Introduction to Computational Neuroscience

Biol698
Math635
Biol498
Math430

How to solve ordinary differential equations (ODEs)

- First order ODEs
- Linear ODEs
- Logistic growth ODE
- Threshold ODE
- Logistic growth with a threshold

Bibliography:

"Elementary Differential Equations and Boundary Value Problems" – W. E. Boyce & R. C. DiPrima (J. Wiley & Sons, Inc, 2001).

Material was taken from this book

First order ODEs

$$\frac{dy}{dt} = f(t, y)$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our object is to determine whether such functions exist and, if so, to develop methods for finding them.

First order ODEs

- Variable coefficients (general case)

$$\frac{dy}{dt} + p(t)y = g(t)$$

where p and g are function of the independent variable t

- Constant coefficients

$$\frac{dy}{dt} = -ay + b$$

where a and b are constants

First order ODEs

$$\frac{dy}{dt} + 2y = 3$$

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = 3\mu(t)$$

μ(t) yet undetermined

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y \quad \Rightarrow \quad \frac{d\mu(t)}{dt} = 2\mu(t)$$

First order ODEs

$$\frac{d\mu(t)/dt}{\mu(t)} = 2$$

$$\frac{d}{dt} \ln|\mu(t)| = 2$$

$$\ln|\mu(t)| = 2t + C$$

$$\mu(t) = ce^{2t}$$

μ(t): integrating factor

C: arbitrary constant of integration

First order ODEs

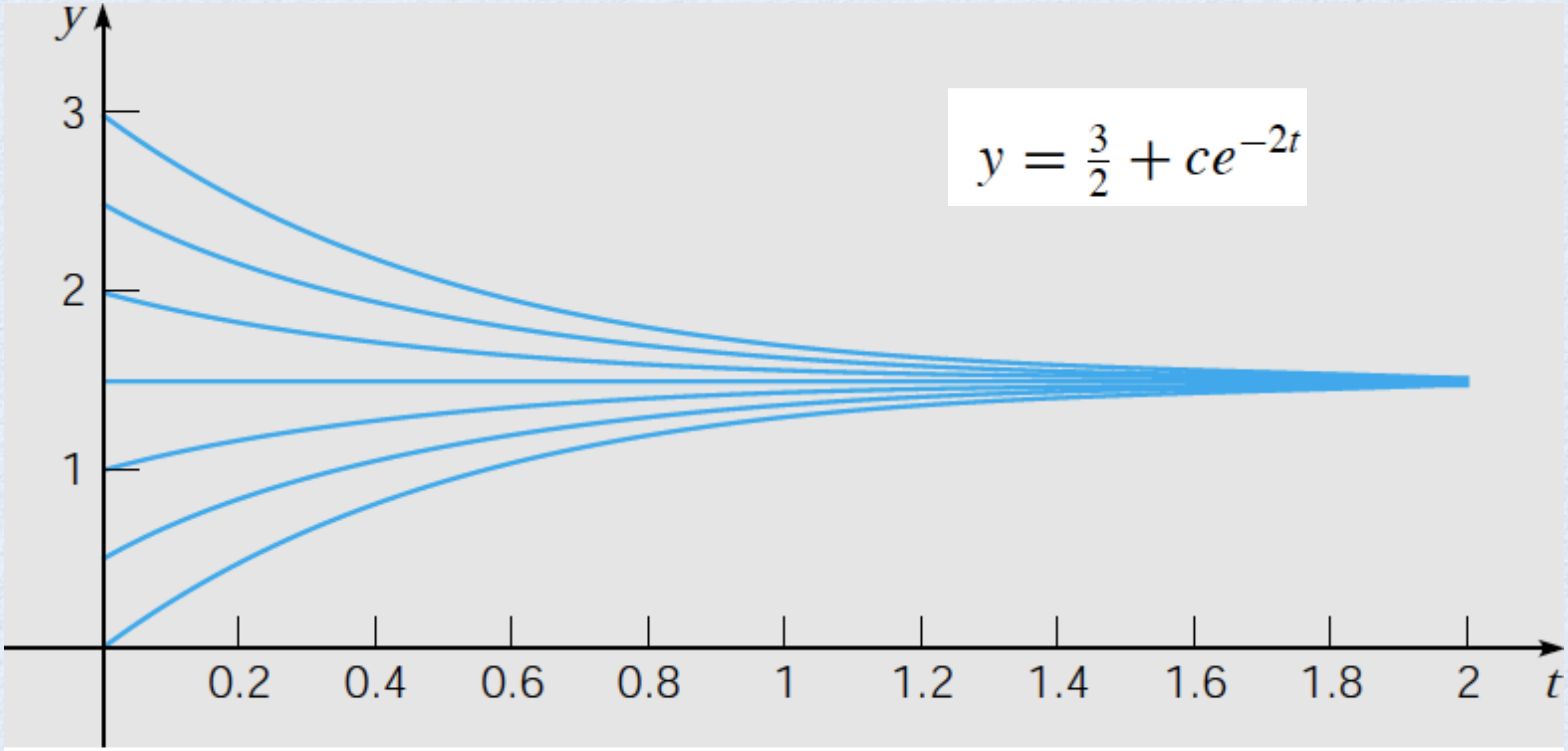
$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = 3e^{2t}$$

$$\frac{d}{dt}(e^{2t}y) = 3e^{2t}$$

$$e^{2t}y = \tfrac{3}{2}e^{2t} + c$$

$$y = \tfrac{3}{2} + ce^{-2t}$$

First order ODEs



First order ODEs

$$\frac{dy}{dt} + ay = b$$

$$\mu(t) = e^{at}$$

First order ODEs

$$\frac{dy}{dt} + ay = g(t)$$

$$\mu(t) = e^{at}$$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

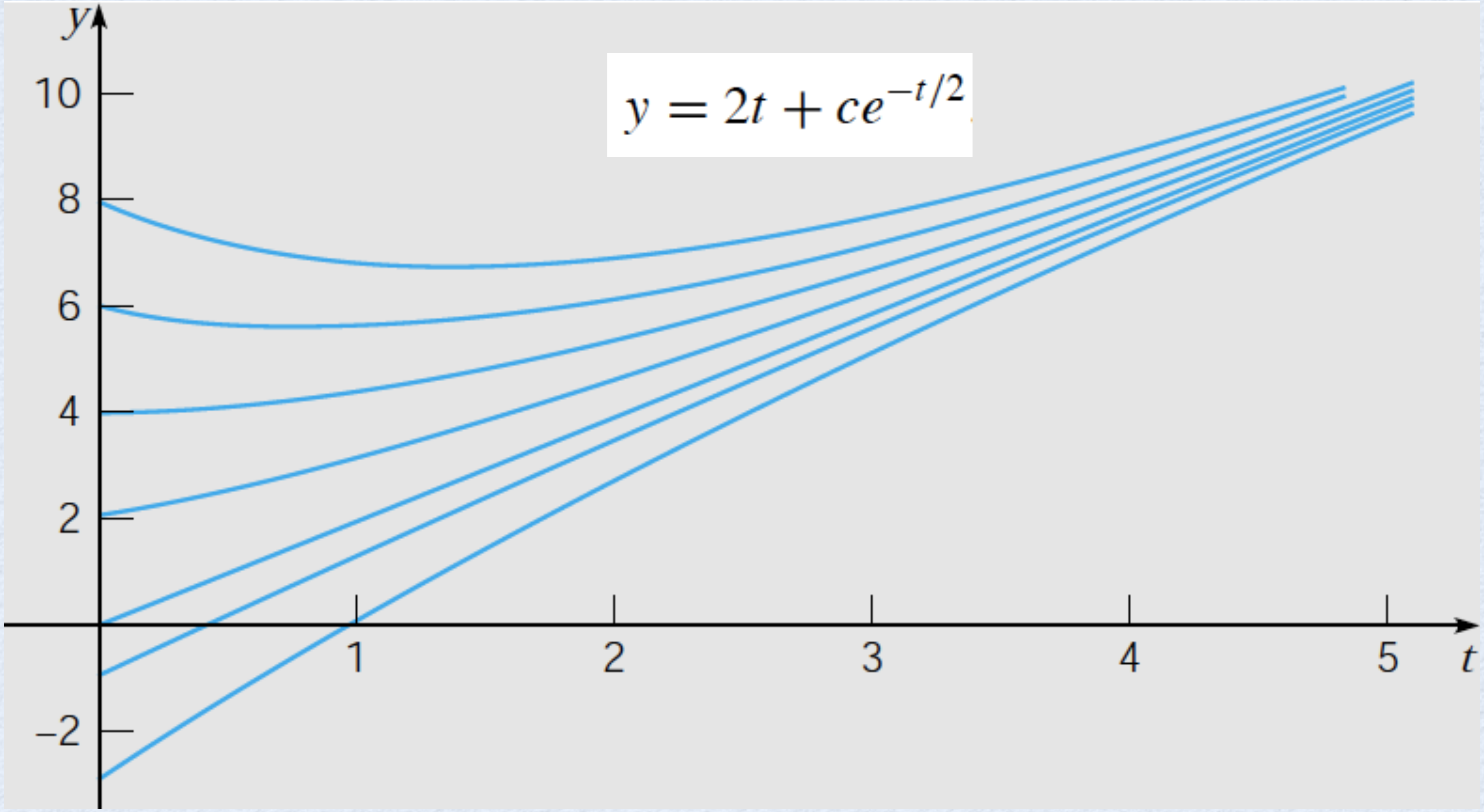
$$\frac{d}{dt}(e^{at}y) = e^{at}g(t)$$

$$e^{at}y = \int e^{as}g(s)\,ds + c$$

$$y = e^{-at}\int e^{as}g(s)\,ds + ce^{-at}$$

First order ODEs

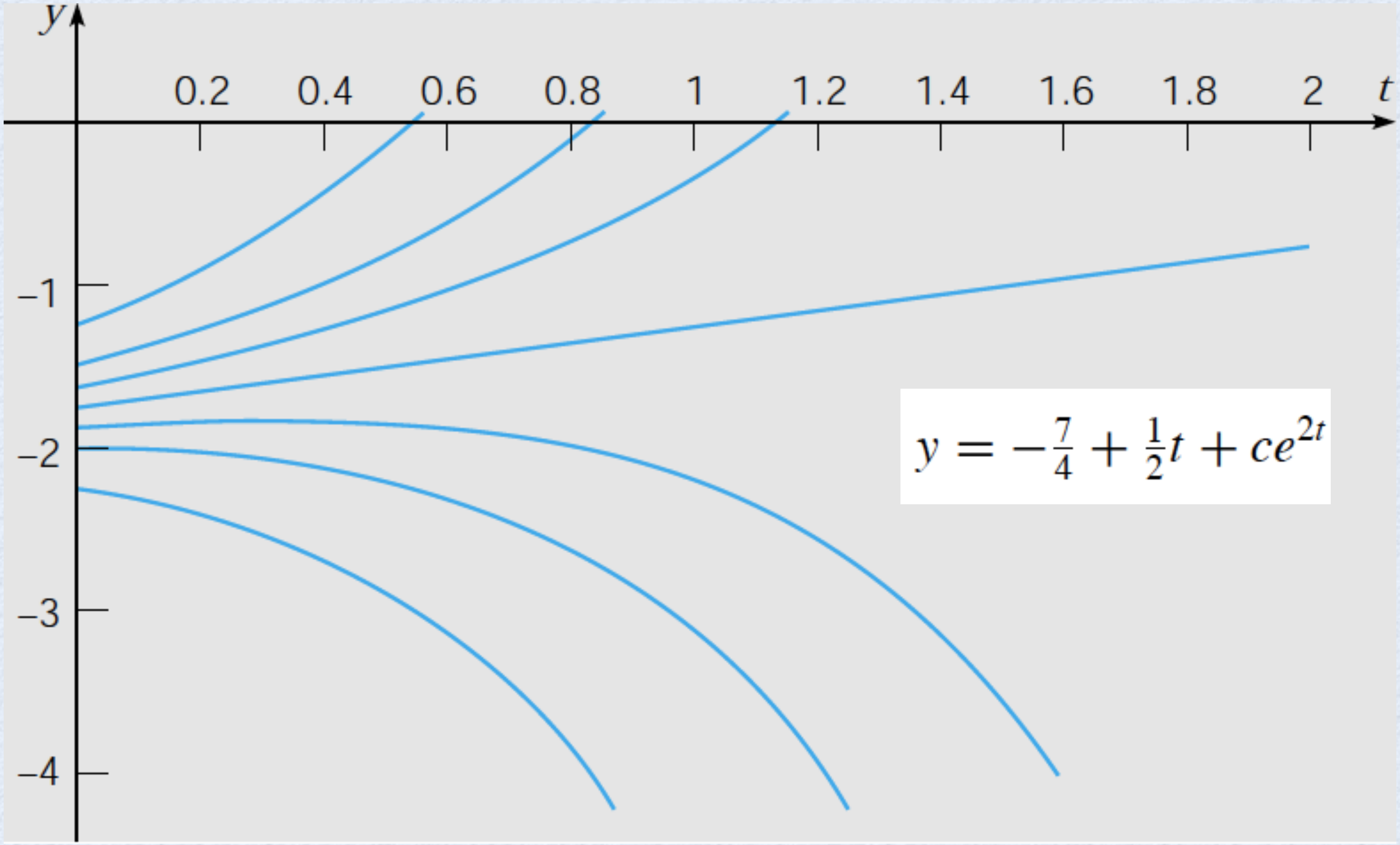
$$\frac{dy}{dt} + \tfrac{1}{2}y = 2 + t$$



$$y = 2t + ce^{-t/2}$$

First order ODEs

$$\frac{dy}{dt} - 2y = 4 - t$$



$$y = -\tfrac{7}{4} + \tfrac{1}{2}t + ce^{2t}$$

First order ODEs

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

$\mu(t)$: integrating factor, yet undetermined

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

First order ODEs

We assume (temporarily) that $\mu(t) > 0$

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t)$$

$$\ln \mu(t) = \int p(t)\,dt + k$$

k: arbitrary constant of integration (we choose k=0)

$$\mu(t) = \exp \int p(t)\,dt$$

First order ODEs

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

$$\mu(t)y = \int \mu(s)g(s)\,ds + c$$

c: arbitrary constant of integration

$$y = \frac{\int \mu(s)g(s)\,ds + c}{\mu(t)}$$

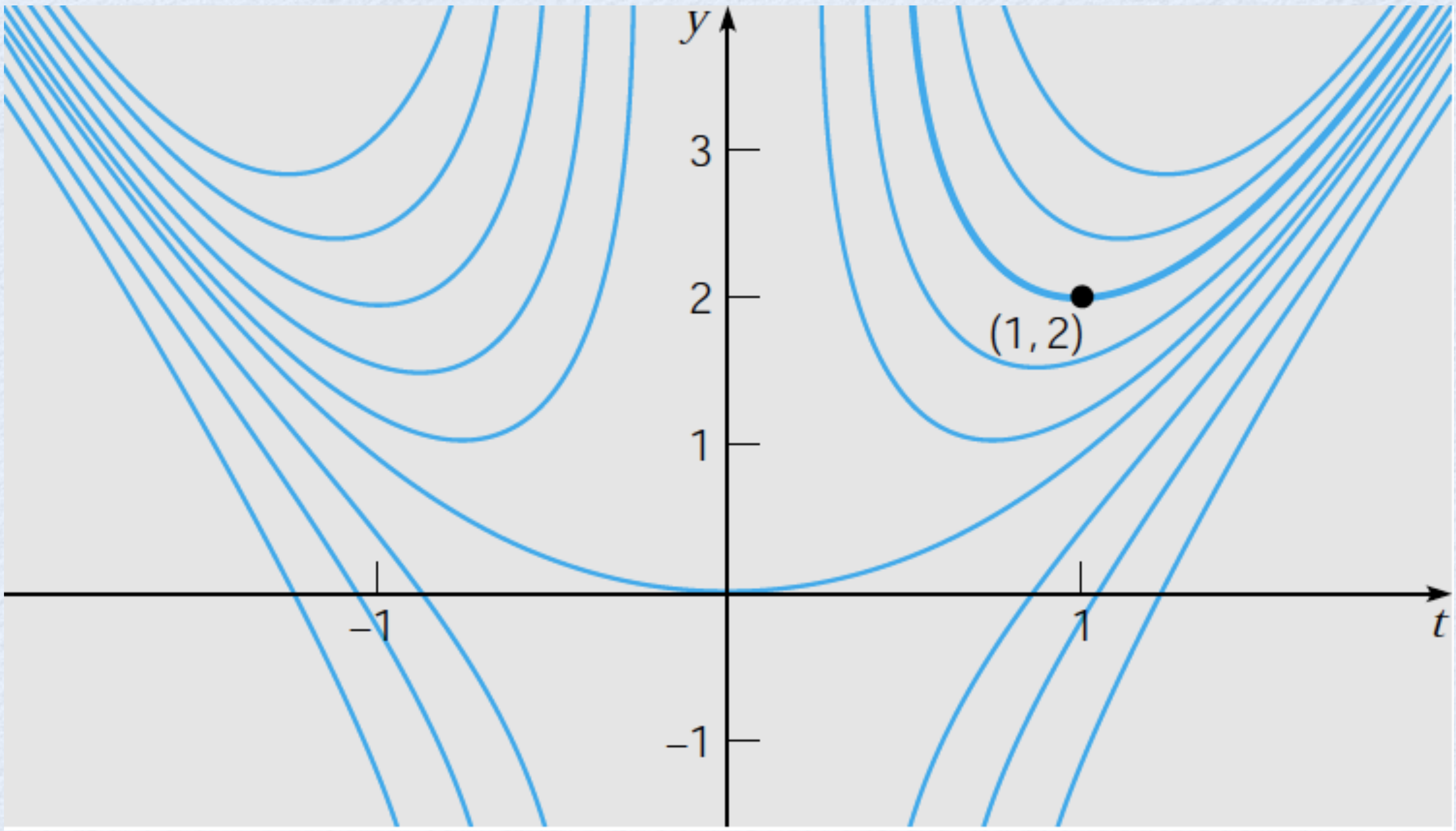
First order ODEs

$$ty' + 2y = 4t^2,$$
$$y(1) = 2.$$

$$\mu(t) = \exp \int \frac{2}{t}\, dt = e^{2 \ln |t|} = t^2$$

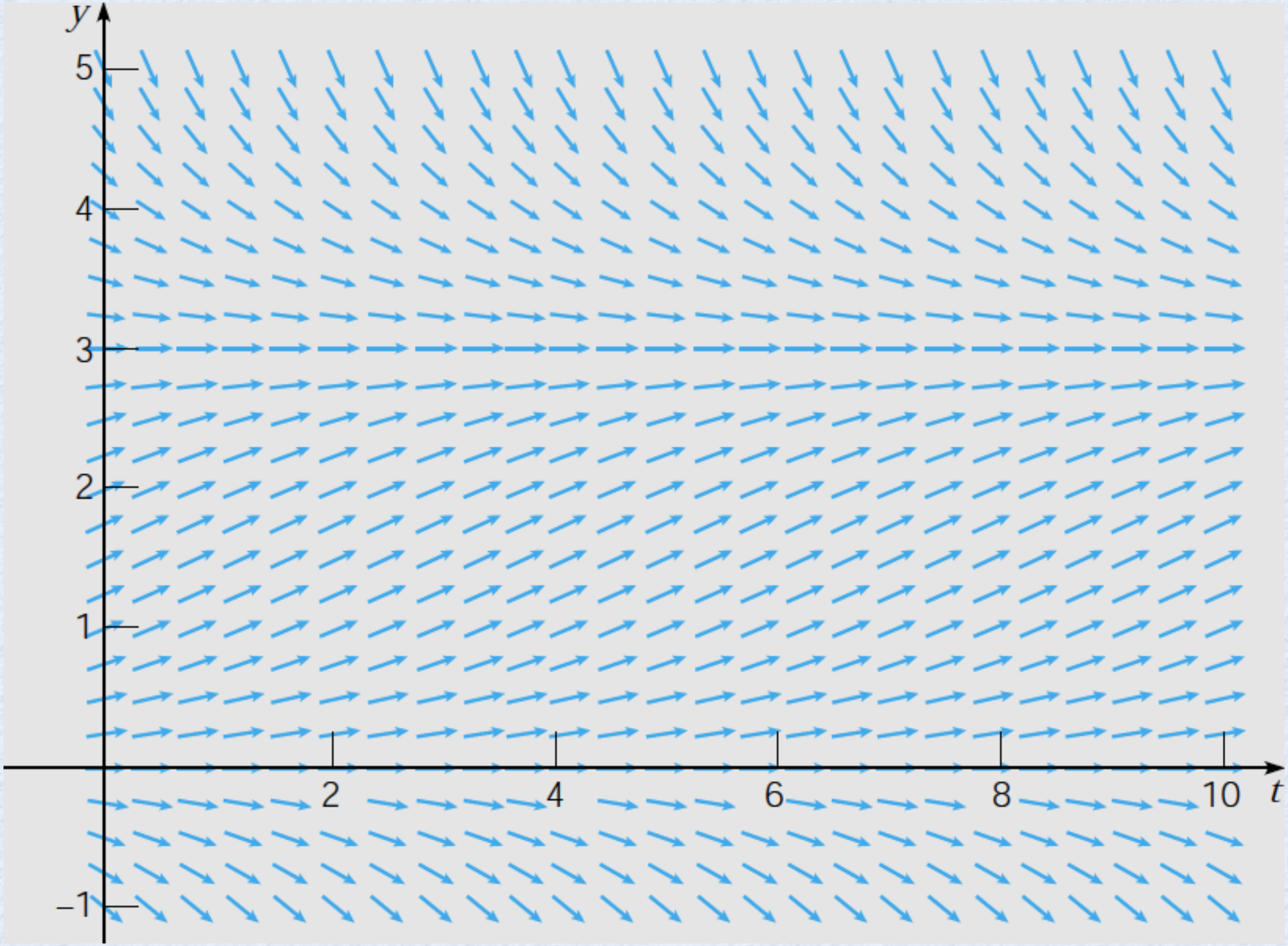
$$y = t^2 + \frac{c}{t^2}$$

First order ODEs



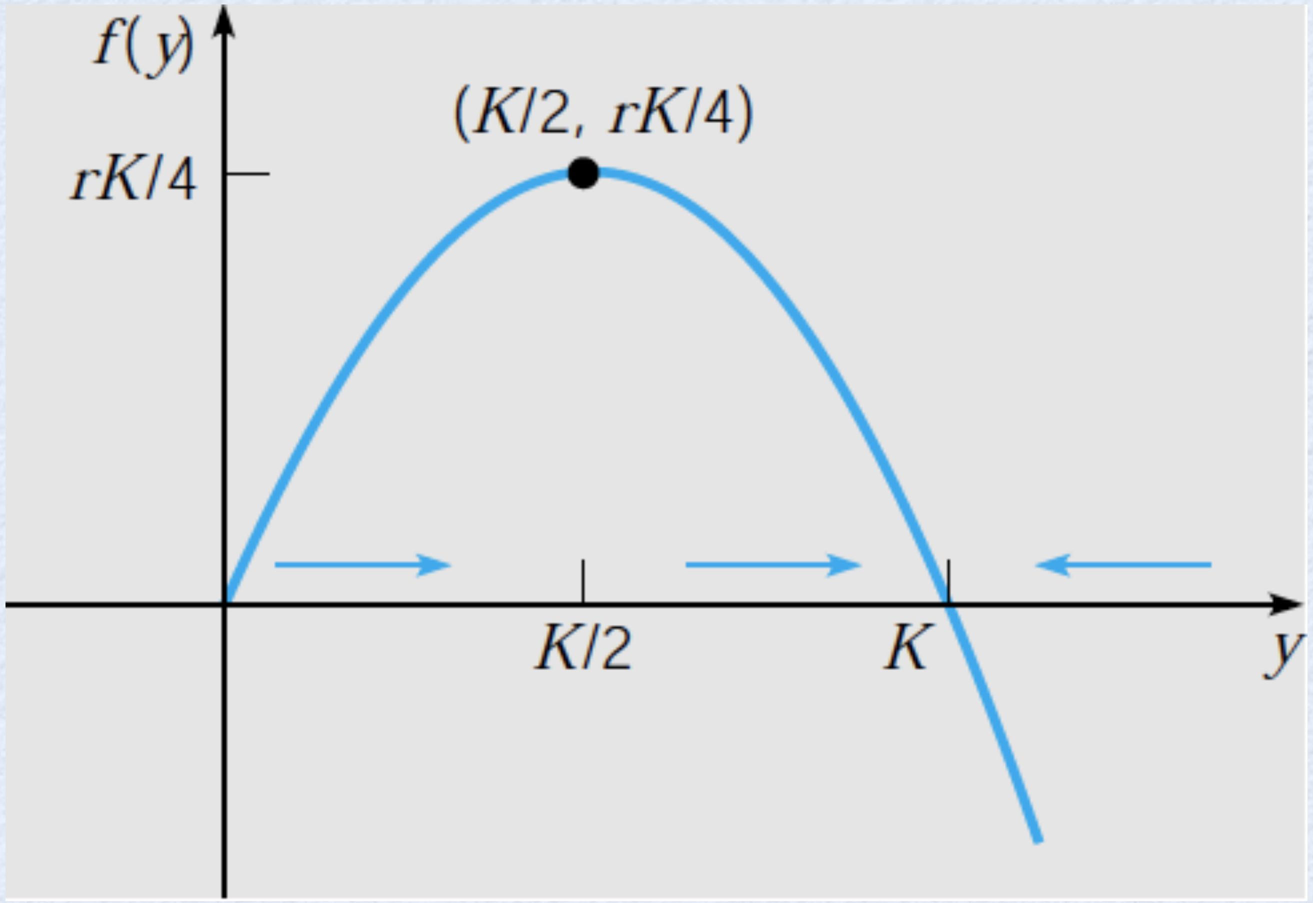
$$y = t^2 + \frac{c}{t^2}$$

Logistic equation



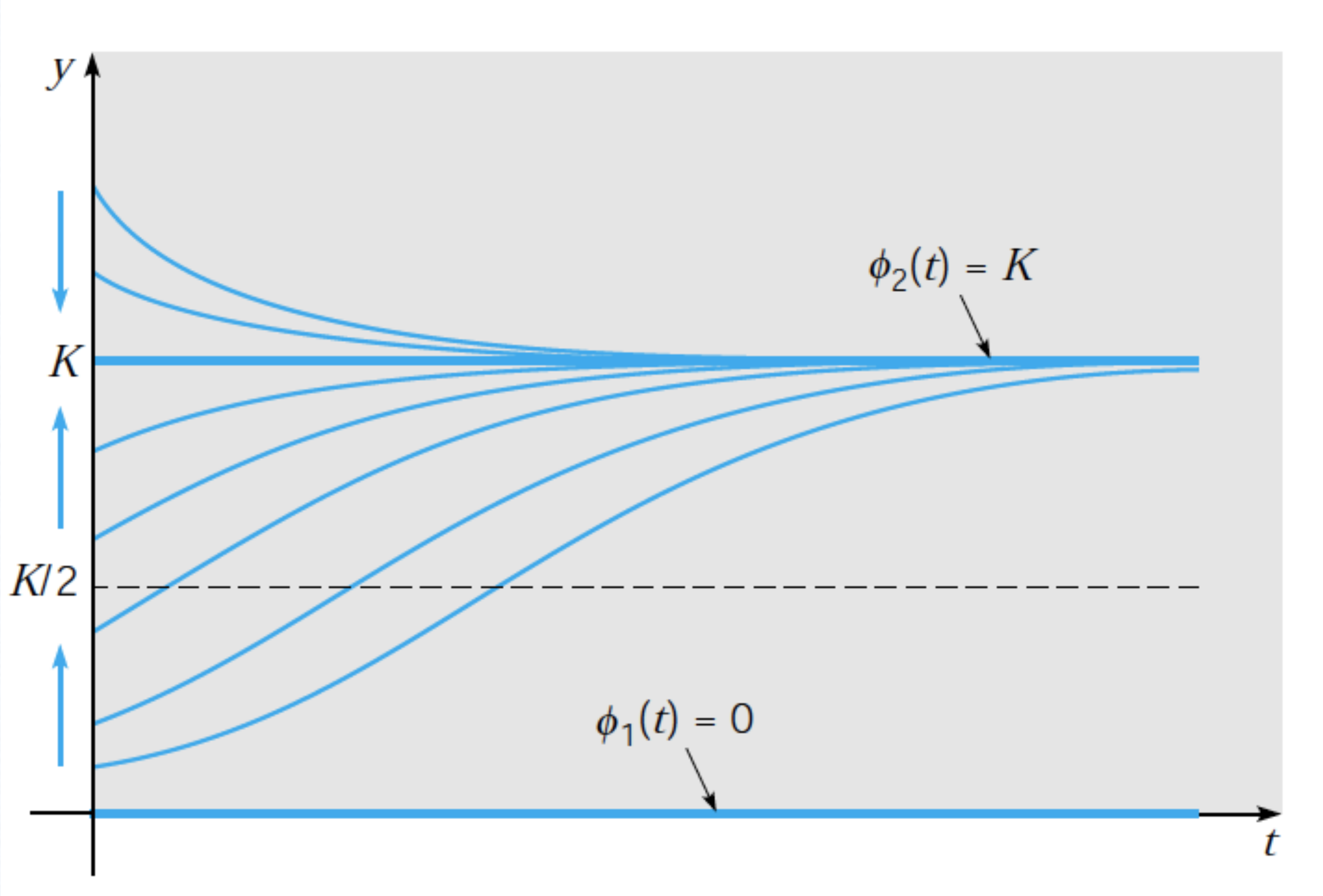
$r = 1/2$ and $K = 3$

Logistic equation



f(y) vs. y for $dy/dt = r\,(1 - y/K)\,y$

Logistic equation



$dy/dt = r\,(1 - y/K)\,y$

Logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

$$y(0) = y_0$$

$$\frac{dy}{(1 - y/K)y} = r\,dt$$

$y \neq 0$ $y \neq K$

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right)dy = r\,dt$$

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + c$$

Logistic equation

$$\frac{y}{1-(y/K)} = Ce^{rt}$$

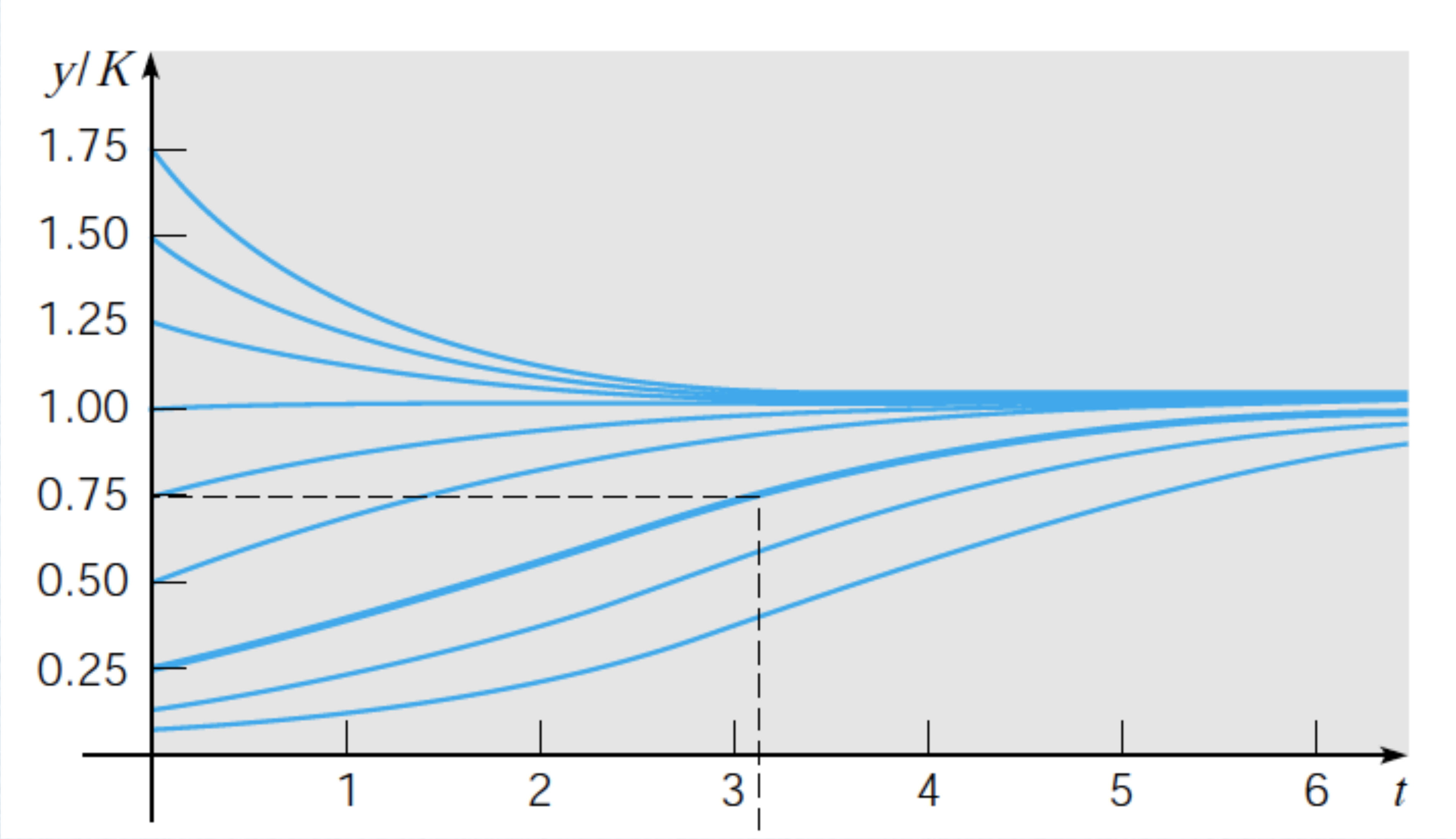
$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

$$\lim_{t\to\infty} y(t) = y_0 K / y_0 = K$$

y = K is an asymptotically stable solution

Logistic equation

$$\frac{y}{K} = \frac{y_0/K}{(y_0/K) + [1 - (y_0/K)]e^{-rt}}$$



Threshold equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$

- r : intrinsic growth rate ($r > 0$)
- T: critical ($T > 0$)

Equilibrium solutions: $y = 0$ & $y = T$

$r(1 - y/T)\,y = 0$ $dy/dt = 0$

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

$$y(0) = y_0$$

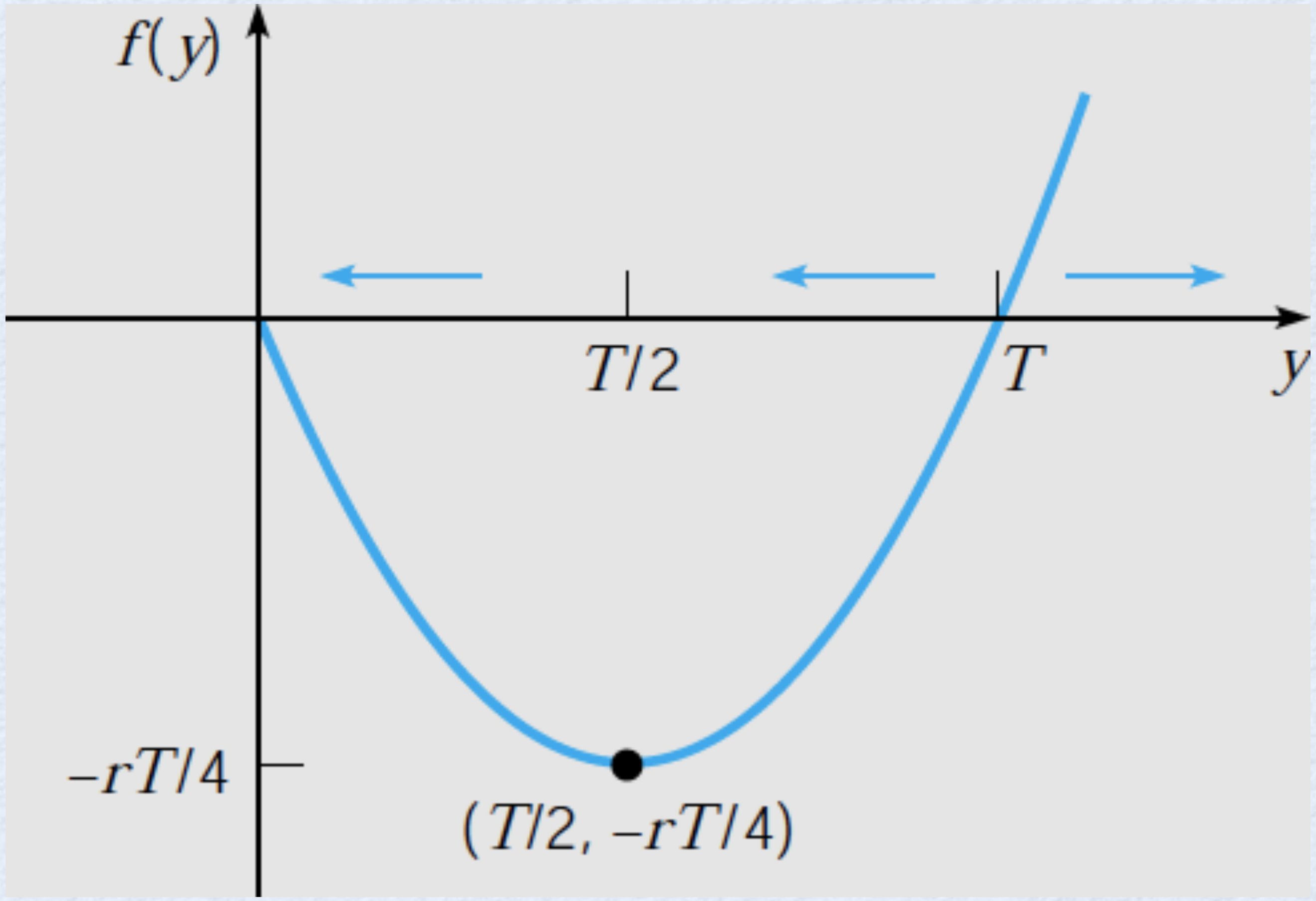
Threshold equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y$$

$$y(0) = y_0$$

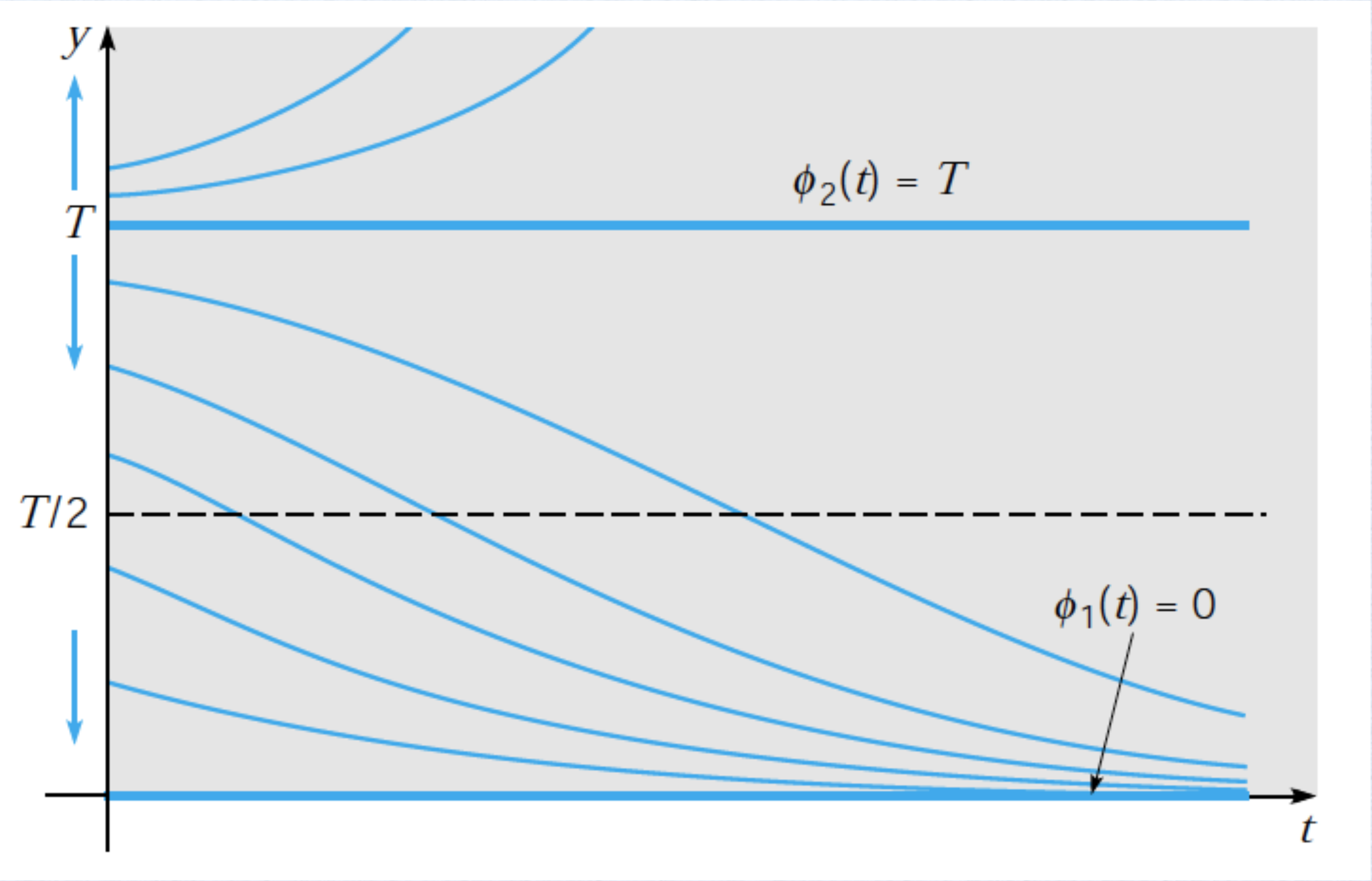
$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

Logistic equation



f(y) vs. y for $dy/dt = -r(1 - y/T)\,y$

Logistic equation



$dy/dt = -\,r\,(1 - y/T)\,y$

Logistic growth with a threshold

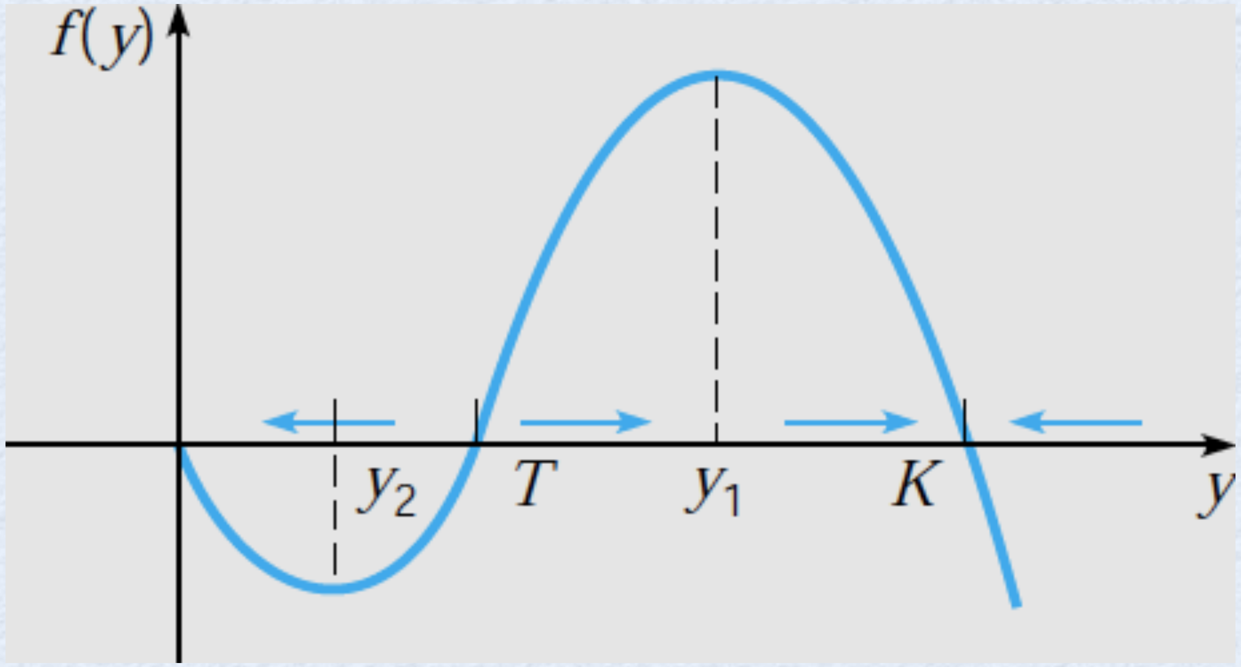
$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

- r : intrinsic growth rate ($r > 0$)
- K: carrying capacity or saturation level ($K > 0$)
- T: critical ($0 < T < K$)

Equilibrium solutions: $y = 0$, $y = T$ & $y = K$

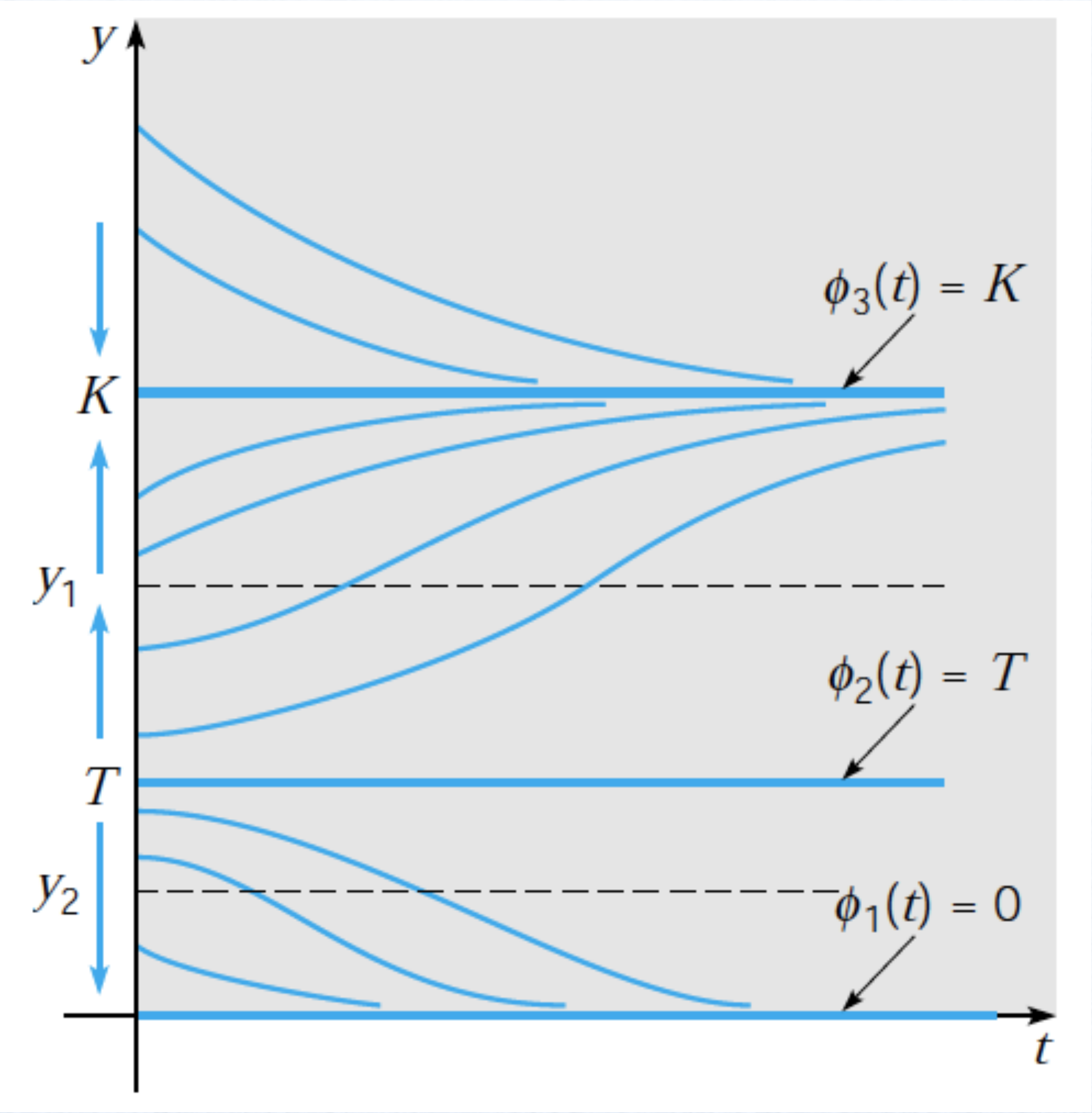
$r\,(1 - y/T)\,(1 - y/K)\,y = 0$ $\qquad$ $dy/dt = 0$

Logistic equation



f(y) vs. y   for $dy/dt = - r\,(1 - y/T)\,(1 - y/K)\,y$

Logistic equation



$dy/dt = - r\,(1 - y/T)\,(1 - y/K)\,y$