Introduction to Computational Neuroscience

Biol 698
Math 635
Biol 498
Math 430

Bibliography:

"Mathematical Foundations of Neuroscience", by G. B. Ermentrout & D. H. Terman - Springer (2010), 1st edition. ISBN 978-0-387-87707-5

* "Foundations of Cellular Neurophysiology", by Daniel Johnston and Samuel M.-S. Wu. The MIT Press, 1995. ISBN 0-262-10053-3

* "Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting", by Eugene M. Izhikevich. The MIT Press, 2007. ISBN 0-262-09043-8

* "Biophysics of Computation - Information processing in single neurons", by Christof Koch. Oxford University Press, 1999. ISBN 0-19-510491-9

* "Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems", by Peter Dayan and Larry F. Abbott. The MIT Press,2001. ISBN 0-262-04199-5

Figures were taken from some of these books

Overview

- The Hodgkin-Huxley model (review)
- The cable equation
- Multiple compartmental approach
- B

Hodgkin-Huxley model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
\dot{n} &= \alpha_n(V)(1-n) - \beta_n(V) n \\
\dot{m} &= \alpha_m(V)(1-m) - \beta_m(V) m \\
\dot{h} &= \alpha_h(V)(1-h) - \beta_h(V) h \,,
\end{aligned}
$$

$$\alpha_n(V) = 0.01 \frac{10 - V}{\exp(\frac{10-V}{10}) - 1}$$

$$\alpha_m(V) = 0.1 \frac{25 - V}{\exp(\frac{25-V}{10}) - 1}$$

$$\alpha_h(V) = 0.07 \exp\left(\frac{-V}{20}\right)$$

$$\beta_n(V) = 0.125 \exp\left(\frac{-V}{80}\right)$$

$$\beta_m(V) = 4 \exp\left(\frac{-V}{18}\right)$$

$$\beta_h(V) = \frac{1}{\exp(\frac{30-V}{10}) + 1}$$

Hodgkin-Huxley model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
\dot{n} &= (n_\infty(V) - n)/\tau_n(V) , \\
\dot{m} &= (m_\infty(V) - m)/\tau_m(V) , \\
\dot{h} &= (h_\infty(V) - h)/\tau_h(V) ,
\end{aligned}
$$

$$
\begin{aligned}
n_\infty &= \alpha_n/(\alpha_n + \beta_n) , & \tau_n &= 1/(\alpha_n + \beta_n) , \\
m_\infty &= \alpha_m/(\alpha_m + \beta_m) , & \tau_m &= 1/(\alpha_m + \beta_m) , \\
h_\infty &= \alpha_h/(\alpha_h + \beta_h) , & \tau_h &= 1/(\alpha_h + \beta_h)
\end{aligned}
$$

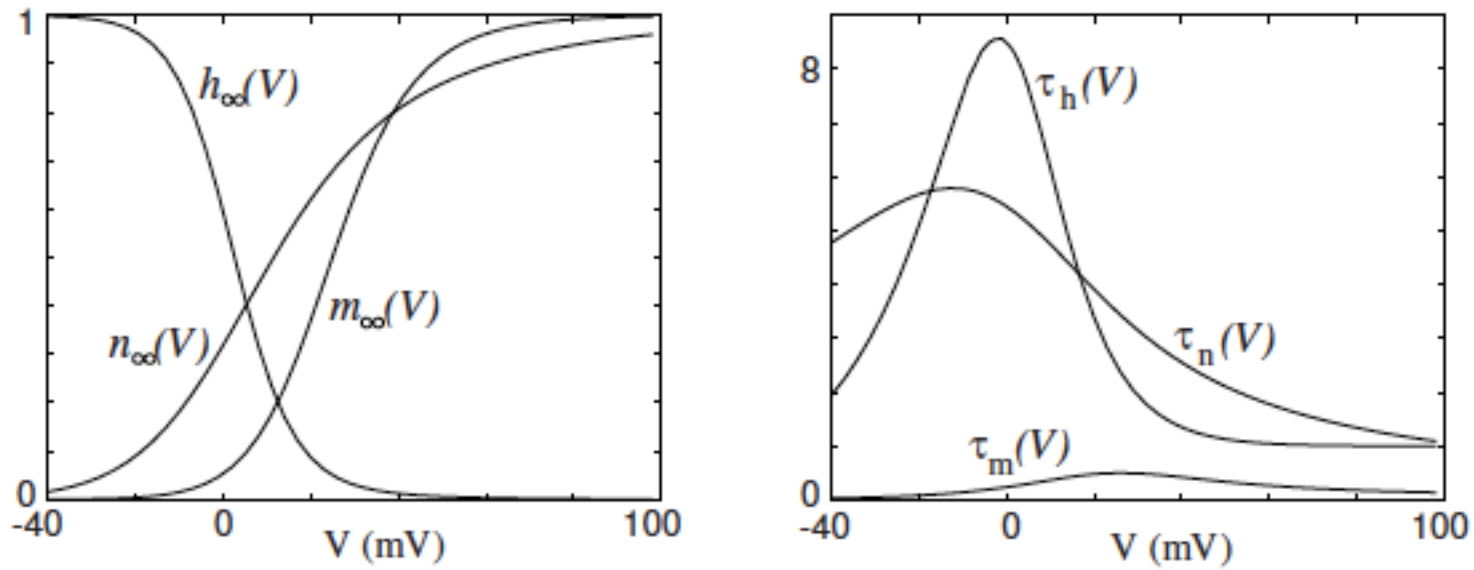


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

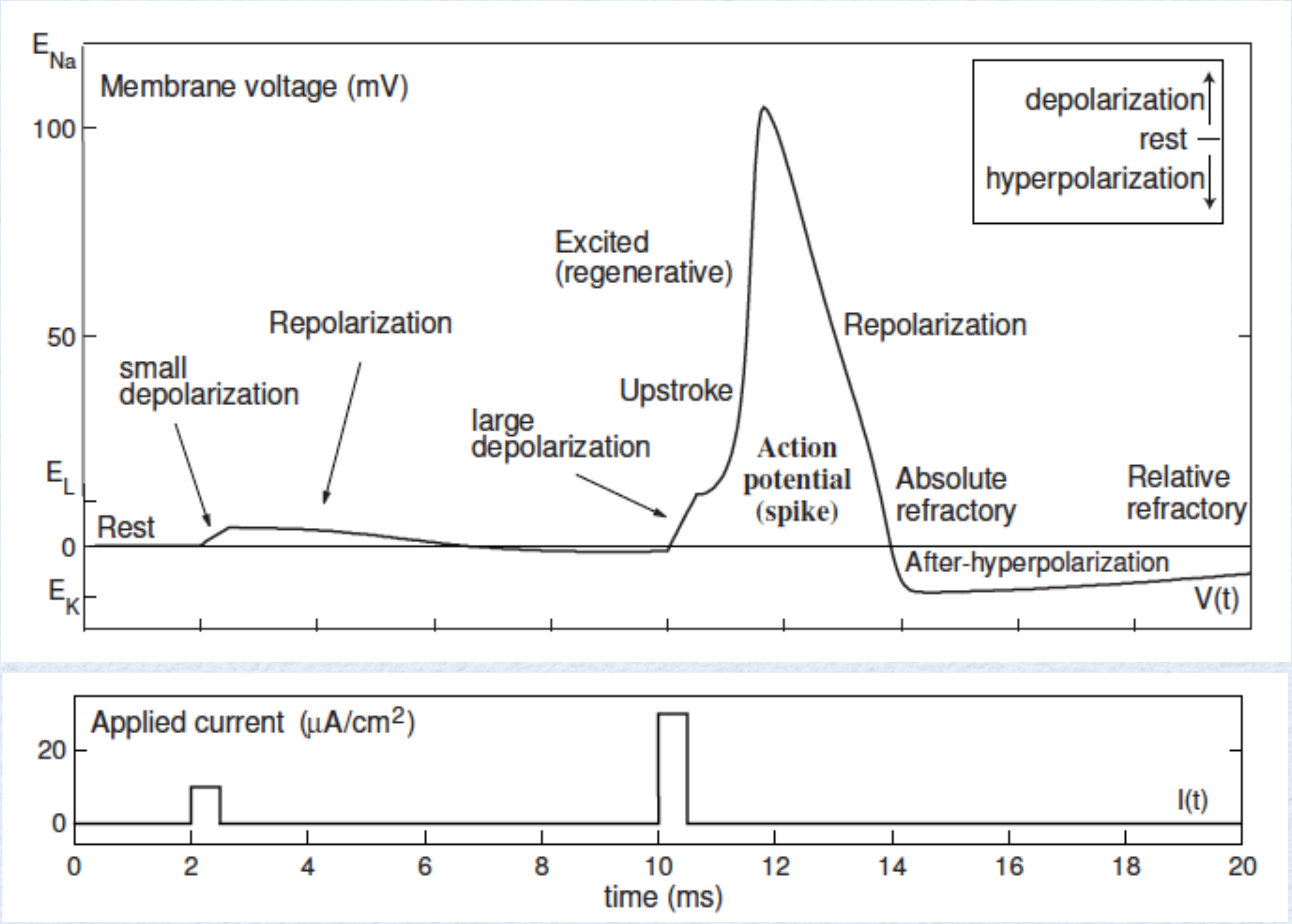
Hodgkin-Huxley model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
\dot{n} &= \alpha_n(V)(1-n) - \beta_n(V)n \\
\dot{m} &= \alpha_m(V)(1-m) - \beta_m(V)m \\
\dot{h} &= \alpha_h(V)(1-h) - \beta_h(V)h \,,
\end{aligned}
$$

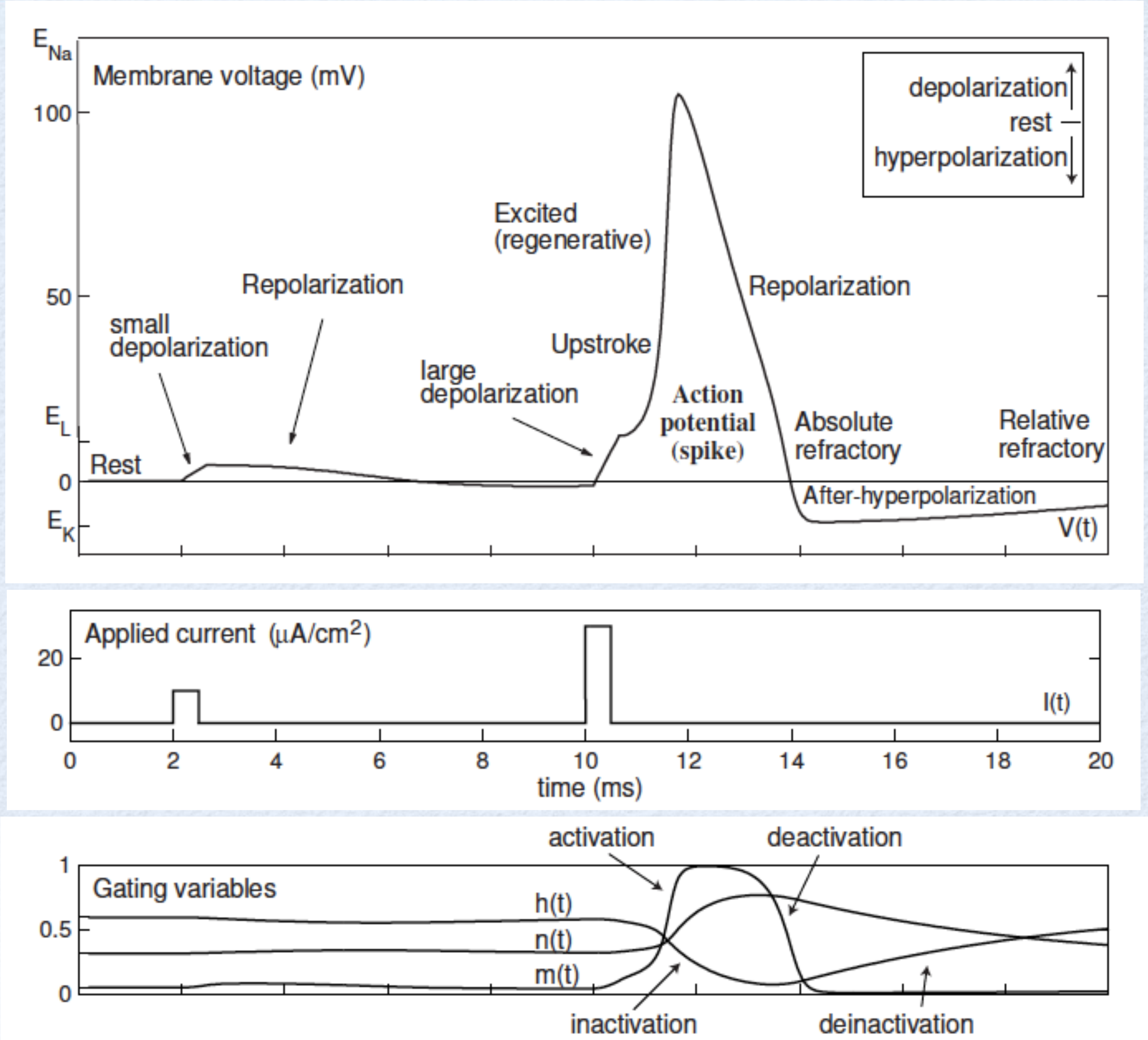
$E_{\mathrm{K}} = -12\ \mathrm{mV}$ $E_{\mathrm{Na}} = 120\ \mathrm{mV}$ $E_{\mathrm{L}} = 10.6\ \mathrm{mV}$

$\bar{g}_{\mathrm{K}} = 36\ \mathrm{mS/cm^2}$ $\bar{g}_{\mathrm{Na}} = 120\ \mathrm{mS/cm^2}$ $g_{\mathrm{L}} = 0.3\ \mathrm{mS/cm^2}$

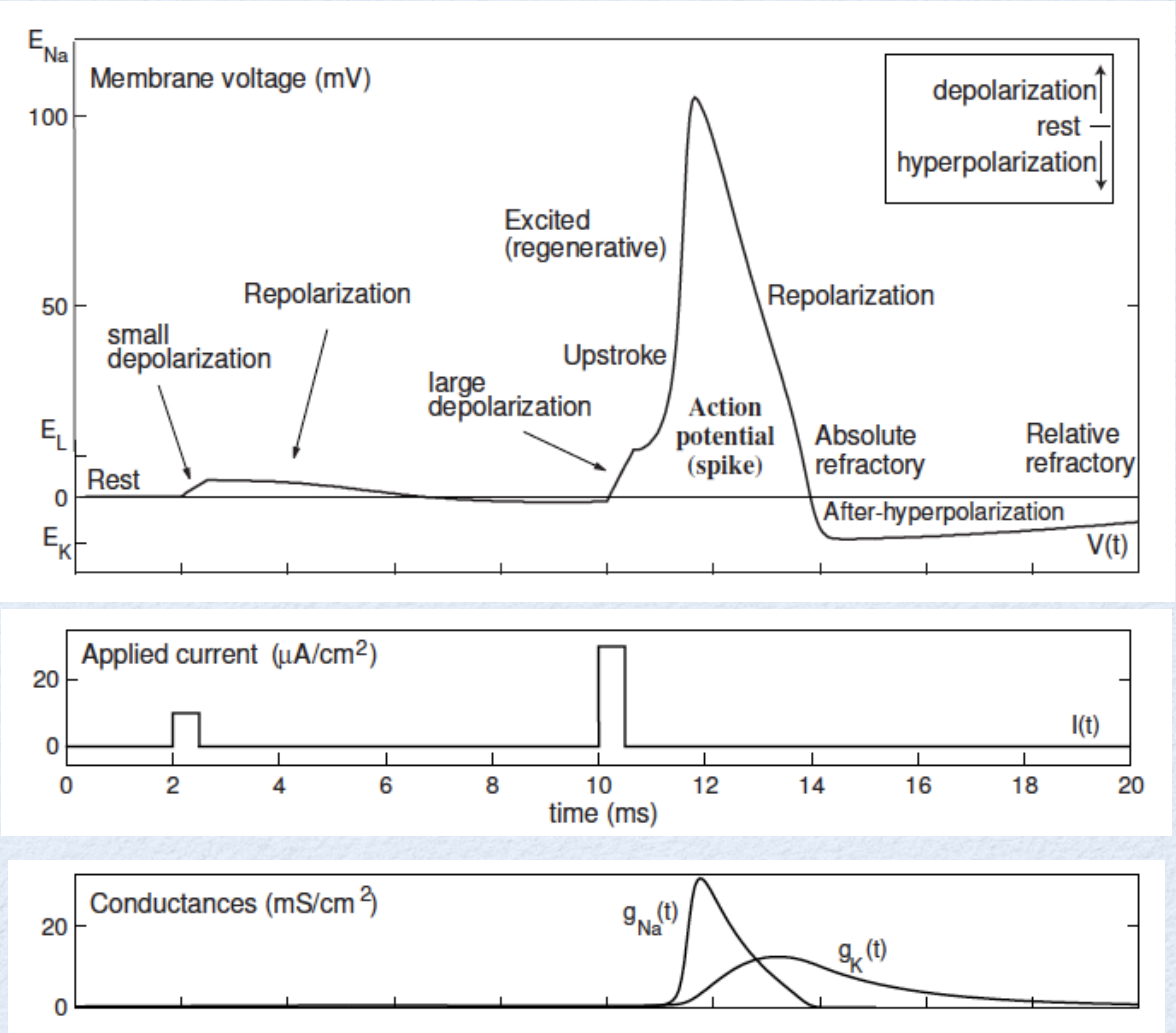
Hodgkin-Huxley model



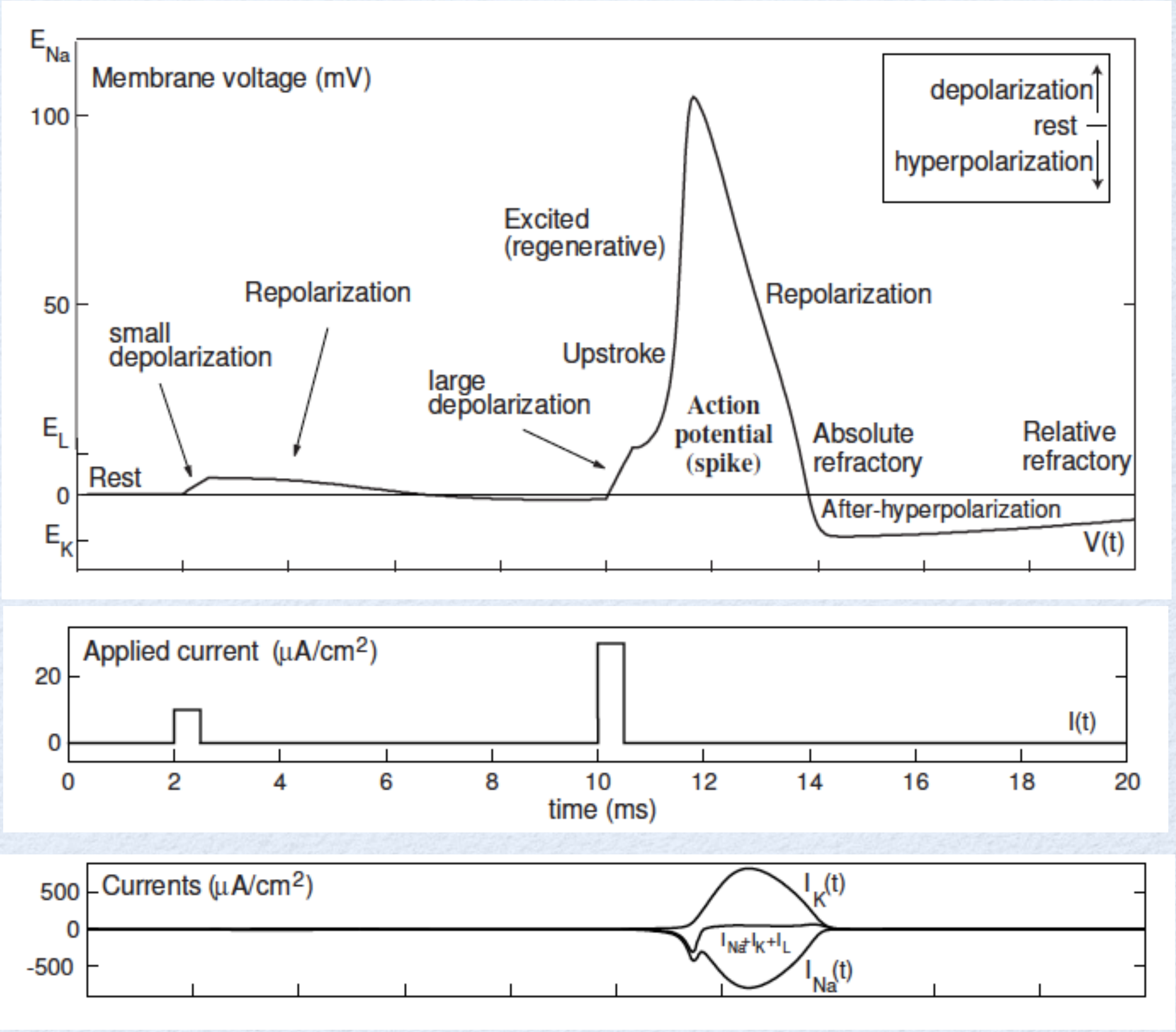
Hodgkin-Huxley model



Hodgkin-Huxley model



Hodgkin-Huxley model



Hodgkin-Huxley model

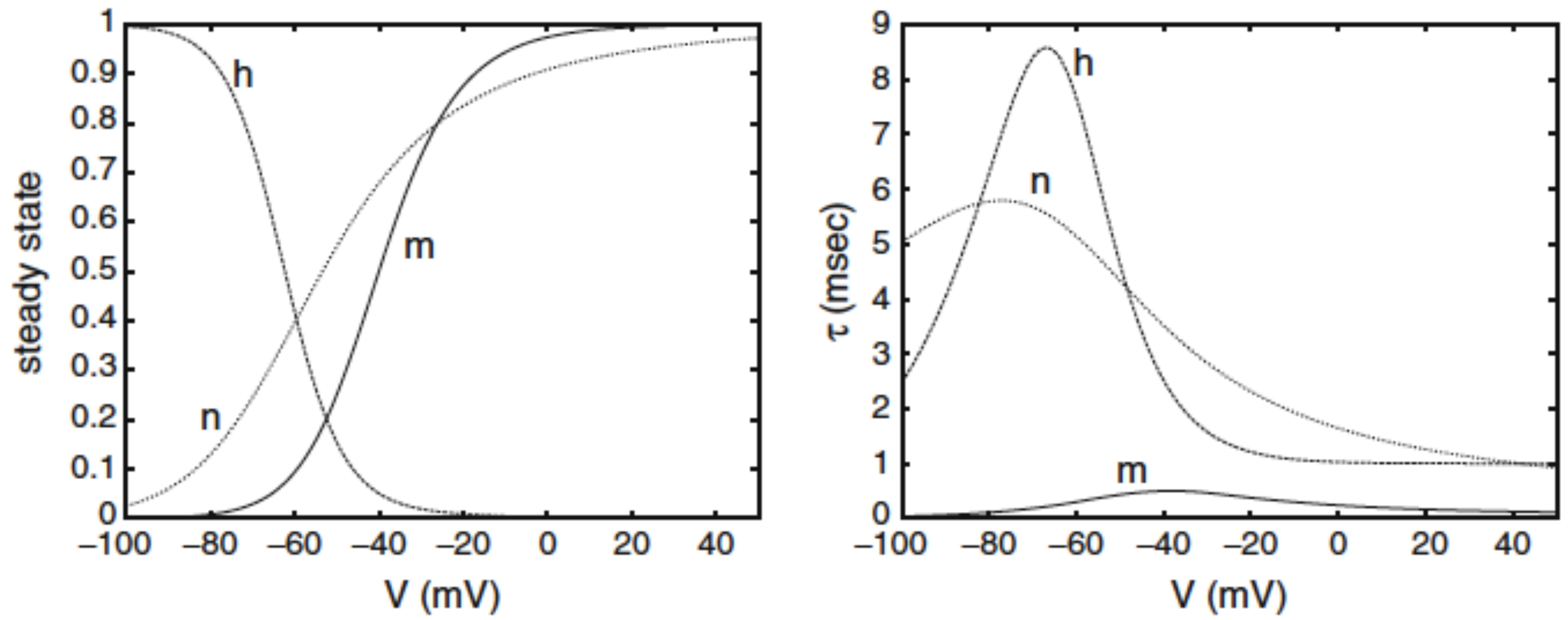


Fig. 1.11 Hodgkin–Huxley functions. *Left* the steady-state opening of the gates and *right* the time constants

Hodgkin-Huxley model

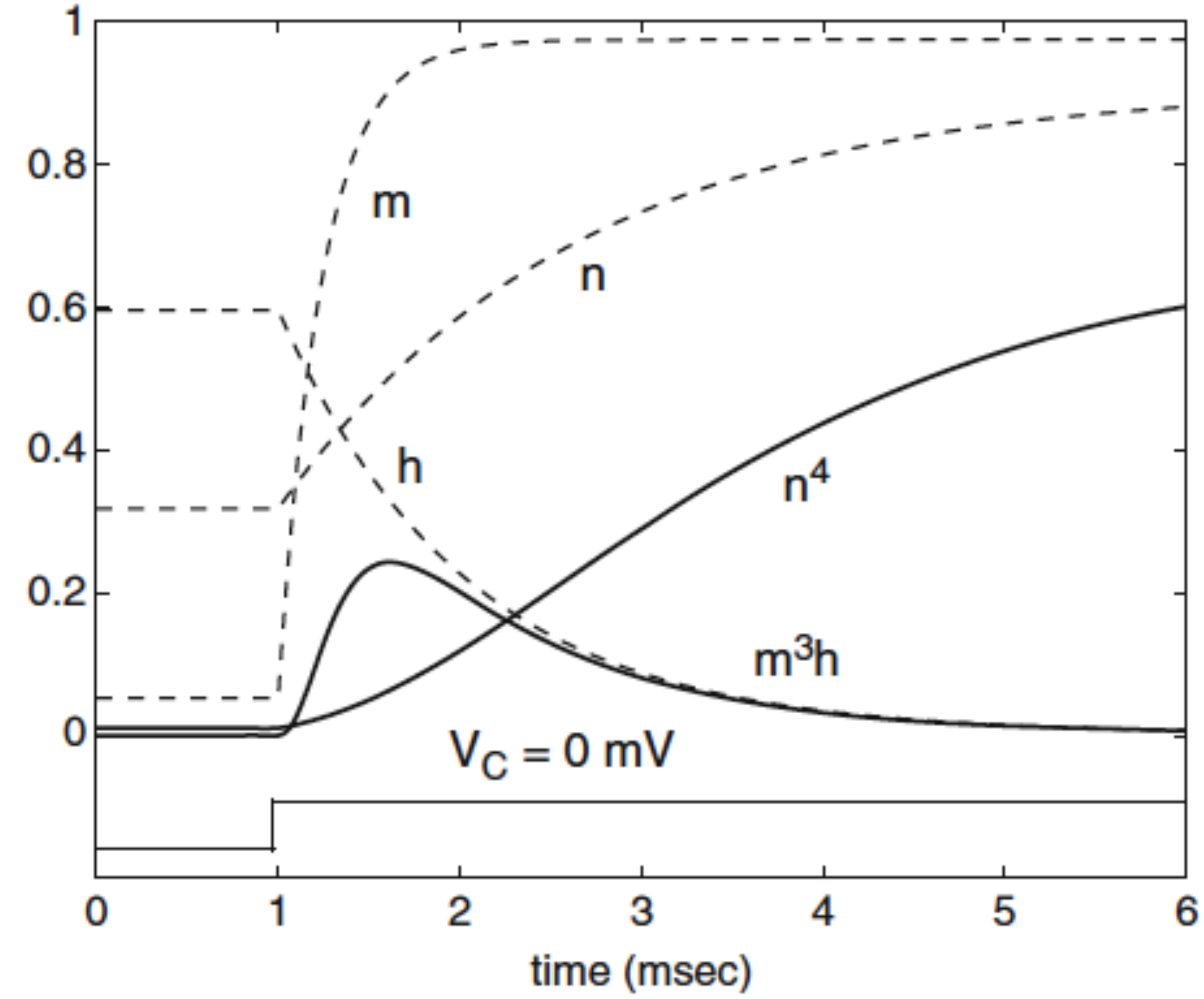


Fig. 1.12 Response of the activation and inactivation variables *m*, *h*, and *n* to a step in voltage

Hodgkin-Huxley model

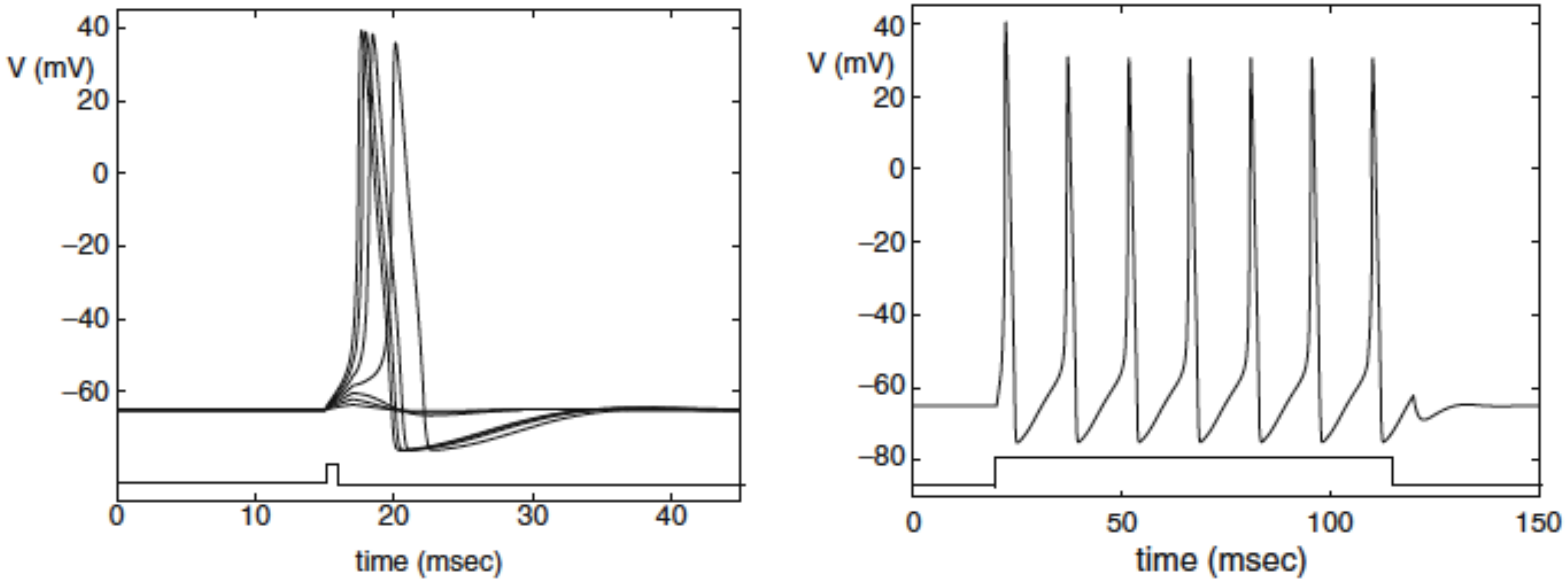


Fig. 1.13 Responses of the Hodgkin–Huxley model to applied currents. *Left* transient responses showing "all-or-none" behavior and *right* sustained periodic response

Hodgkin-Huxley model

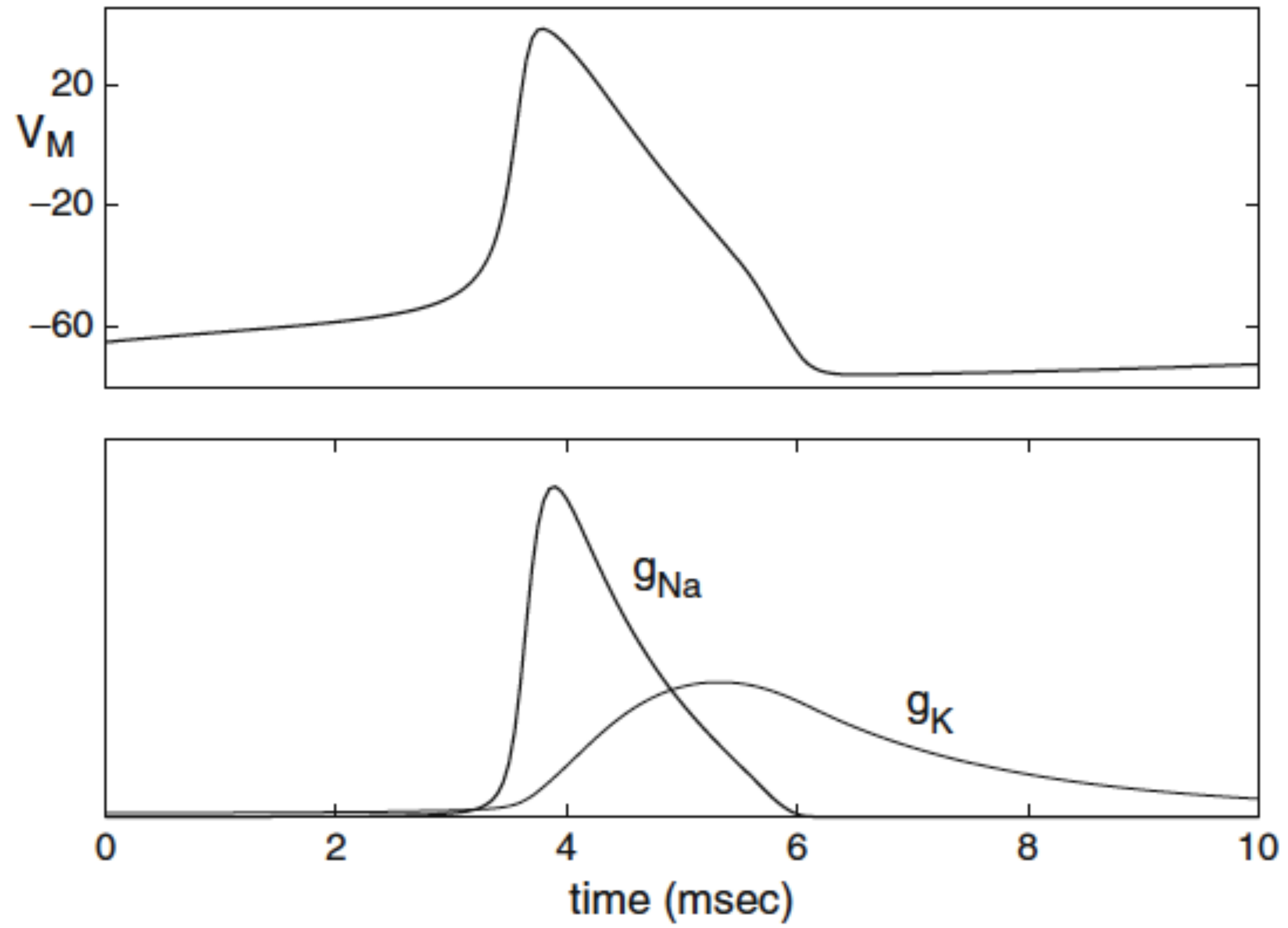
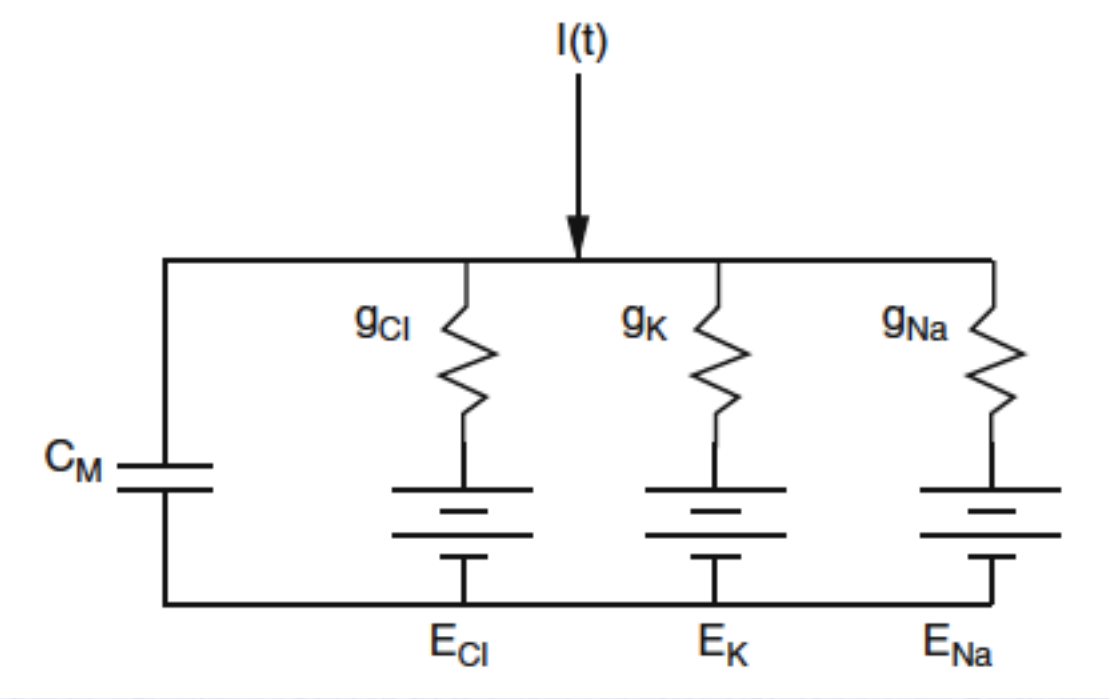


Fig. 1.14 Solution of the Hodgkin–Huxley equations showing an action potential. Also shown are the Na^+ and K^+ conductances

Hodgkin-Huxley model

Notation:

- C_M: membrane capacitance (c_M: specific membrane capacitance)
- R_M: membrane resistance (r_M: specific membrane resistance)
- i_{cap} (= $C_M\, dV_M/dt$): capacitive current per unit area
- I_{cap}: total capacitive current
- $I(t)$: source current
- i_{ion}: ionic current per unit area
- I_{ion}: Total ionic current
- A: area



$$i_{\mathrm{ion}} = -g_{\mathrm{Cl}}(V_{\mathrm{M}} - E_{\mathrm{Cl}}) - g_{\mathrm{K}}(V_{\mathrm{M}} - E_{\mathrm{K}}) - g_{\mathrm{Na}}(V_{\mathrm{M}} - E_{\mathrm{Na}})$$

$$c_{\mathrm{M}}\frac{\mathrm{d}V_{\mathrm{M}}}{\mathrm{d}t} = -g_{\mathrm{Cl}}(V_{\mathrm{M}} - E_{\mathrm{Cl}}) - g_{\mathrm{K}}(V_{\mathrm{M}} - E_{\mathrm{K}}) - g_{\mathrm{Na}}(V_{\mathrm{M}} - E_{\mathrm{Na}}) + I(t)/A$$

Hodgkin-Huxley model

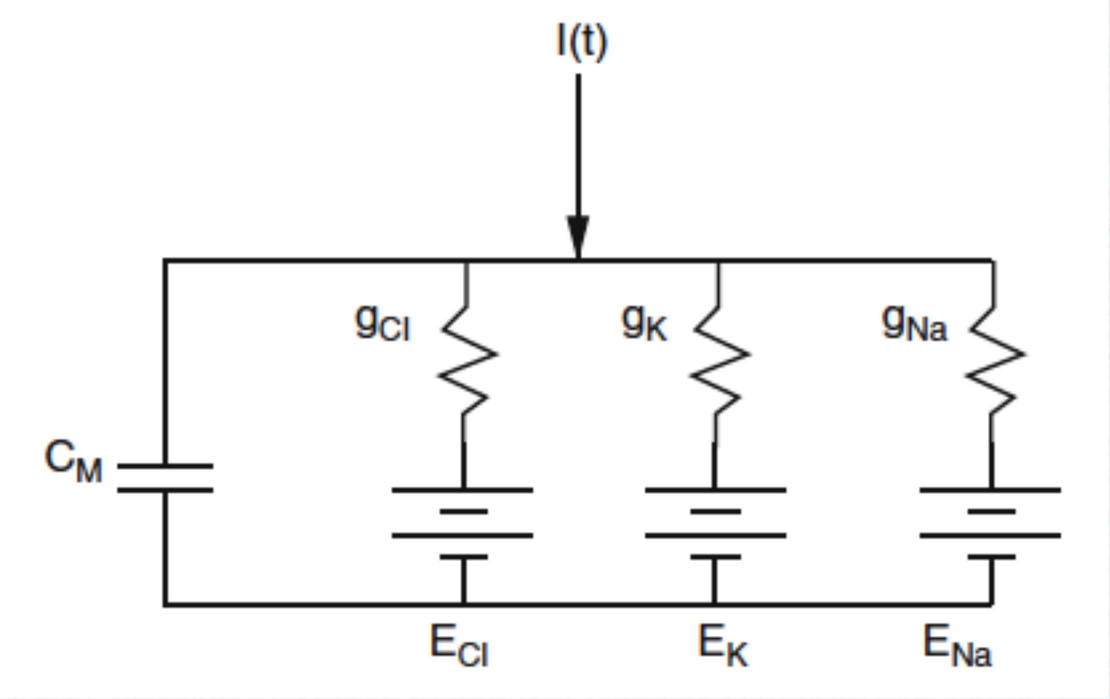
$$c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + I(t)/A$$

$$c_M \frac{dV_M}{dt} = -\frac{(V_M - E_R)}{r_M} + I(t)/A$$

$$E_R = (g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na})r_M$$

$$r_M = \frac{1}{g_{Cl} + g_K + g_{Na}}$$

$$V_{ss} = \frac{g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na} + I/A}{g_{Cl} + g_k + g_{Na}}$$



Spherical cell - passive membrane

Assumptions:

- Membrane is passive
- Spherical cell of radius ρ
- $E_r = 0$: V_M measures the deviation of the membrane potential from rest

Notation:

- $I_M(t)$: current flowing across a unit area of the membrane (injected current distributes uniformly across the surface)
- τ_M: time constant

$$c_M \frac{dV_M}{dt} = -\frac{V_M}{r_M} + I_M(t)$$

$$I_M(t) = \frac{I(t)}{4\pi\rho^2} = \begin{cases} \frac{I_0}{4\pi\rho^2} & \text{if } 0 < t < T \\ 0 & \text{otherwise.} \end{cases}$$

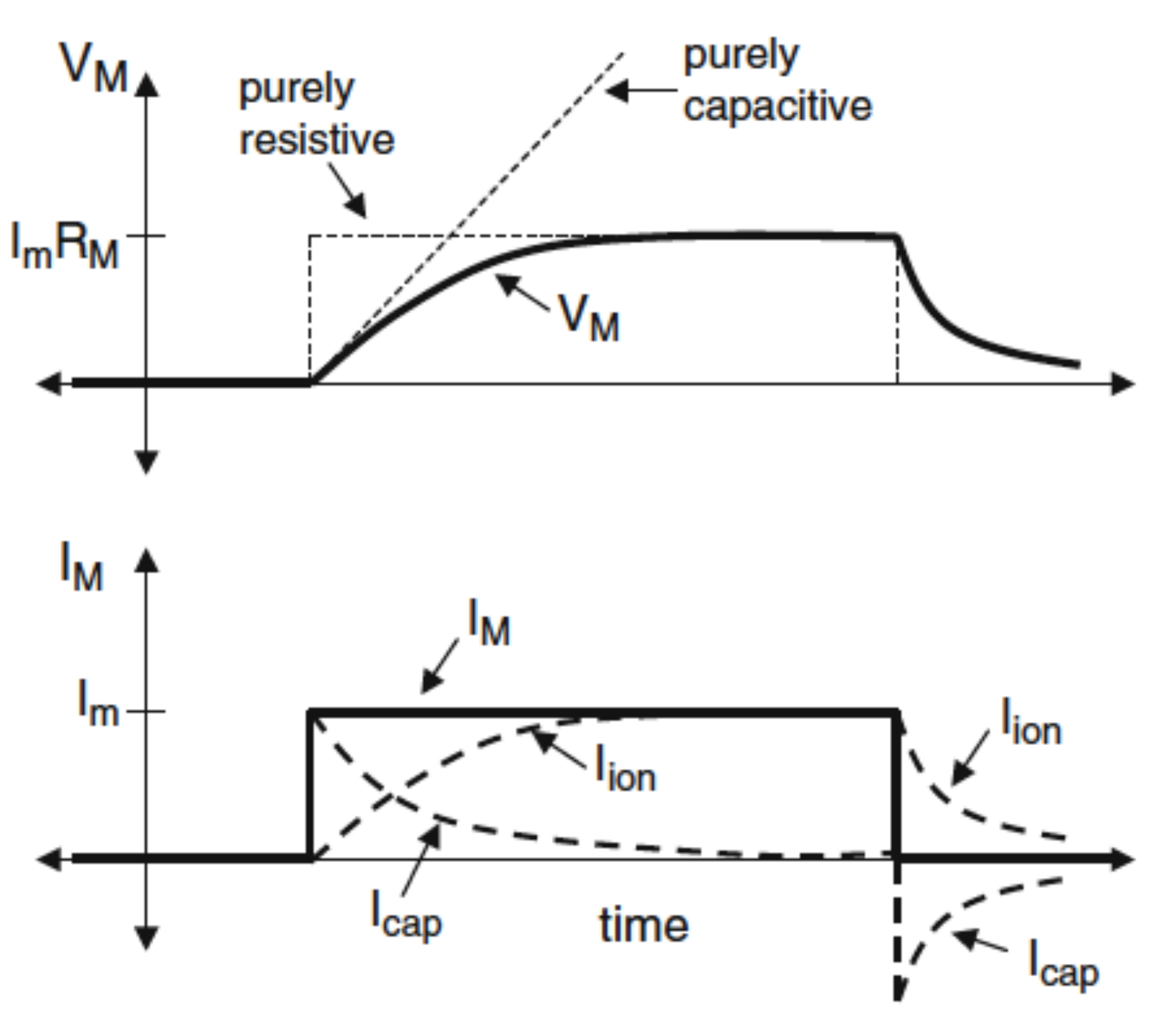
Spherical cell - passive membrane

$$c_M \frac{dV_M}{dt} = -\frac{V_M}{r_M} + I_M(t)$$

$$I_M(t) = \frac{I(t)}{4\pi\rho^2} = \begin{cases} \frac{I_0}{4\pi\rho^2} & \text{if } 0 < t < T \\ 0 & \text{otherwise.} \end{cases}$$

- $V_M(t) = \frac{r_M I_0}{4\pi\rho^2}\left(1 - e^{-\frac{t}{\tau_M}}\right) \quad \text{for } 0 < t < T$
- $V_M(t) = V_M(T) e^{-\frac{t}{\tau_M}} \quad \text{for } t > T$

Fig. 1.4 The change of membrane potential in response to a step of current. The membrane potential is shown with a *solid line*. The *dashed lines* show the time courses of the purely capacitive and resistive elements. The *bottom panel* shows the time course of the total membrane current, the ionic current, and the capacitive current

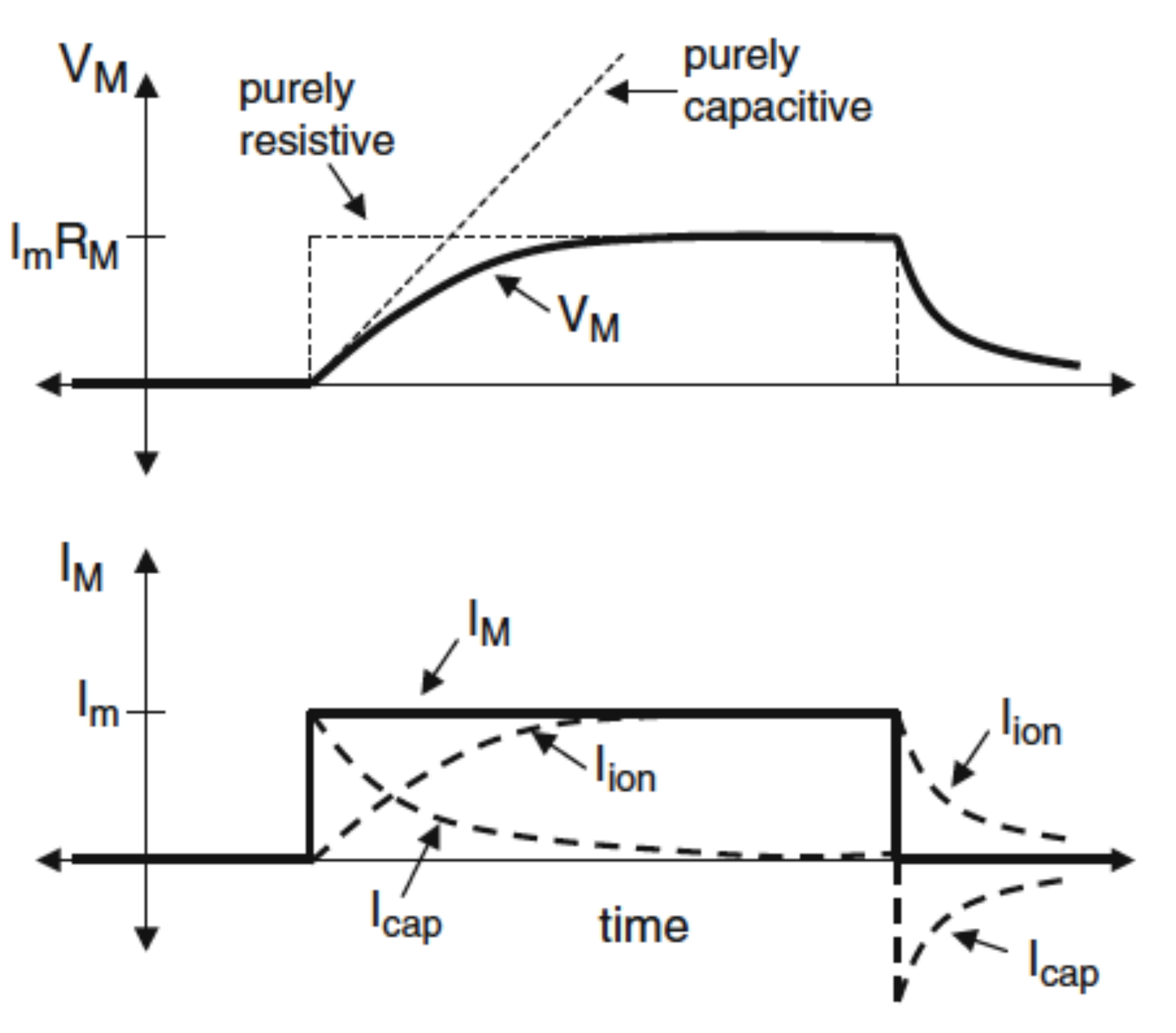


Spherical cell - passive membrane

$$c_M \frac{dV_M}{dt} = -\frac{V_M}{r_M} + I_M(t)$$

$$I_M(t) = \frac{I(t)}{4\pi\rho^2} = \begin{cases} \frac{I_0}{4\pi\rho^2} & \text{if } 0 < t < T \\ 0 & \text{otherwise.} \end{cases}$$

- $V_M(t) = \frac{r_M I_0}{4\pi\rho^2}\left(1 - e^{-\frac{t}{\tau_M}}\right) \quad \text{for } 0 < t < T,$
- $V_M(t) = V_M(T)e^{-\frac{t}{\tau_M}} \quad \text{for } t > T$



- R_{INP}: Input resistance of the cell

$$I_0 \frac{r_M}{4\pi\rho^2} \equiv I_0 R_{INP}$$

R_{INP} is the slope of the I-V curve obtained by plotting the steady-state voltage against the injected current

The cable equation

- Neurons are not isopotential: soma, dendrites, axon and spatial extension
- Isopotential approach: appropriate for the study of signal generation but not for the investigation of signal propagation.
- Axons and dendrites are better approximated by cylinders than by spheres
- Goal: understanding how geometry affects the spread of the signal

The cable equation

Assumptions:

- Membrane is passive (applicable to dendrites rather than axons)
- Cell shaped as a long cylinder (or cable)
- Current flows along a single spatial dimension (x)
- Membrane potential depends only on x, not on the radial or angular components: $V_M(x,t)$
- Cable equation: Partial differential equation (PDE) that describes how $V_M(x,t)$ depends on currents entering, leaving, and flowing within the neuron.
- Extracellular space is isopotential

The cable equation

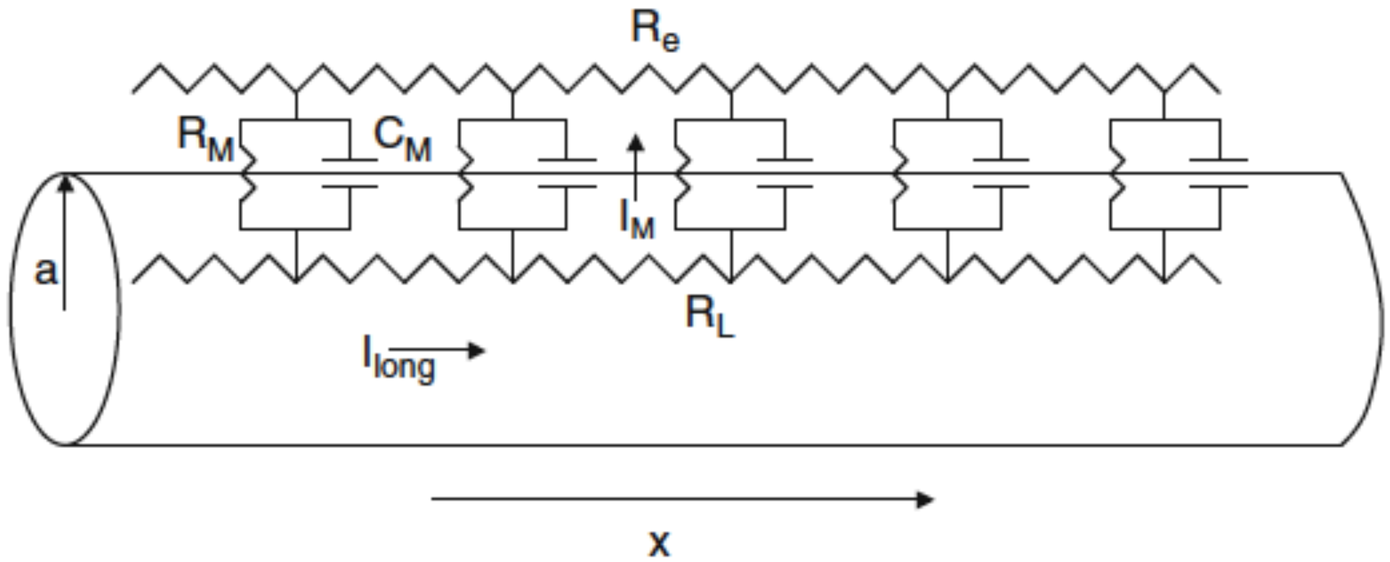


Fig. 1.5 Equivalent circuit for a uniform passive cable. I_{long} is the current along the inside of the cable, I_M is the current across the membrane, R_L is the resistance of the cytoplasm, R_e is the resistance of the extracellular space, R_M is the membrane resistance, and C_M is the membrane capacitance

I_{long}: current along the inside of the cable

I_M: current across the membrane

R_L: resistance of the cytoplasm

R_e: resistance of the extracellular space

C_M: membrane capacitance

R_M: membrane resistance

a: radius of the cable

Δx: length of the cable

The cable equation

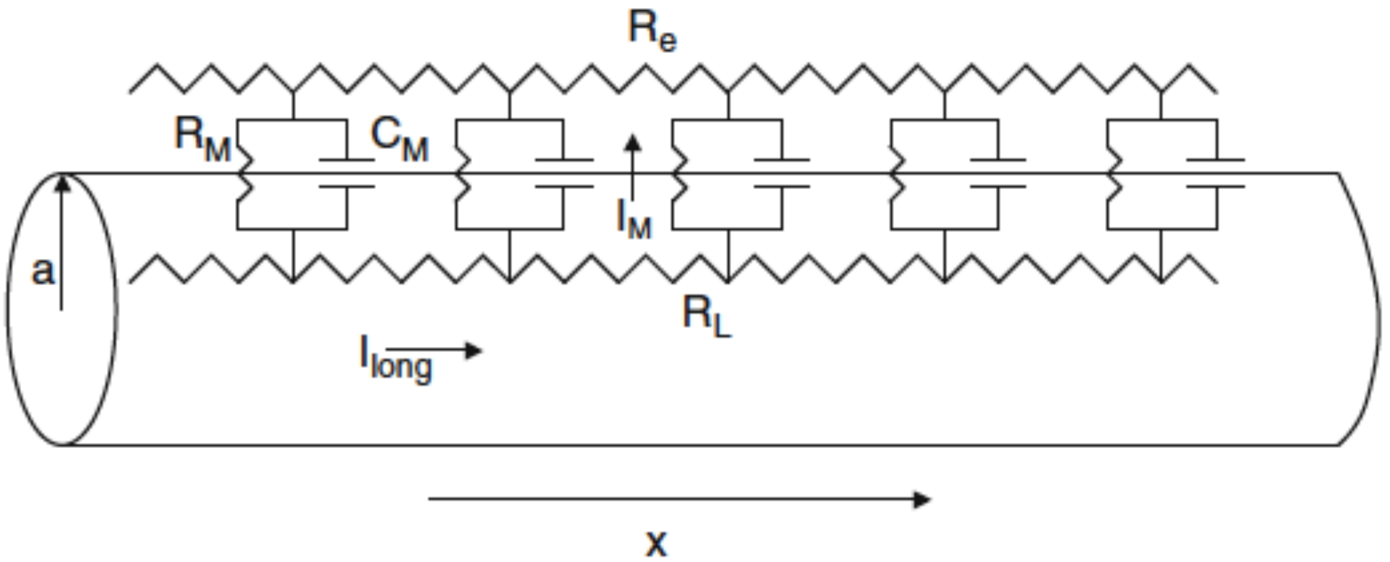


Fig. 1.5 Equivalent circuit for a uniform passive cable. I_{long} is the current along the inside of the cable, I_{M} is the current across the membrane, R_{L} is the resistance of the cytoplasm, R_{e} is the resistance of the extracellular space, R_{M} is the membrane resistance, and C_{M} is the membrane capacitance

Axial current:

- current flowing along the neuron due to current gradients

- the total resistance of the cytoplasm grows proportionally to the length of the cable
- the total resistance of the cytoplasm is inversely proportional to the cross-sectional area of the cable

$$R_{\text{L}} = r_{\text{L}} \Delta x / (\pi a^2)$$

The cable equation

Axial current:

$$V_M(x + \Delta x, t) - V_M(x, t) = -I_{long}(x, t) R_L = -I_{long}(x, t) \frac{\Delta x}{\pi a^2} r_L$$

Ohm's law

If voltage decreases with increasing x, then the current is positive

Δx → 0

$$I_{long}(x, t) = -\frac{\pi a^2}{r_L} \frac{\partial V_M}{\partial x}(x, t)$$

ionic current: $I_{ion} = (2\pi a \Delta x) i_{ion}$

capacitive current: $C_M = (2\pi a \Delta x) c_M$ $I_{cap}(x, t) = (2\pi a \Delta x) c_M \frac{\partial V_M}{\partial t}$

$$I_{cap}(x, t) + I_{ion}(x, t) = -I_{long}(x + \Delta x, t) + I_{long}(x, t)$$

Kirchhoff's law

The cable equation

$$(2\pi a\Delta x)c_M\frac{\partial V_M}{\partial t} + (2\pi a\Delta x)i_{ion} = \frac{\pi a^2}{r_L}\frac{\partial V_M}{\partial x}(x+\Delta x,t) - \frac{\pi a^2}{r_L}\frac{\partial V_M}{\partial x}(x,t)$$

$\Delta x \rightarrow 0$

$$c_M\frac{\partial V_M}{\partial t} = \frac{a}{2r_L}\frac{\partial^2 V_M}{\partial x^2} - i_{ion}$$

$$i_{ion} = V_M(x,t)/r_M$$

$$c_M\frac{\partial V_M}{\partial t} = \frac{a}{2r_L}\frac{\partial^2 V_M}{\partial x^2} - \frac{V_M}{r_M}$$

$$\tau_M\frac{\partial V_M}{\partial t} = \lambda^2\frac{\partial^2 V_M}{\partial x^2} - V_M$$

$\tau_M = c_M r_M$ membrane time constant

$\lambda = \sqrt{\frac{a r_M}{2 r_L}}$ space (length) constant

The cable equation

$$\tau_M \frac{\partial V_M}{\partial t} = \lambda^2 \frac{\partial^2 V_M}{\partial x^2} - V_M$$

$\tau_M = c_M r_M$ membrane time constant $\lambda = \sqrt{\frac{a r_M}{2 r_L}}$ space (length) constant

Steady state solution (semi-infinite cable):

$$\lambda^2 \frac{d^2 V_{ss}}{dx^2} - V_{ss} = 0$$ $t \rightarrow \infty$

$$\frac{dV_{ss}}{dx}(0) = -\frac{r_L}{\pi a^2} I_0$$ boundary condition

$$V_{ss}(x) = \frac{\lambda r_L}{\pi a^2} I_0 e^{-x/\lambda}$$ solution

The cable equation

- The thicker the cable the larger the space constant
- Thicker processes transmit signals for greater distances

$$\lambda = \sqrt{\frac{a r_M}{2 r_L}}$$

Input resistance:

$$R_{inp} = V_{ss}(0)/I_0 = \frac{r_L \lambda}{\pi a^2} = \frac{1}{\pi a^{3/2}} \sqrt{r_M r_L / 2}$$

R_{inp} & λ can be measured experimentally → r_M & R_L can be computed from experimental data

The cable equation

Hodgkin-Huxley model:

$$\frac{a}{2r_L}\frac{\partial^2 V_M}{\partial x^2} = c_M \frac{\partial V_M}{\partial t} + I_K + I_{Na} + I_L$$

$$c_M \frac{\partial V_M}{\partial t} = \frac{a}{2r_L}\frac{\partial^2 V_M}{\partial x^2} - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) - g_L(V_M - E_L)$$

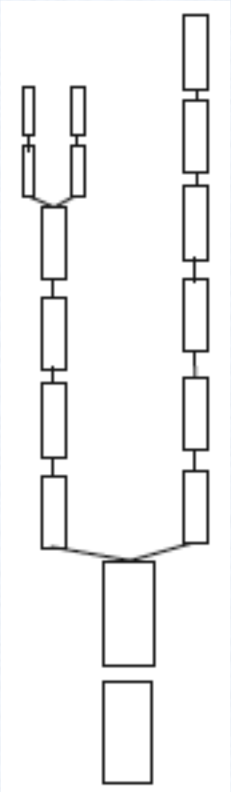
Multiple compartments

- Neurons are not isopotential (soma, dendrites, axon and spatial extension)
- The majority of the total area of many neurons is occupied by the dendritic tree
- Dendrites have a tree-like structure
- Dendrites enable neurons to connect to thousands of other cells
- Many dendrites have spines (fine structures at the ends of dendrites
- During development, animals that are raised in rich environments have more extensive dendritic trees and more spines

Multiple compartments

Compartmental approach:

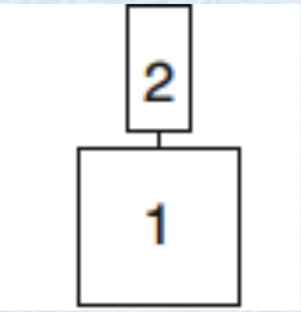
- Dendritic tree is divided into small segments or compartments that are linked together
- Each compartment is assumed to be isopotential
- Each compartment is viewed as a cylinder
- Each compartment is assumed to be spatially uniform in its properties (including diameter)
- Differences in voltage and nonuniformity in membrane properties occur between compartments



Multiple compartments

Two-compartment model:

- a_i: radius of the compartment i (=1,2)
- L_i: length of the compartment i (=1,2)
- A_i: area of the compartment i (=1,2) ($A_i = 2 \pi a_i L_i$)
- V_i: membrane potential of the compartment i (=1,2)
- c_i: specific membrane capacitance of the compartment i (=1,2)
- $r_{M,i}$: specific membrane resistivity of the compartment i (=1,2)
- $I^i_{electrode}$: Electrode current of the compartment i (=1,2)
- r_L: Intracellular (or longitudinal) resistivity
- i^i_{cap}: capacitive current per unit area of membrane for compartment i (=1,2)
- i^i_{ion}: ionic current per unit area of membrane for compartment i (=1,2)

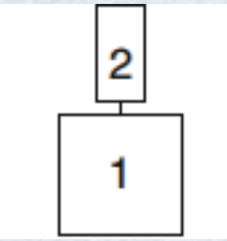


Multiple compartments

Two-compartment model:

$$i^i_{\text{cap}} + i^i_{\text{ion}} = i^i_{\text{long}} + i^i_{\text{electrode}}$$

$$i^i_{\text{cap}} = c_i \frac{\mathrm{d}V_i}{\mathrm{d}t} \qquad i^i_{\text{ion}} = \frac{V_i}{r_{\mathrm{M}i}}$$



$$R_{\text{long}} = \frac{r_{\mathrm{L}} L_1}{2\pi a_1^2} + \frac{r_{\mathrm{L}} L_2}{2\pi a_2^2}$$

$$i^1_{\text{long}} = g_{1,2}(V_2 - V_1) \quad \text{and} \quad i^2_{\text{long}} = g_{2,1}(V_1 - V_2)$$

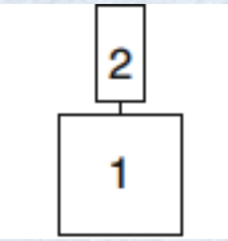
$$g_{1,2} = \frac{a_1 a_2^2}{r_{\mathrm{L}} L_1 (a_2^2 L_1 + a_1^2 L_2)} \qquad g_{2,1} = \frac{a_2 a_1^2}{r_{\mathrm{L}} L_1 (a_2^2 L_1 + a_1^2 L_2)}$$

Multiple compartments

Two-compartment model:

$$i^i_{\text{cap}} + i^i_{\text{ion}} = i^i_{\text{long}} + i^i_{\text{electrode}}$$

$$i^i_{\text{electrode}} = \frac{I^i_{\text{electrode}}}{A_i} \qquad A_i = 2\pi a_i L_i$$

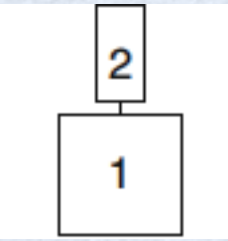


$$c_1 \frac{\mathrm{d}V_1}{\mathrm{d}t} + \frac{V_1}{r_{\mathrm{M}1}} = g_{1,2}(V_2 - V_1) + \frac{I^1_{\text{electrode}}}{A_1}$$
$$c_2 \frac{\mathrm{d}V_2}{\mathrm{d}t} + \frac{V_2}{r_{\mathrm{M}2}} = g_{2,1}(V_1 - V_2) + \frac{I^2_{\text{electrode}}}{A_2}$$

Multiple compartments

Two-compartment model:

$$c_1 \frac{dV_1}{dt} + \frac{V_1}{r_{M1}} = g_{1,2}(V_2 - V_1) + \frac{I^1_{\text{electrode}}}{A_1}$$
$$c_2 \frac{dV_2}{dt} + \frac{V_2}{r_{M2}} = g_{2,1}(V_1 - V_2) + \frac{I^2_{\text{electrode}}}{A_2}$$



$$c_1 \frac{dV_1}{dt} + \frac{V_1}{r_{M1}} = \frac{V_2 - V_1}{r_1} + i_1$$
$$c_2 \frac{dV_2}{dt} + \frac{V_2}{r_{M2}} = \frac{V_1 - V_2}{r_2} + i_2$$

$$r_1 = 1/g_{1,2}$$

$$r_2 = 1/g_{2,1}$$

$$i_i = I^i_{\text{electrode}}/A_i$$

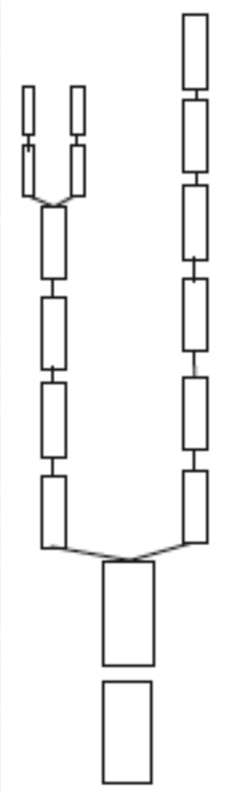
Multiple compartments

Cable equation:

- For each cylinder, j, with radius and length a_j and L_j in micrometers, compute the surface area, $A_j = 2\pi a_j L_j$, and the axial resistance factor, $Q_j = L_j/(\pi a_j^2)$.
- The membrane capacitance is $C_j = c_j A_j \times 10^{-8}$ and the membrane resistance is $R_j = (r_{Mj}/A_j) \times 10^8$.
- The coupling resistance between compartments j and k is $R_{jk} = \frac{r_L}{2}(Q_j + Q_k) \times 10^4$.
- The equations are then

$$C_j \frac{dV_j}{dt} = -\frac{V_j}{R_j} + \sum_{k \text{ connected } j} \frac{V_k - V_j}{R_{jk}} + I_j .$$

The factors of $10^{\pm 8}$ and 10^4 are the conversion from micrometers to centimeters.



Multiple compartments

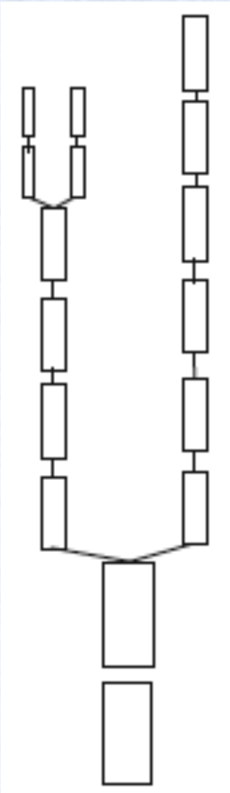
Cable equation:

Assumptions:

- Cable defined on the interval (0,l), l > 0
- Cable has circular cross-section and diameter d(x)

Partition:

- Break the cable into n pieces and define $x_j = j h$ where $h = l / n$
- Call $d_j = d(x_j)$
- Surface area: $A_j = h$
- Cross-sectional area: $\pi d^2_j / 4$
- Neglect the end points



Multiple compartments

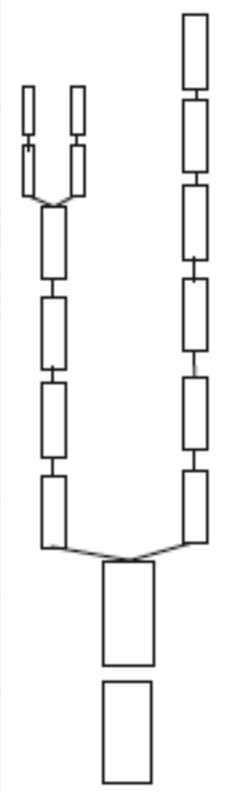
$$c_M A_j \frac{dV_j}{dt} = -\frac{V_j}{r_M/A_j} + \frac{V_{j+1} - V_j}{4r_L h/(\pi d_{j+1}^2)} + \frac{V_{j-1} - V_j}{4r_L h/(\pi d_j^2)}$$

Dividing by h

$$\frac{\pi}{h}\left(\frac{d_{j+1}^2(V_{j+1} - V_j)}{4r_L h} - \frac{d_j^2(V_j - V_{j-1})}{4r_L h}\right)$$

As $h \to 0$,

$$\frac{\pi}{4r_L}\frac{\partial}{\partial x}\left(d^2(x)\frac{\partial V}{\partial x}\right)$$



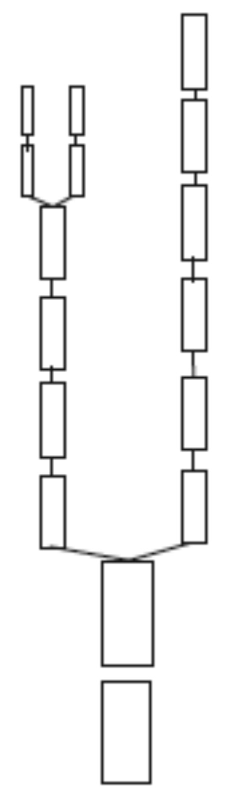
Multiple compartments

dividing by $\pi d(x)$

$$c_M \frac{\partial V}{\partial t} = -\frac{V}{r_M} + \frac{1}{4 r_L d(x)} \frac{\partial}{\partial x}\left(d^2(x) \frac{\partial V}{\partial x}\right)$$

$$\frac{\pi d_j^2 (V_{j-1} - V_j)}{4 r_L h}$$ has dimensions of current

as $h \rightarrow 0$ $$I_L = -\frac{\pi d^2(x)}{4 r_L} \frac{\partial V}{\partial x}$$ *longitudinal* *current*



$d(x) = d$ is constant $$\tau \frac{\partial V}{\partial t} = -V + \lambda^2 \frac{\partial^2 V}{\partial x^2}$$ $\tau = r_M c_M$ $$\lambda = \sqrt{\frac{d r_M}{4 r_L}}$$

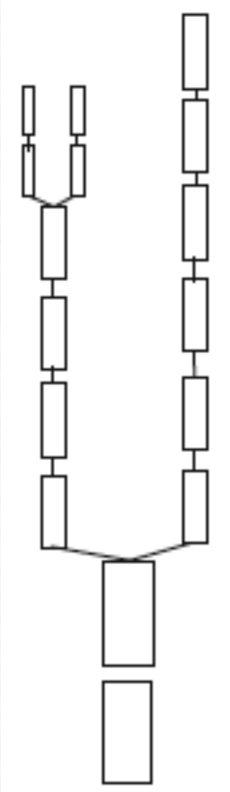
Multiple compartments

$d(x) = d$ is constant.

$$\tau \frac{\partial V}{\partial t} = -V + \lambda^2 \frac{\partial^2 V}{\partial x^2}$$

$$\tau = r_M c_M$$

$$\lambda = \sqrt{\frac{d r_M}{4 r_L}}$$



$c_M = 1\,\mu F/cm^2$

$r_M = 20{,}000\,\Omega\,cm^2$

$r_L = 100\,\Omega\,cm$

$d(x) = 2\,\mu m$

$\rightarrow$ $\tau = 20\,ms$ and $\lambda = 1\,mm$.