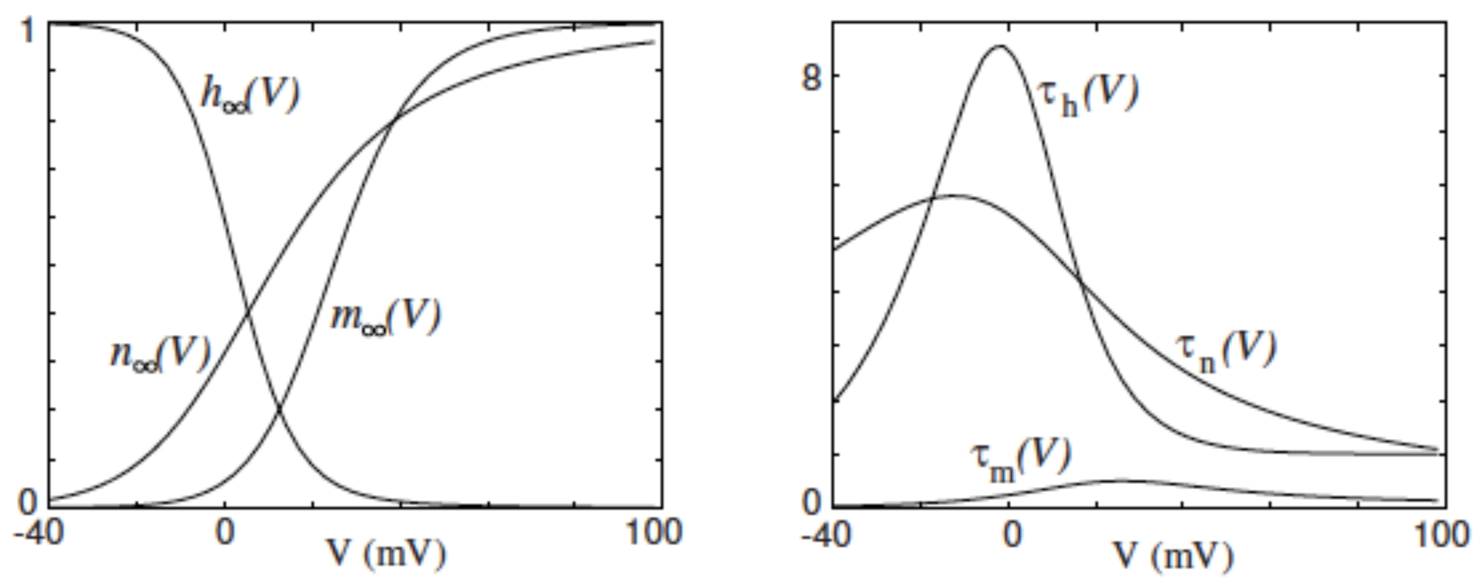


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = (n_\infty(V) - n)/\tau_n(V)\,,$$
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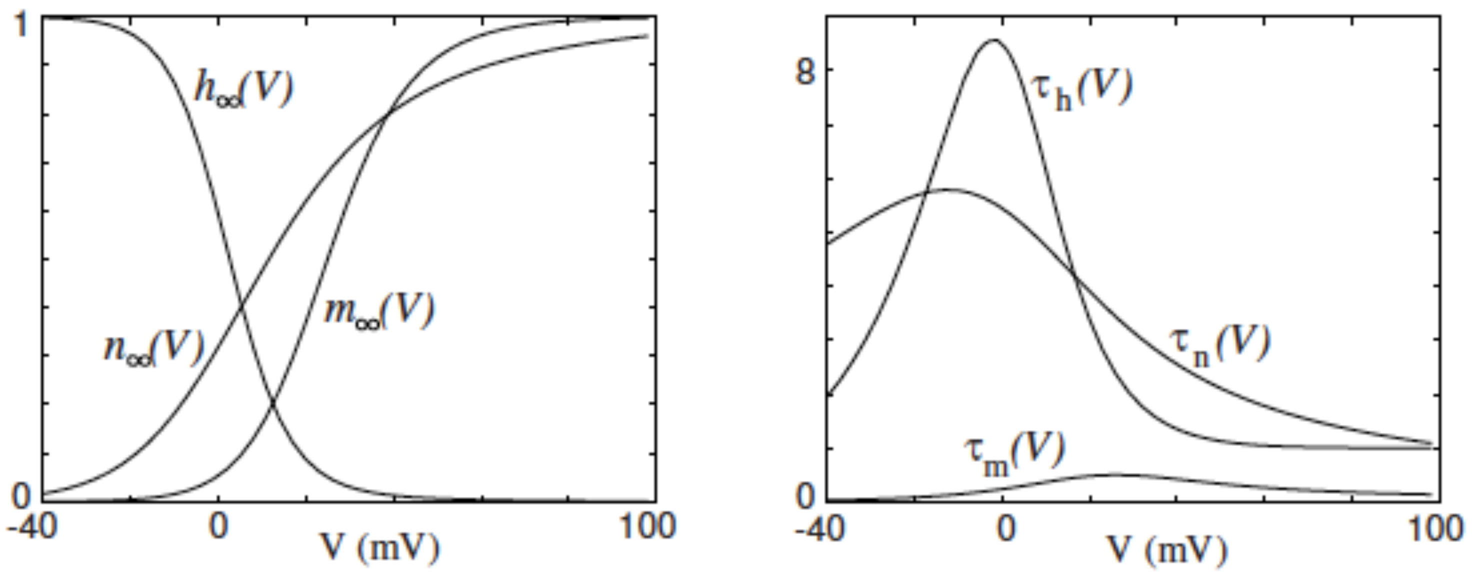

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

- The passive membrane equation

$$C\frac{dV}{dt} = -G_L(V - E_L) + I_{app}$$

One-dimensional neural model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
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\end{aligned}
$$

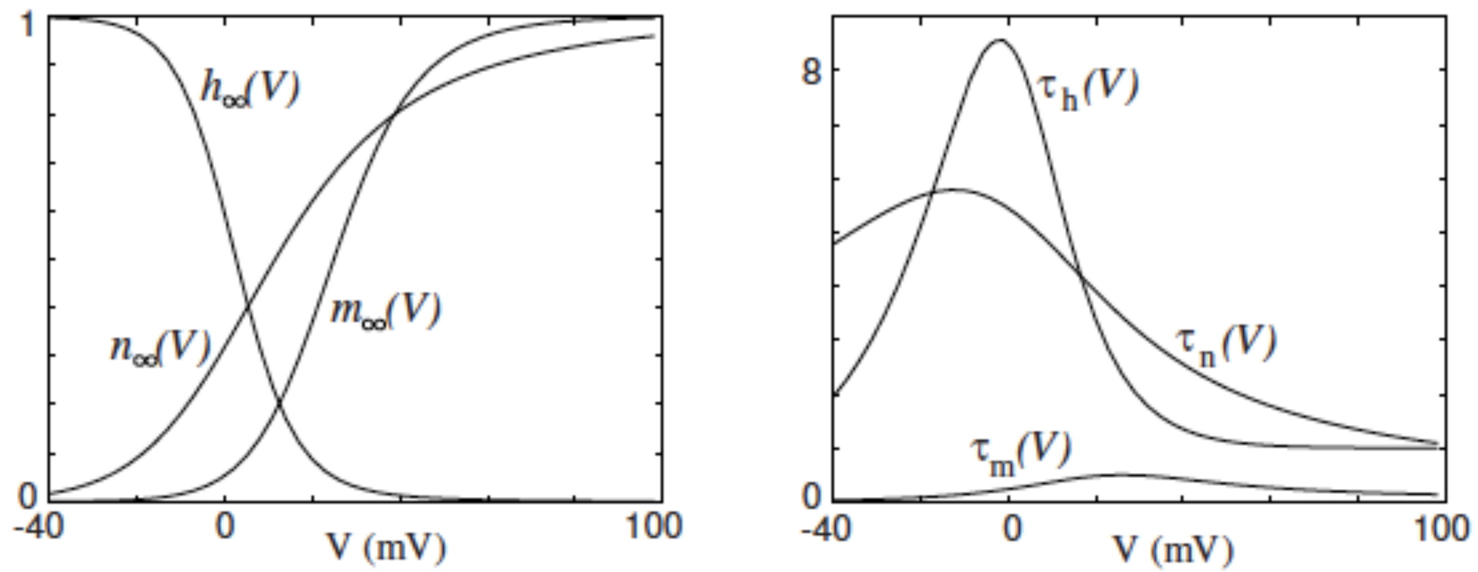

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_{K} n^{4}(V - E_{K})}^{I_{K}} - \overbrace{\bar{g}_{Na} m^{3} h (V - E_{Na})}^{I_{Na}} - \overbrace{g_{L}(V - E_{L})}^{I_{L}}$$

$$\dot{n} = (n_{\infty}(V) - n)/\tau_{n}(V)\,,$$

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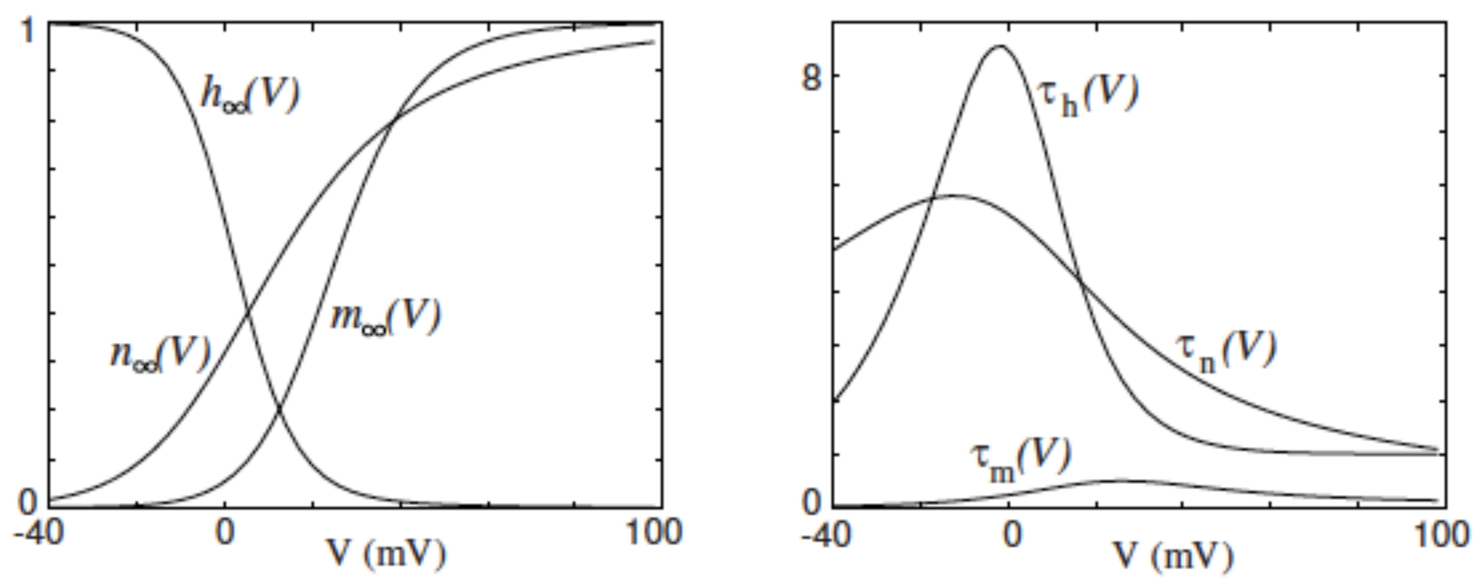

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

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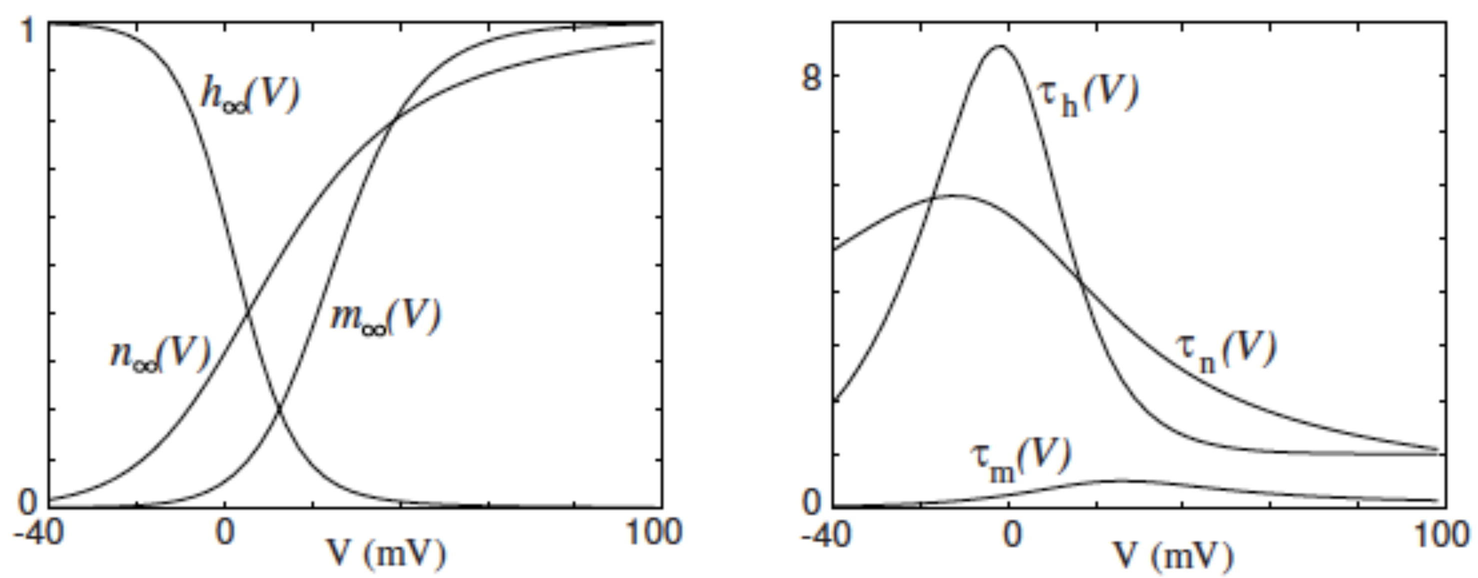
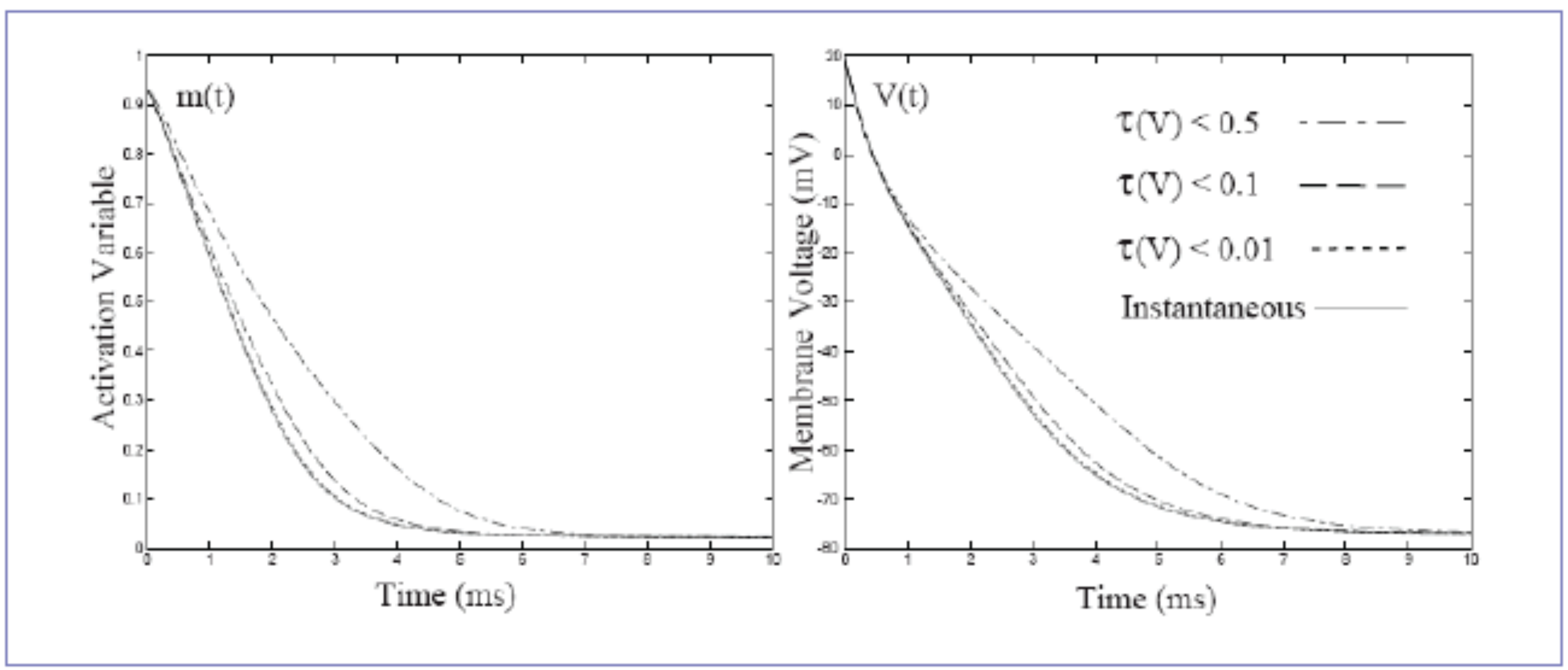


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\frac{dV}{dt} = -G_{Na}m_{\infty}^{3}(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I_{app}$$

$$\frac{dm}{dt} = \frac{m_{\infty}(V) - m}{\tau(V)}$$



One-dimensional neural model

- Persistent sodium model

$$C\frac{dV}{dt} = -G_{Na}m_{\infty}^{3}(V)(V - E_{Na}) - G_{L}(V - E_{L}) + I_{app}$$

One-dimensional neural model

- Persistent sodium model

$$C\frac{dV}{dt} = -G_{Na}p_{\infty}^{3}(V)(V - E_{Na}) - G_{L}(V - E_{L}) + I_{app}$$

Two-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

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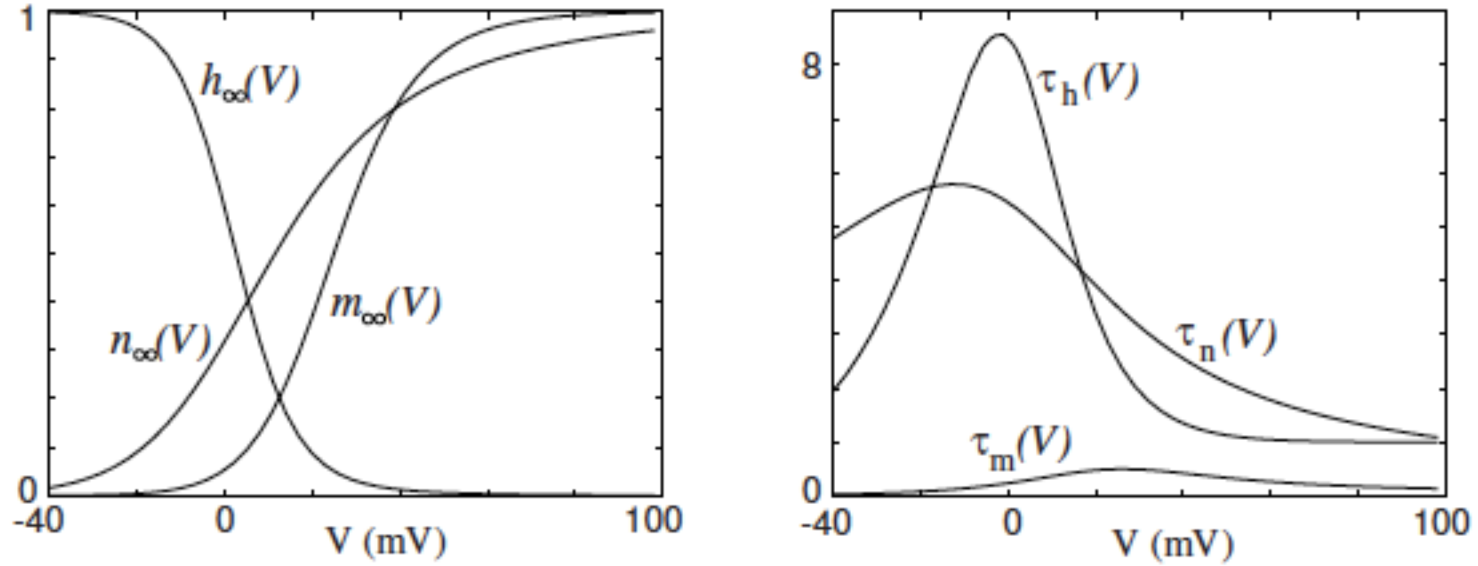

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Two-dimensional neural model

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\begin{aligned}
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$$

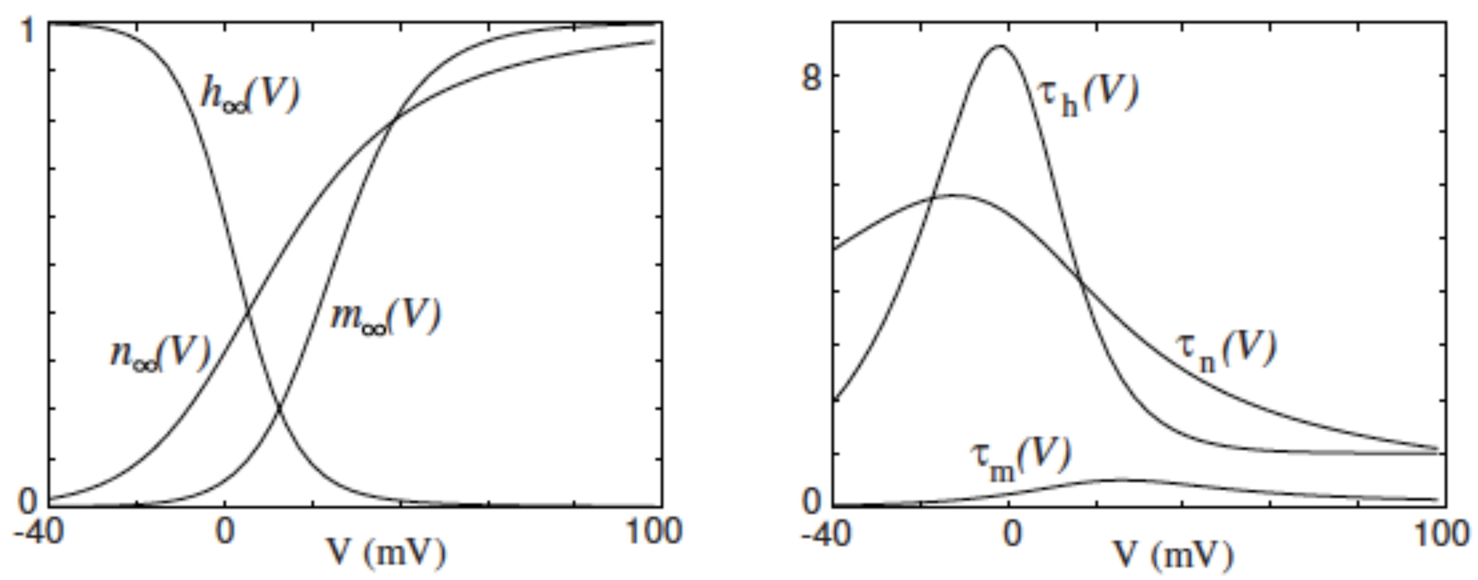

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Two-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} \overbrace{- \bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} \overbrace{- g_L (V - E_L)}^{I_L}$$

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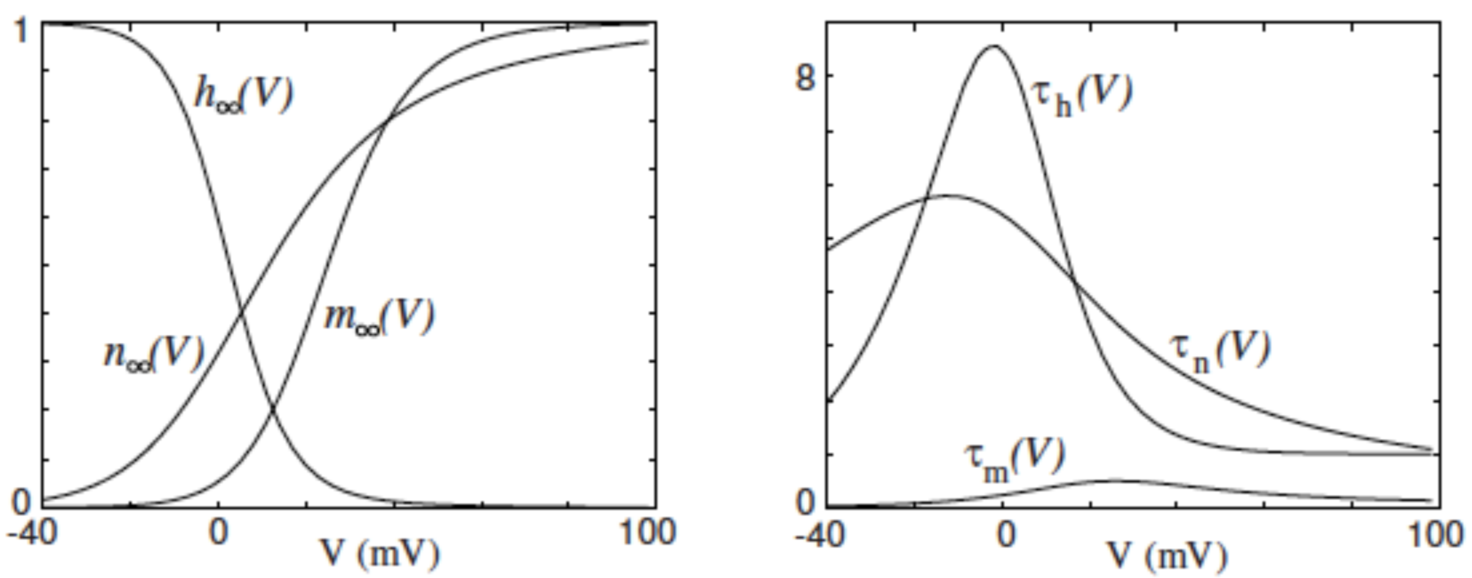


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Two-dimensional neural model

- Persistent sodium & potassium model

$$C\frac{dV}{dt} = - G_{Na} m_{\infty}^{3}(V)\,(V - E_{Na}) - G_K n^4\,(V - E_K) -$$

$$- G_L\,(V - E_L) + I_{app}$$

$$\frac{dn}{dt} = \frac{n_{\infty}(V) - n}{\tau_n(V)}$$

Two-dimensional neural model

- Persistent sodium & potassium model

$$C\frac{dV}{dt} = -G_{Na}p_{\infty}^{3}(V)(V - E_{Na}) - G_{K}n^{4}(V - E_{K}) - $$
$$- G_{L}(V - E_{L}) + I_{app}$$

$$\frac{dn}{dt} = \frac{n_{\infty}(V) - n}{\tau_{n}(V)}$$

Two-dimensional neural model

- Morris-Lecar model

$$C\frac{dV}{dt} = -G_{Ca}m_{\infty}(V)(V - E_{Ca}) - G_K w (V - E_K) - $$
$$- G_L (V - E_L) + I_{app}$$

$$\frac{dw}{dt} = \frac{w_{\infty}(V) - w}{\tau_w(V)}$$

Two-dimensional neural model

- Morris-Lecar model

$$m_\infty(V) = 0.5\left(1 + \tanh\frac{V + 1}{15}\right)$$

$$w_\infty(V) = 0.5\left(1 + \tanh\frac{V}{30}\right)$$

$$\tau_w(V) = \frac{5}{\cosh(V/60)}$$

Two-dimensional neural model

- FitzHugh-Nagumo (FHN) model

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$$

$$\frac{dW}{dt} = \phi\,(V + a - bW)$$

$a,\ b,\ \phi$: dimensionless & positive

$\phi\ (<< 1)$: inverse of a time constant

Two-dimensional neural model

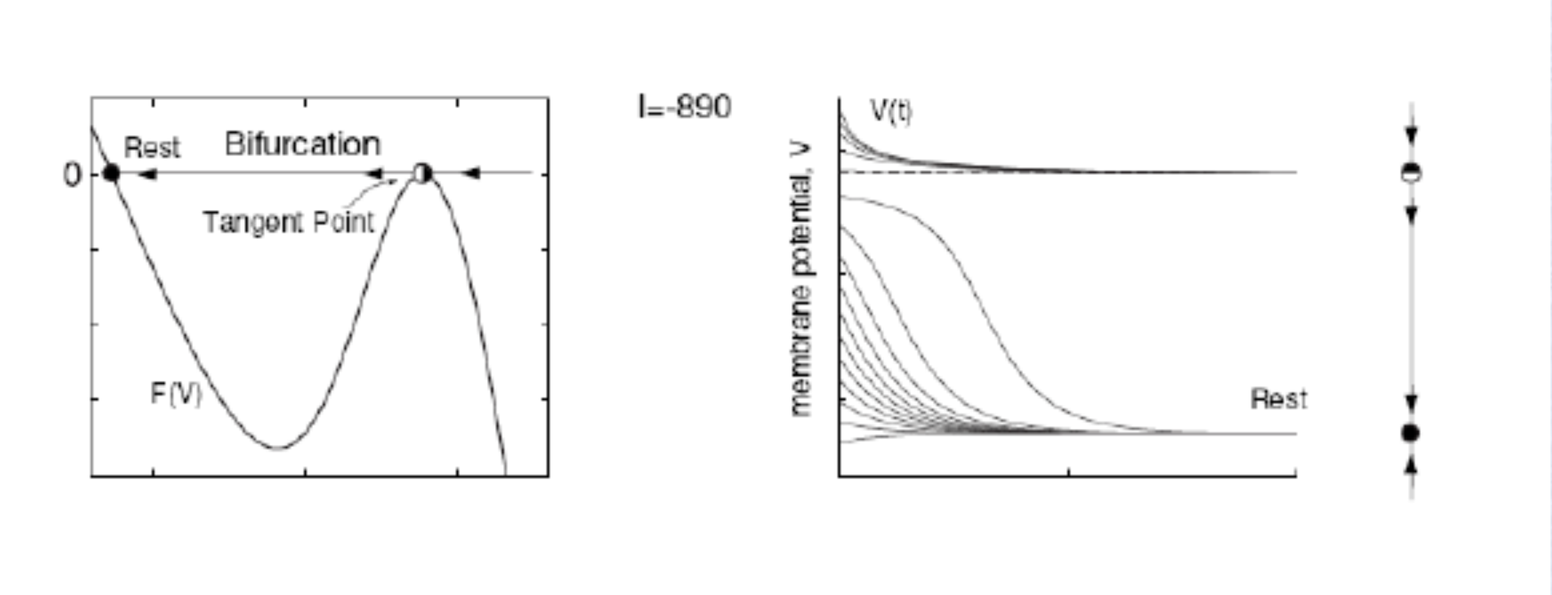
- Quadratic integrate-and-fire model

$$\frac{dV}{dt} = I + V^2 \qquad \text{if } V \geq V_{peak}, \text{ then } V \leftarrow V_{reset}$$

$$\text{Equilibria:} \quad V_{rest} = -\sqrt{I} \quad \text{and} \quad V_{thresh} = +\sqrt{I}$$

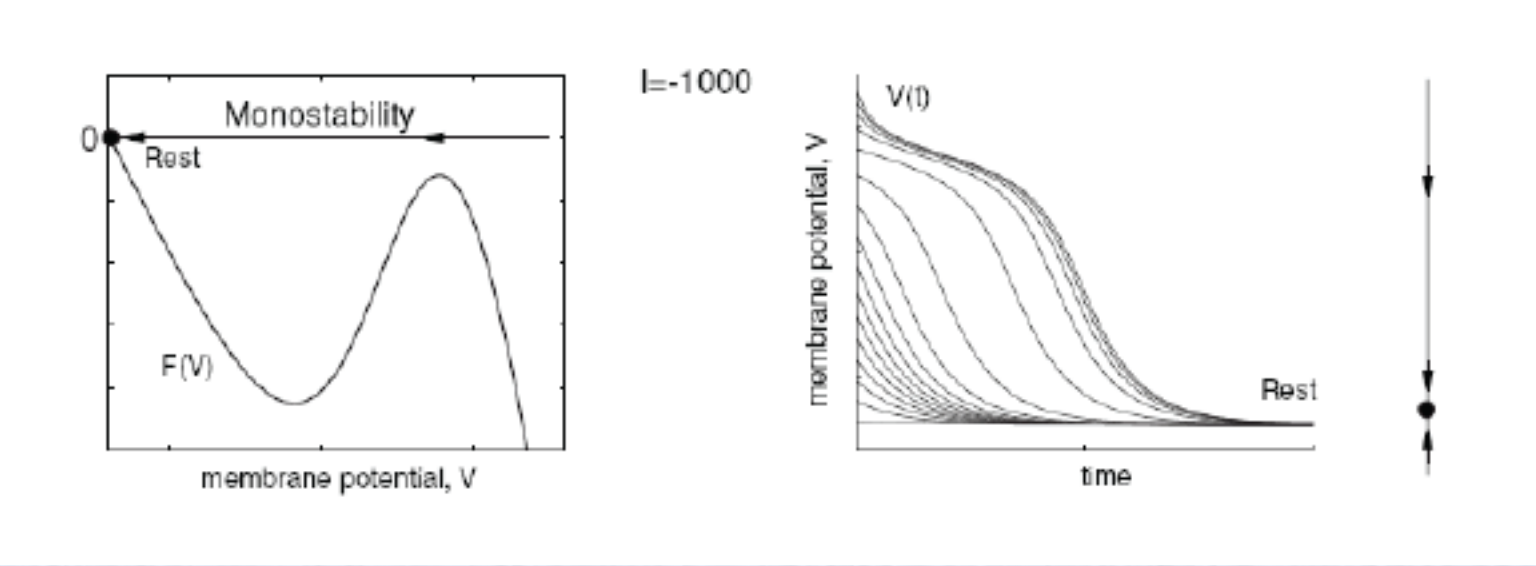
Two-dimensional neural model

- Quadratic integrate-and-fire model



Two-dimensional neural model

- Quadratic integrate-and-fire model



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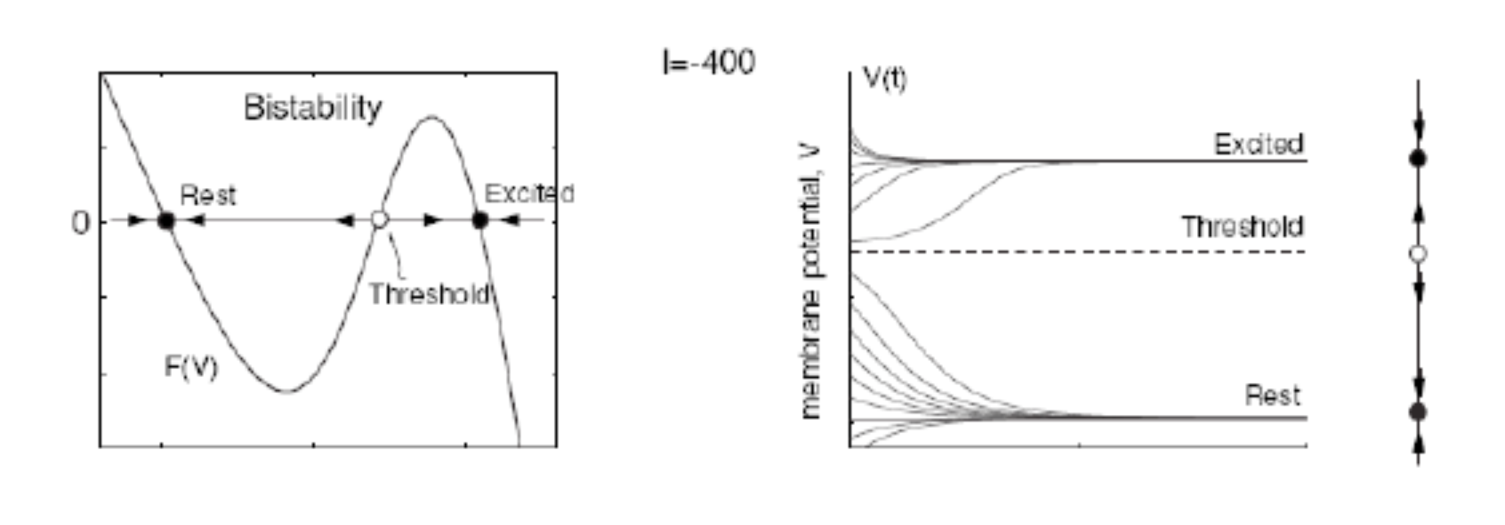
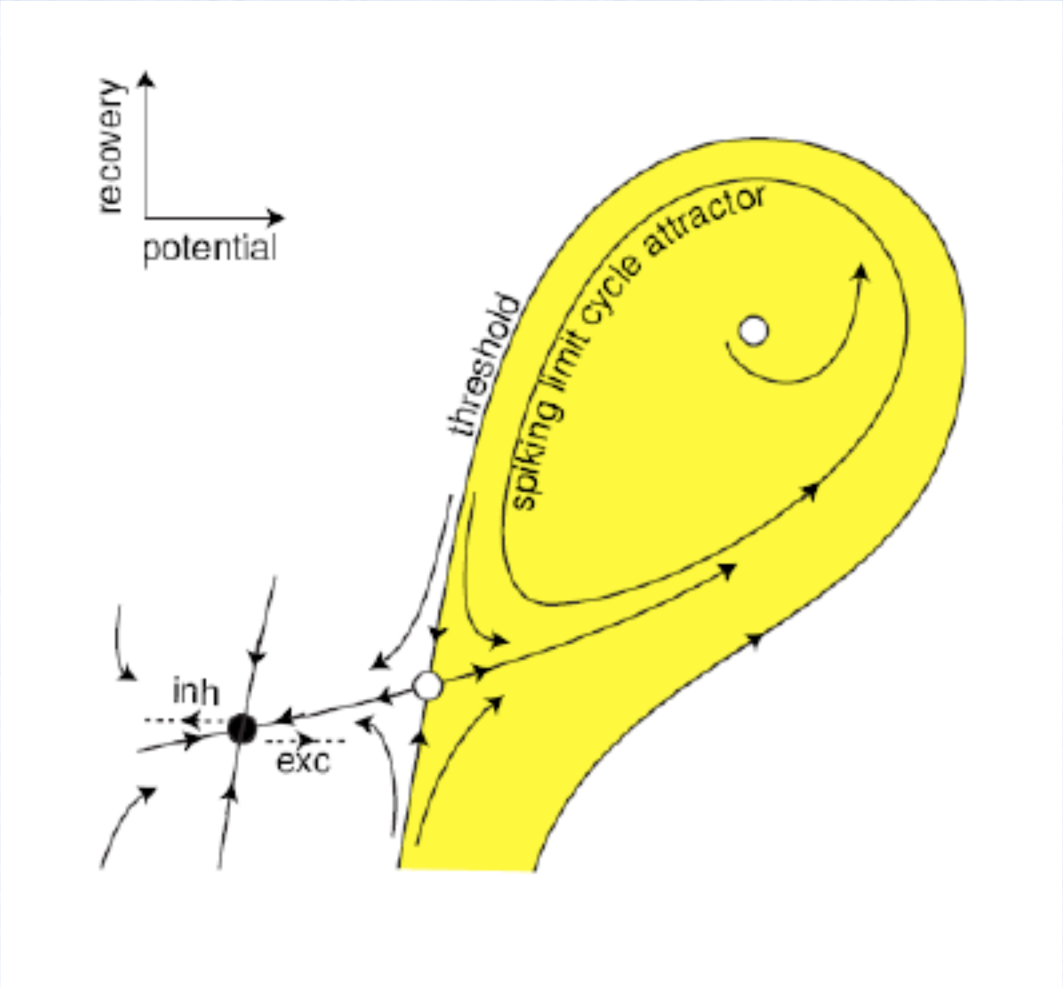
- Quadratic integrate-and-fire model



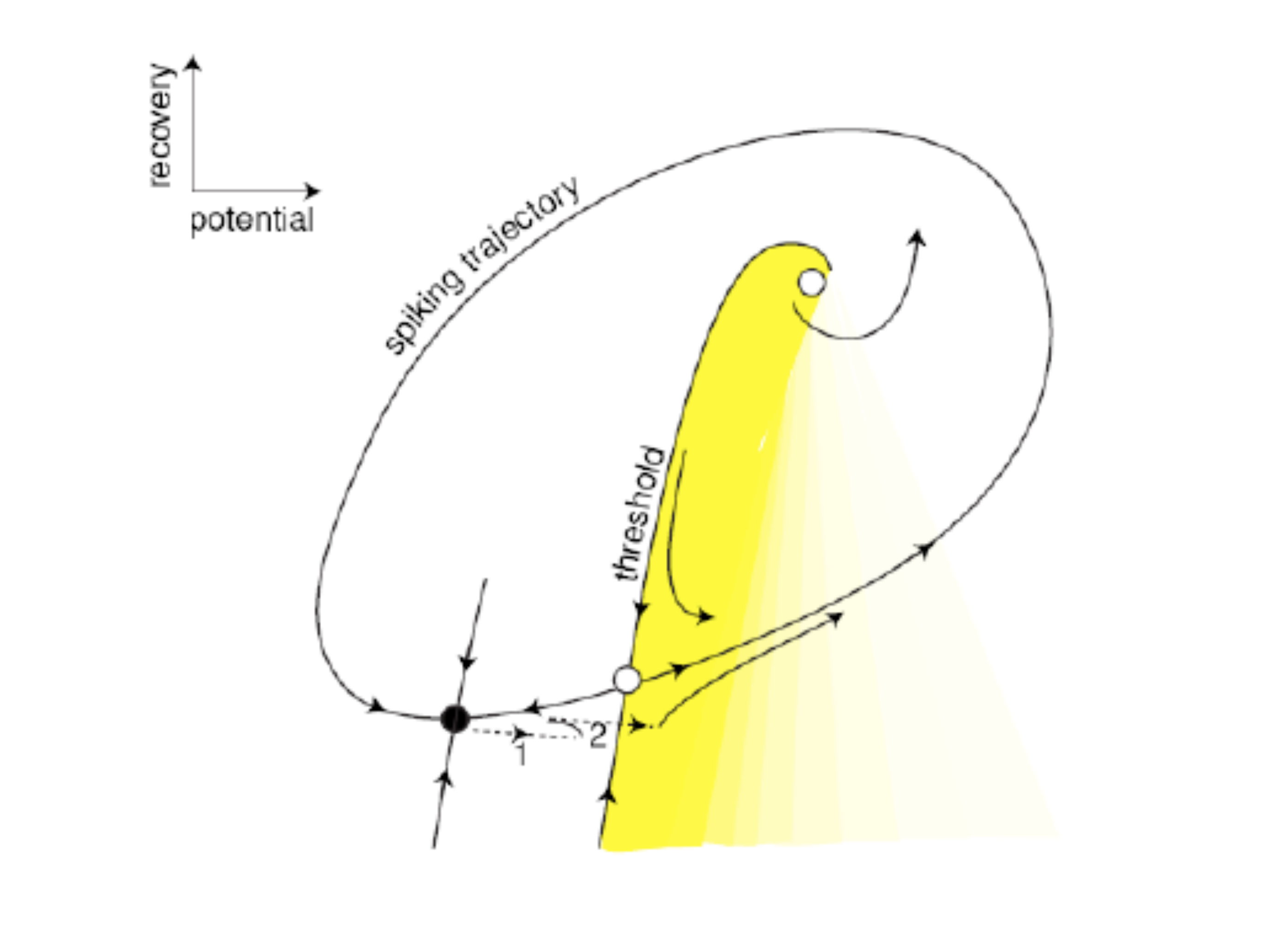
Bifurcations

- Saddle-node



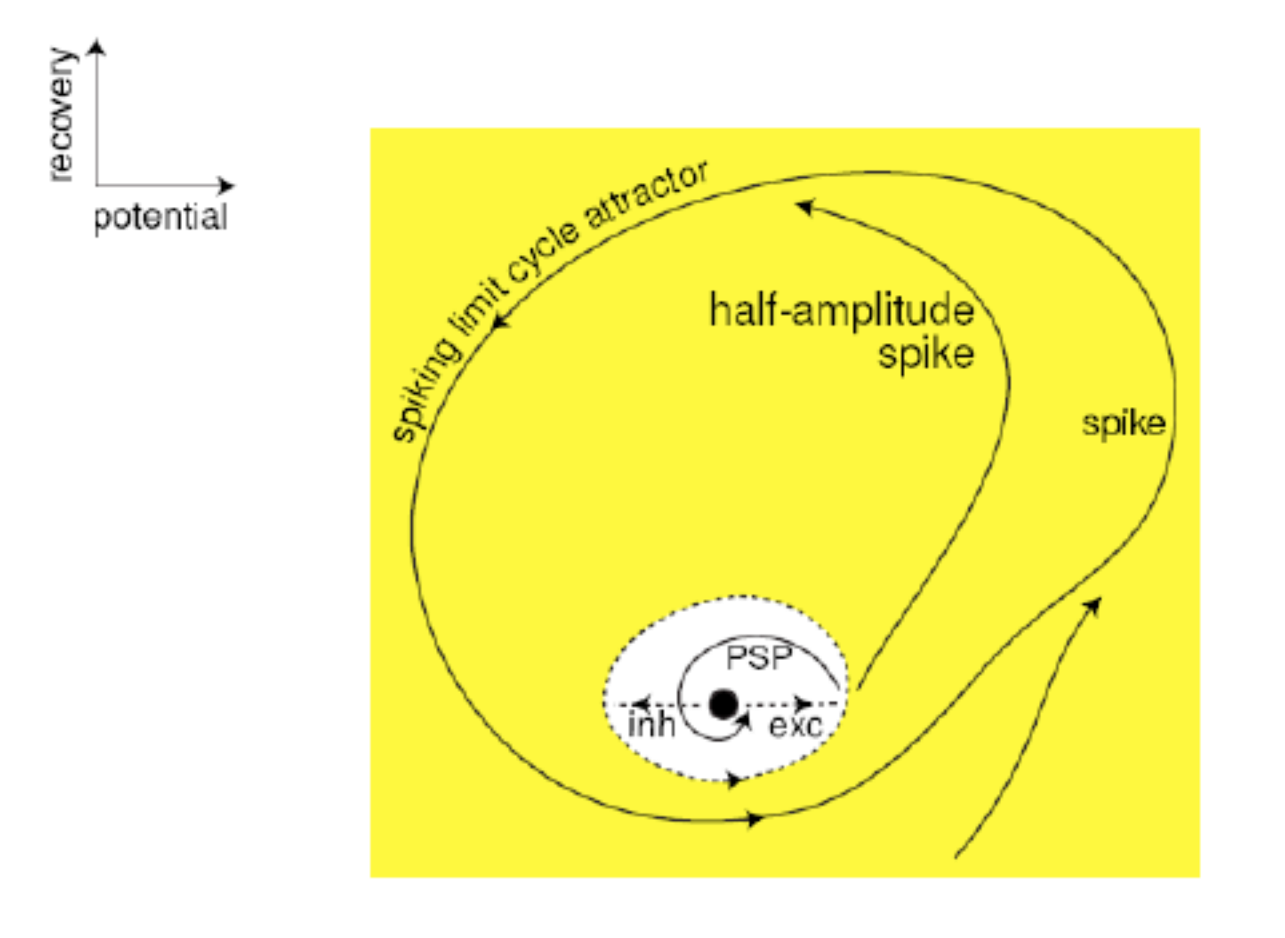
Bifurcations

- Saddle-node on an invariant circle (SNIC)



Bifurcations

- Subcritical Hopf (Andronov-Hopf)



Bifurcations

- Supercritical Hopf (Andronov-Hopf)