

Introduction to Computational Neuroscience

Biol 698

Math 635

Biol 498

Math 430

Bibliography:

"Mathematical Foundations of Neuroscience", by G. B. Ermentrout & D. H. Terman - Springer (2010), 1st edition. ISBN 978-0-387-87707-5

* "Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting", by Eugene M. Izhikevich. The MIT Press, 2007. ISBN 0-262-09043-8

Overview

- The Hodgkin-Huxley model (review)
- Dynamical systems: Basic concepts
- Reduced one-dimensional models
- Reduced two-dimensional models
- Morris-Lecar and FitzHugh-Nagumo models
- Quadratic integrate-and-fire models
- Bifurcations: basic concepts

Hodgkin-Huxley model

$$\begin{aligned}C\dot{V} &= I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L} \\ \dot{n} &= \alpha_n(V)(1 - n) - \beta_n(V)n \\ \dot{m} &= \alpha_m(V)(1 - m) - \beta_m(V)m \\ \dot{h} &= \alpha_h(V)(1 - h) - \beta_h(V)h ,\end{aligned}$$

$$\alpha_n(V) = 0.01 \frac{10 - V}{\exp\left(\frac{10 - V}{10}\right) - 1}$$

$$\alpha_m(V) = 0.1 \frac{25 - V}{\exp\left(\frac{25 - V}{10}\right) - 1}$$

$$\alpha_h(V) = 0.07 \exp\left(\frac{-V}{20}\right)$$

$$\beta_n(V) = 0.125 \exp\left(\frac{-V}{80}\right)$$

$$\beta_m(V) = 4 \exp\left(\frac{-V}{18}\right)$$

$$\beta_h(V) = \frac{1}{\exp\left(\frac{30 - V}{10}\right) + 1}$$

Hodgkin-Huxley model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$
$$\dot{n} = (n_\infty(V) - n) / \tau_n(V),$$
$$\dot{m} = (m_\infty(V) - m) / \tau_m(V),$$
$$\dot{h} = (h_\infty(V) - h) / \tau_h(V),$$

$$n_\infty = \alpha_n / (\alpha_n + \beta_n), \quad \tau_n = 1 / (\alpha_n + \beta_n),$$
$$m_\infty = \alpha_m / (\alpha_m + \beta_m), \quad \tau_m = 1 / (\alpha_m + \beta_m),$$
$$h_\infty = \alpha_h / (\alpha_h + \beta_h), \quad \tau_h = 1 / (\alpha_h + \beta_h)$$

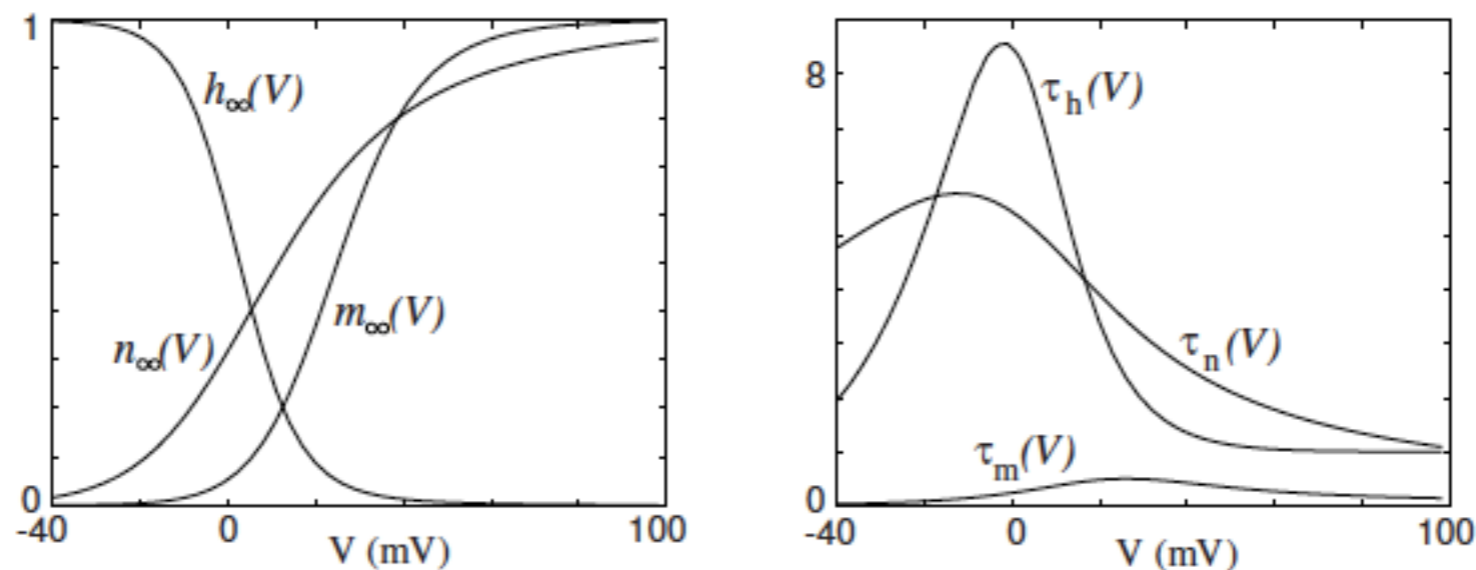


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Hodgkin-Huxley model

$$\begin{aligned}C\dot{V} &= I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L} \\ \dot{n} &= \alpha_n(V)(1 - n) - \beta_n(V)n \\ \dot{m} &= \alpha_m(V)(1 - m) - \beta_m(V)m \\ \dot{h} &= \alpha_h(V)(1 - h) - \beta_h(V)h ,\end{aligned}$$

$$E_K = -12 \text{ mV}$$

$$E_{Na} = 120 \text{ mV}$$

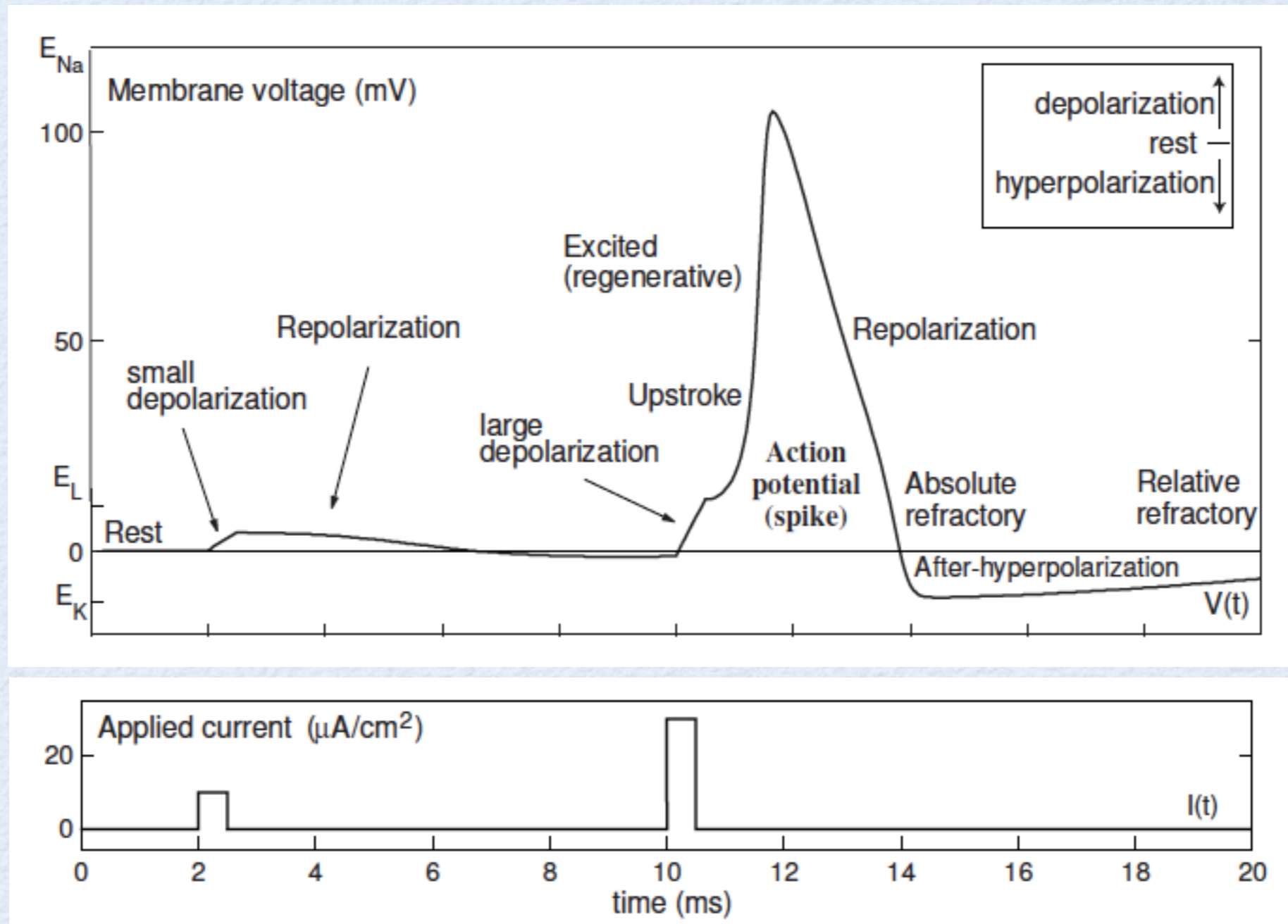
$$E_L = 10.6 \text{ mV}$$

$$\bar{g}_K = 36 \text{ mS/cm}^2$$

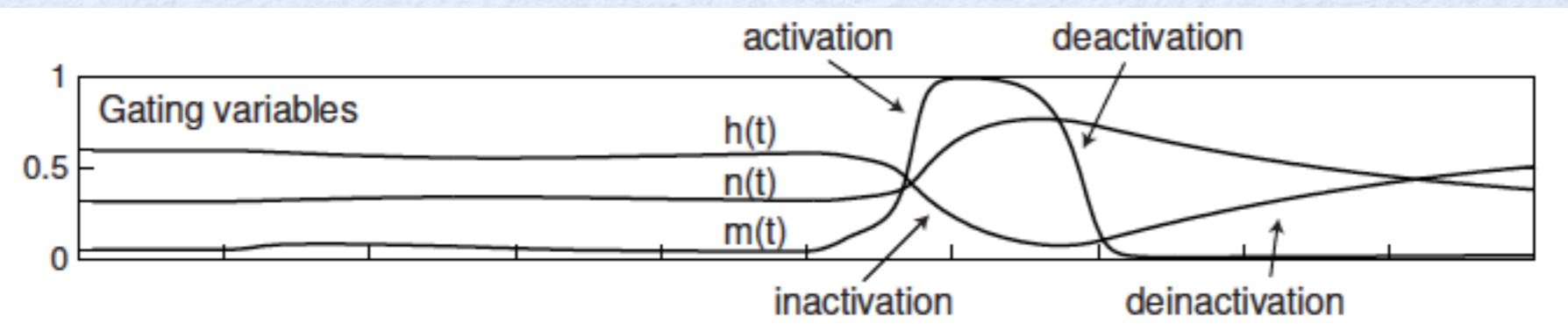
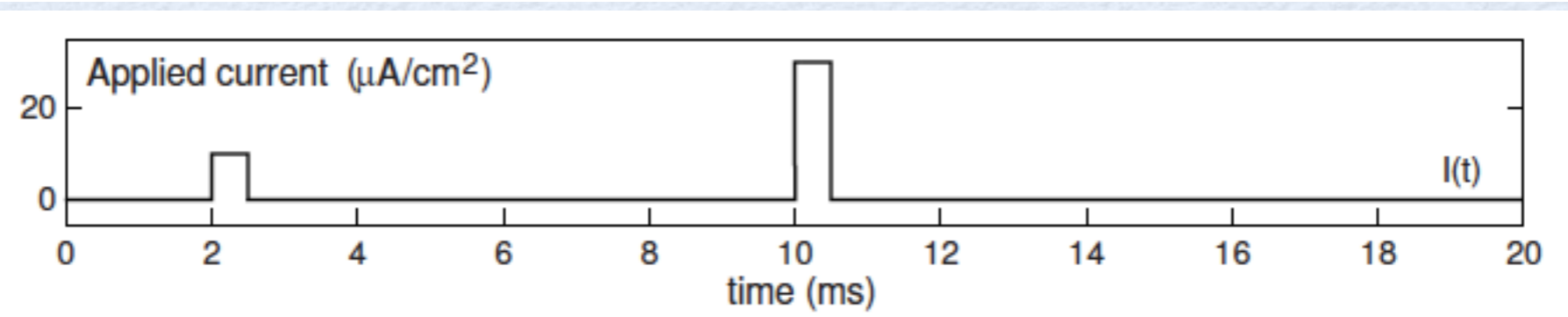
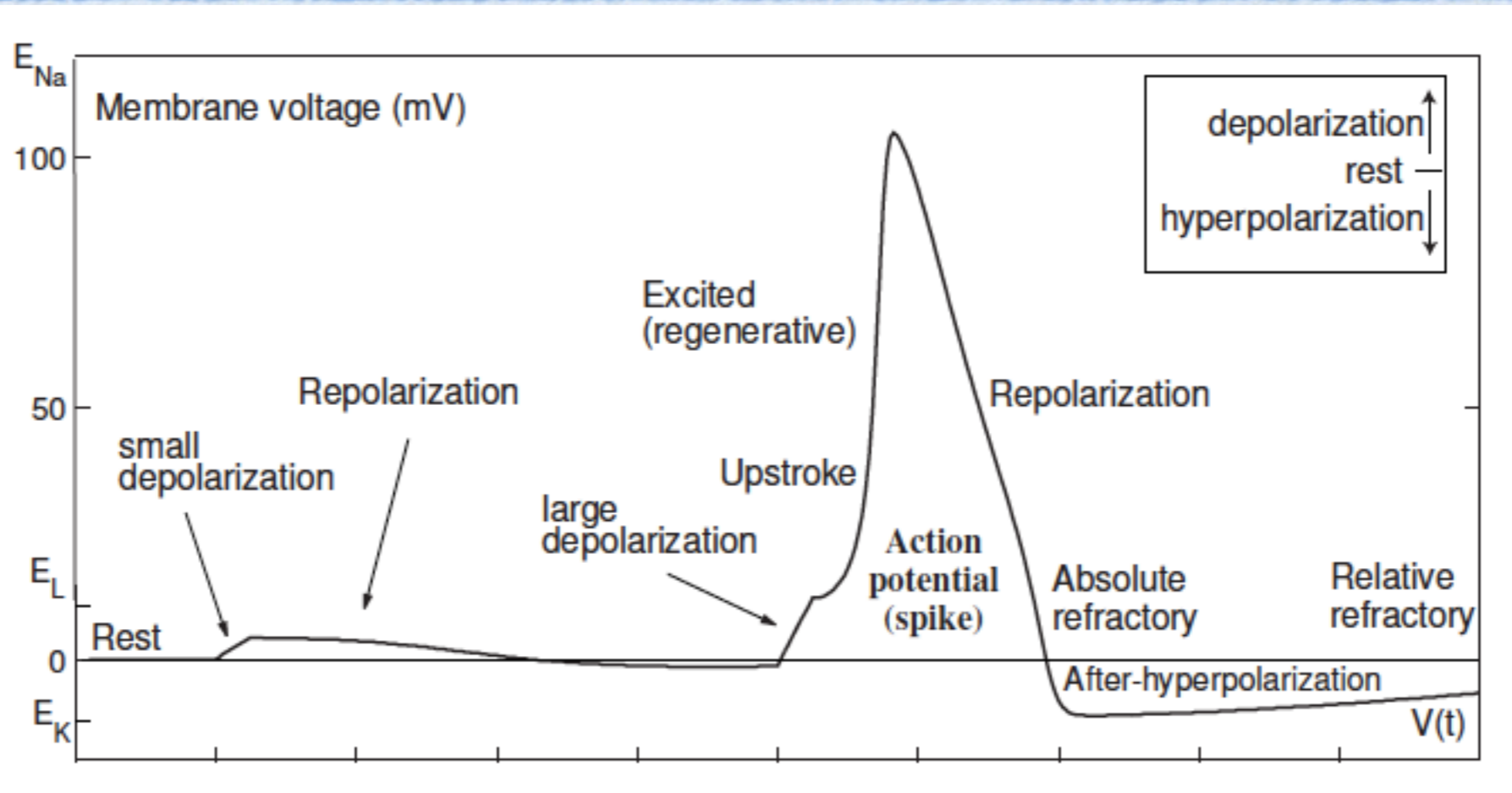
$$\bar{g}_{Na} = 120 \text{ mS/cm}^2$$

$$g_L = 0.3 \text{ mS/cm}^2$$

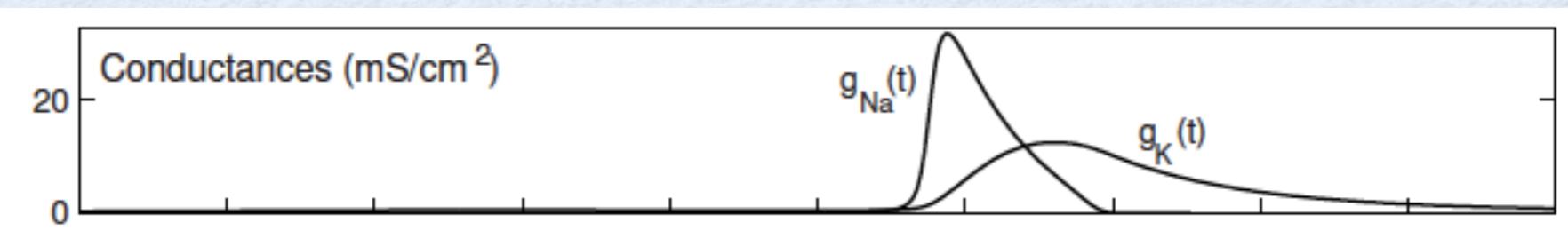
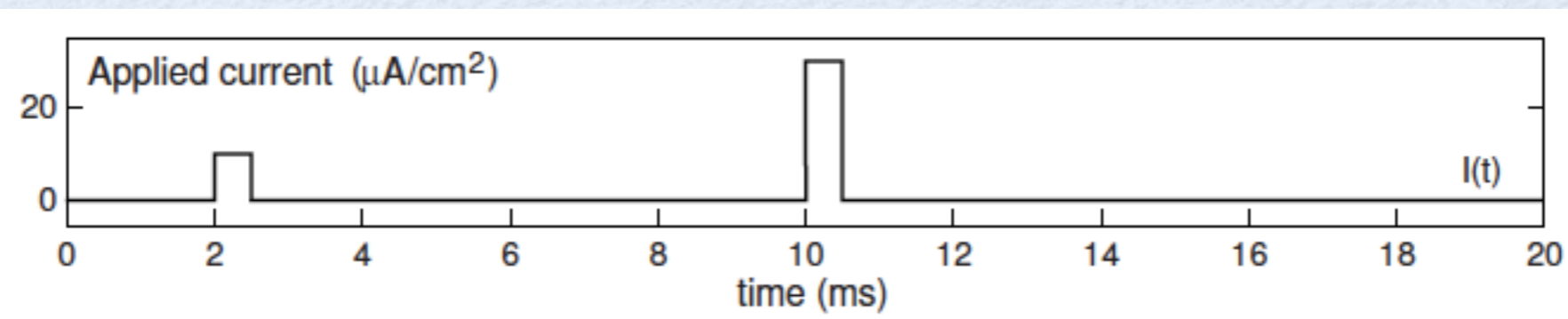
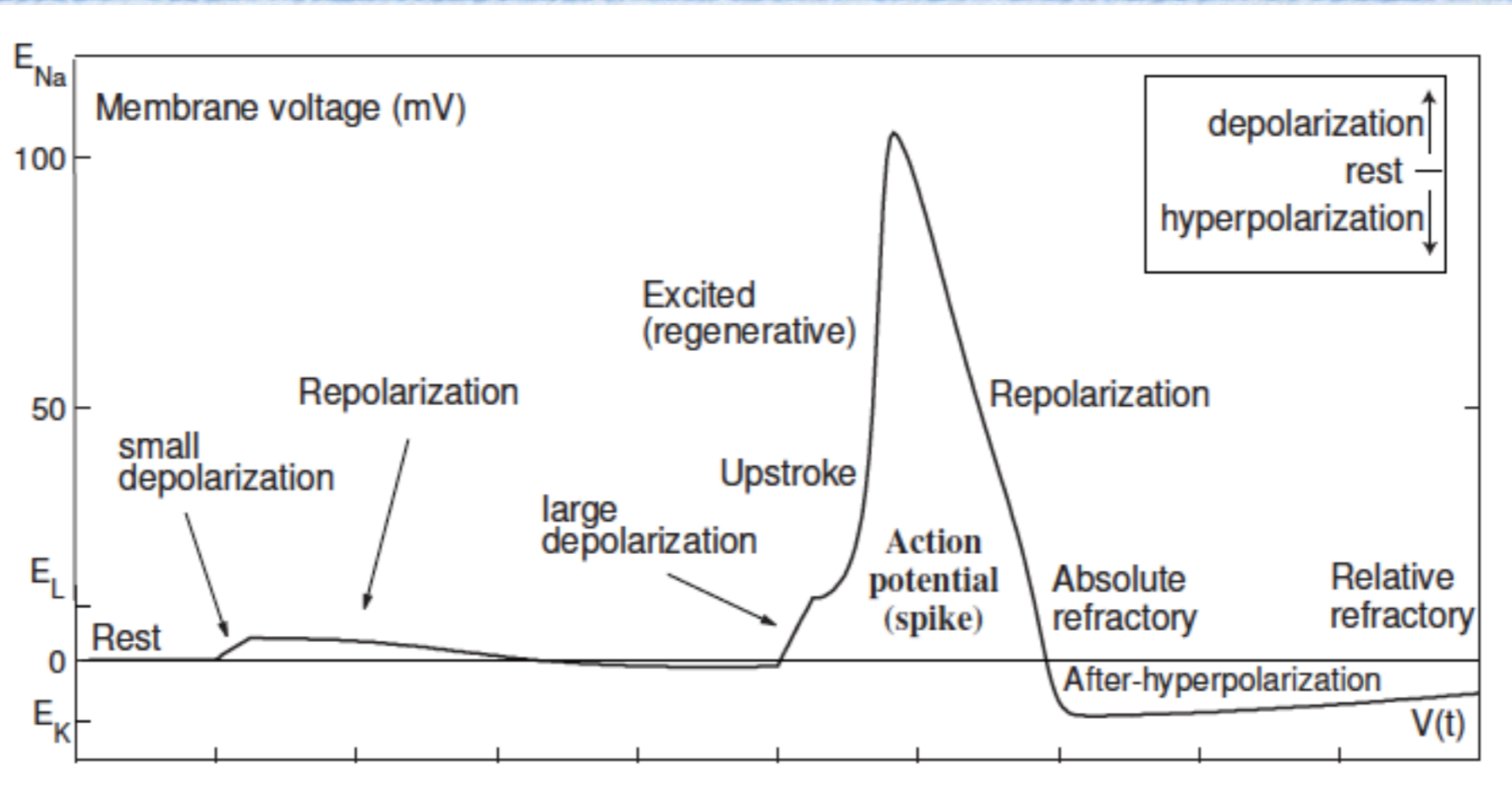
Hodgkin-Huxley model



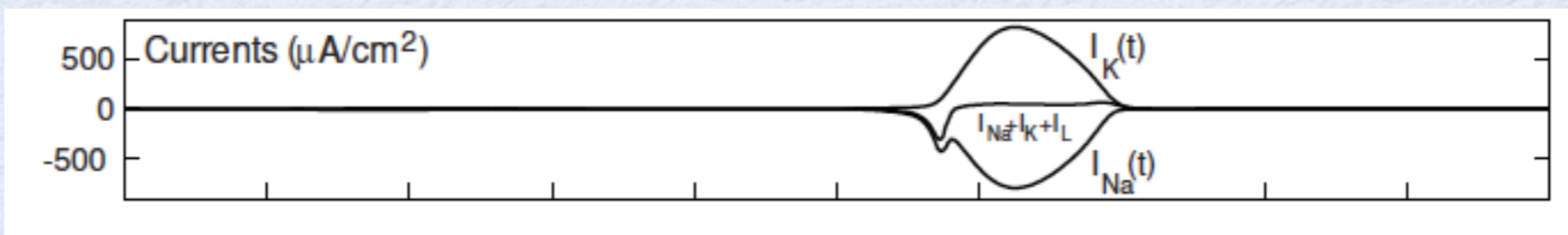
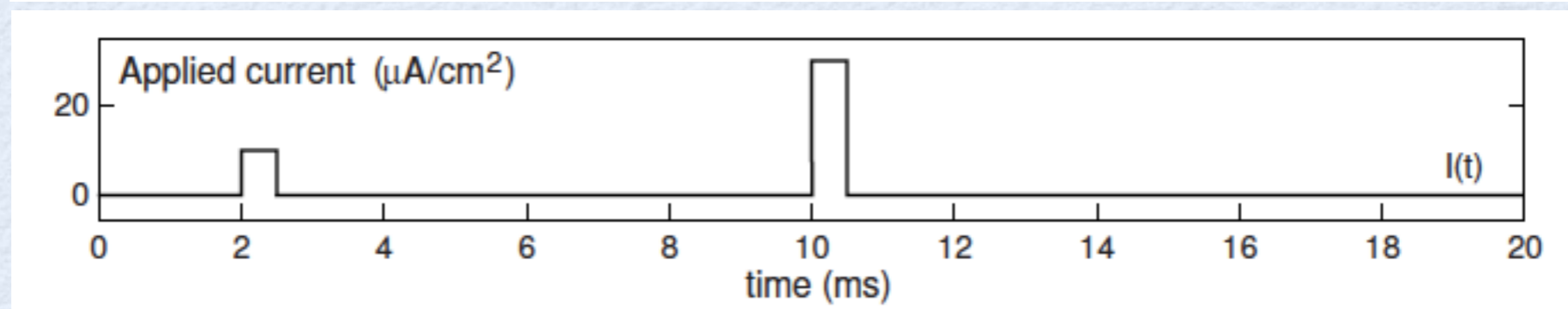
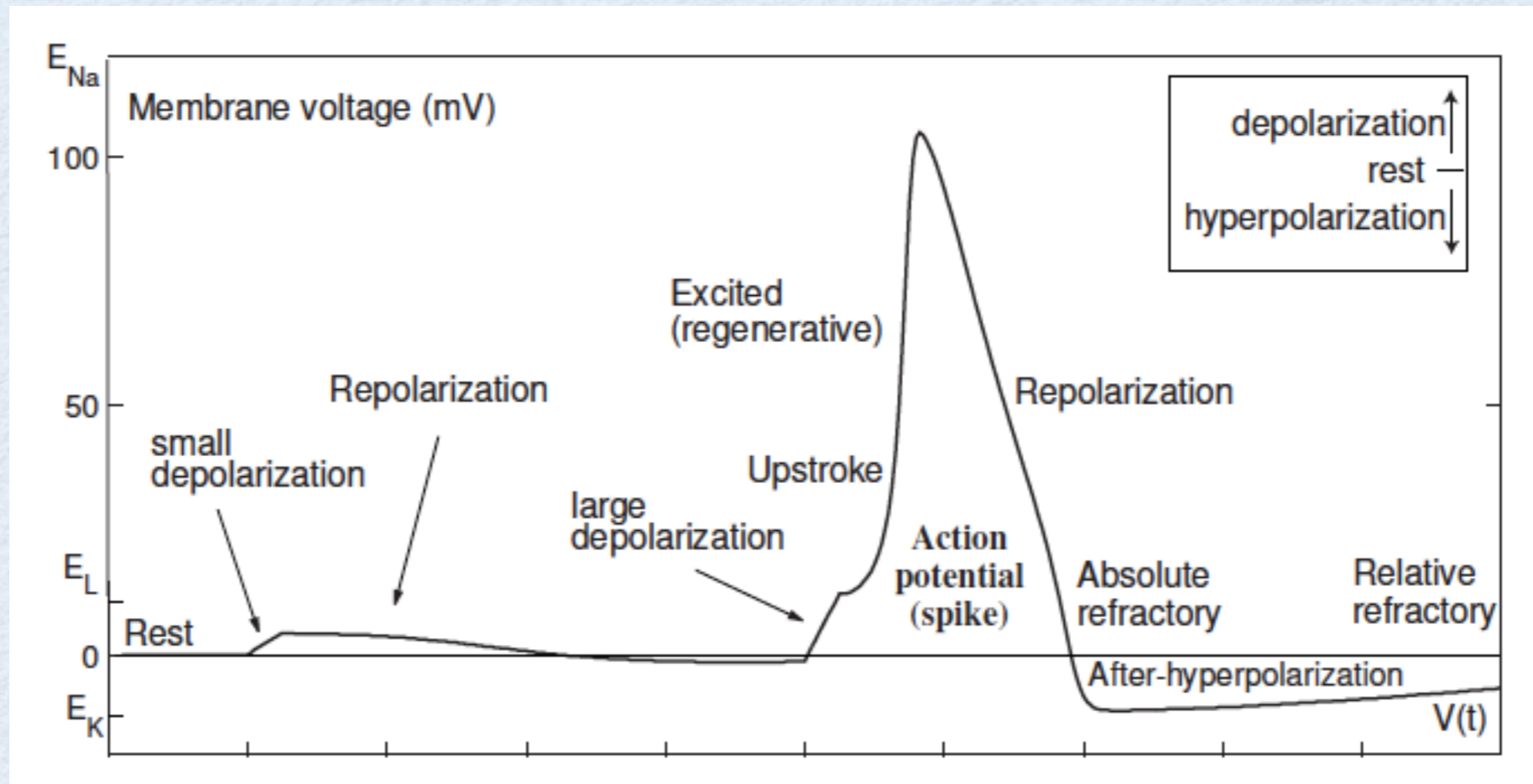
Hodgkin-Huxley model



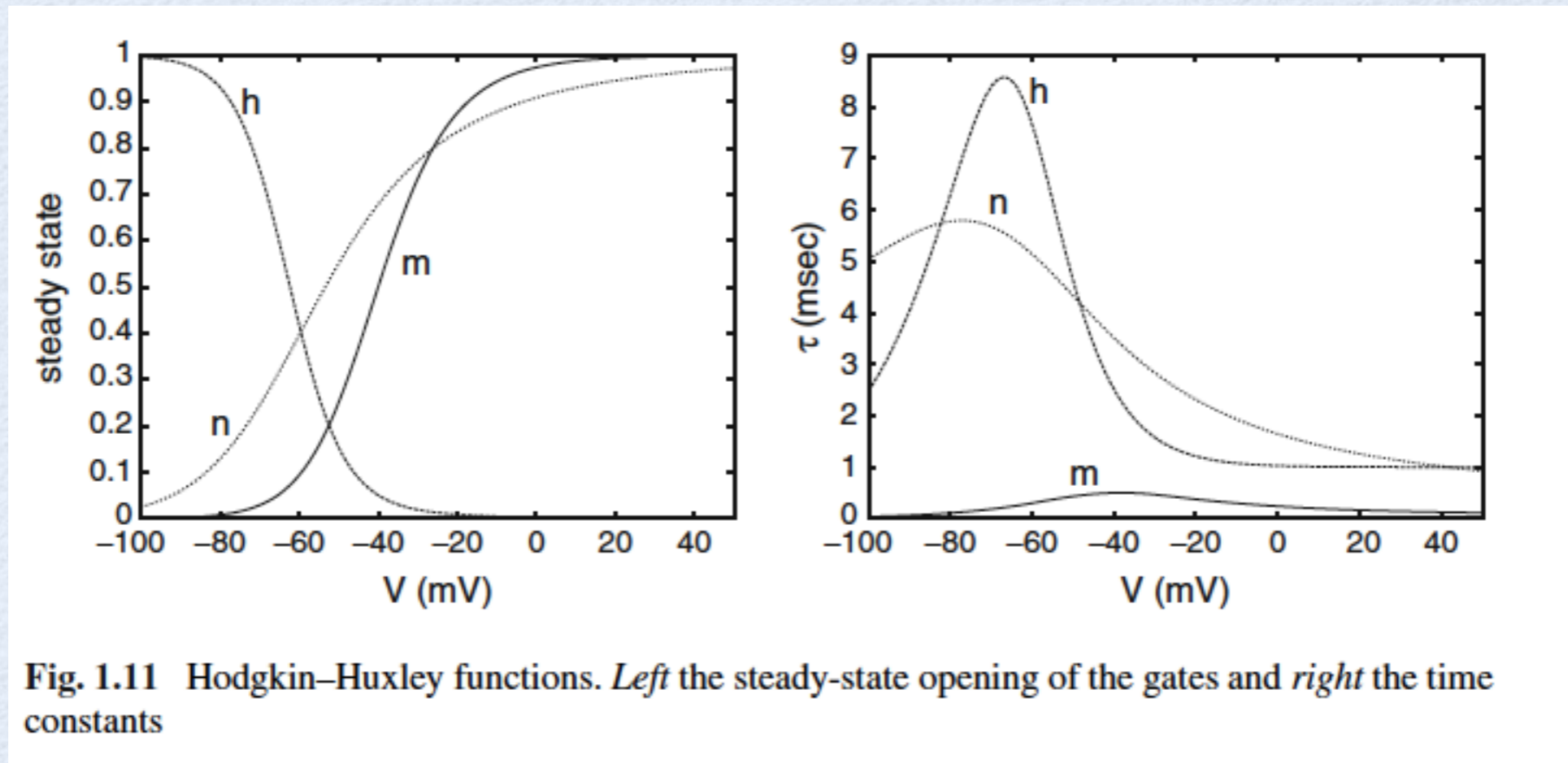
Hodgkin-Huxley model



Hodgkin-Huxley model



Hodgkin-Huxley model



Hodgkin-Huxley model

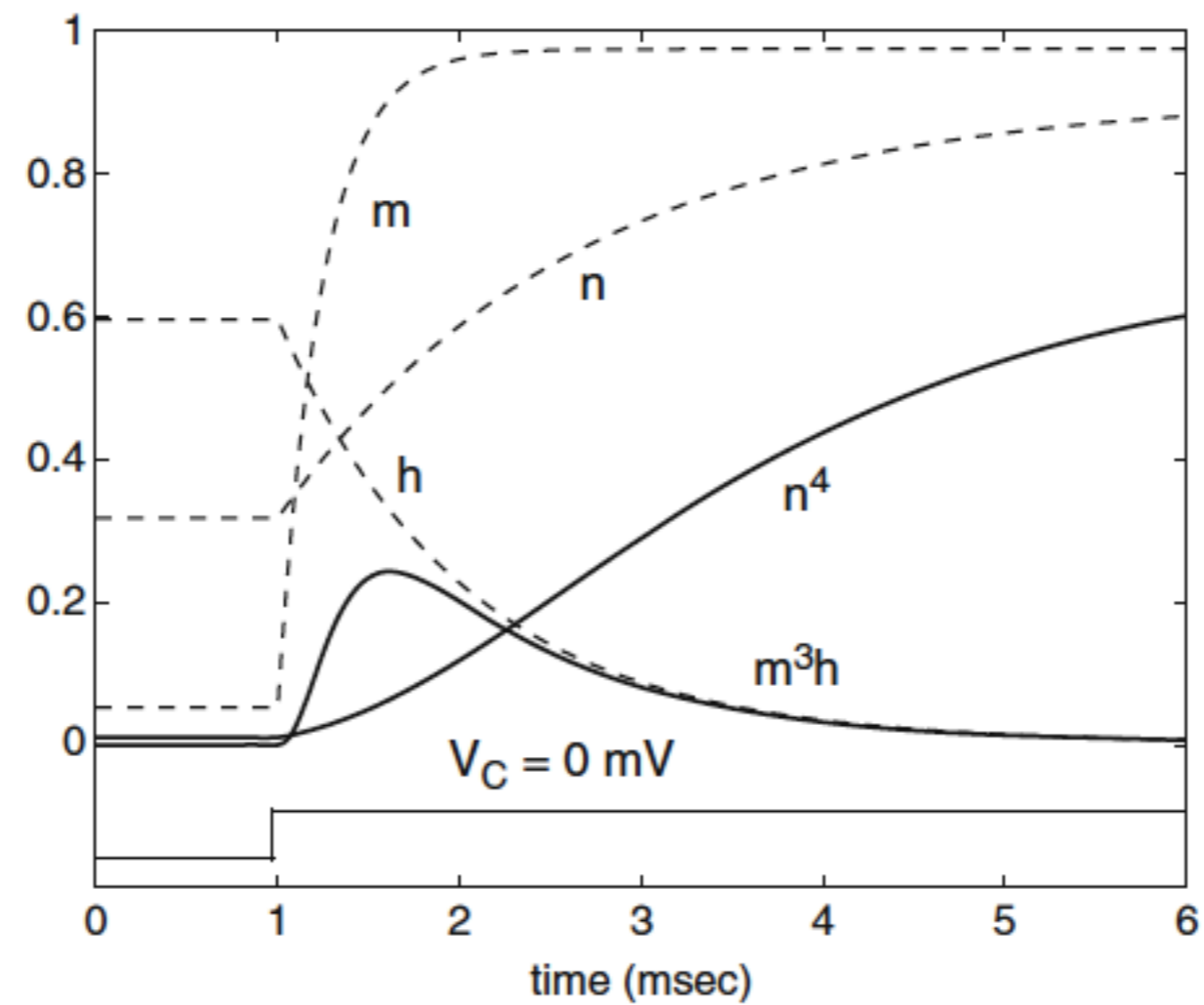


Fig. 1.12 Response of the activation and inactivation variables m , h , and n to a step in voltage

Hodgkin-Huxley model

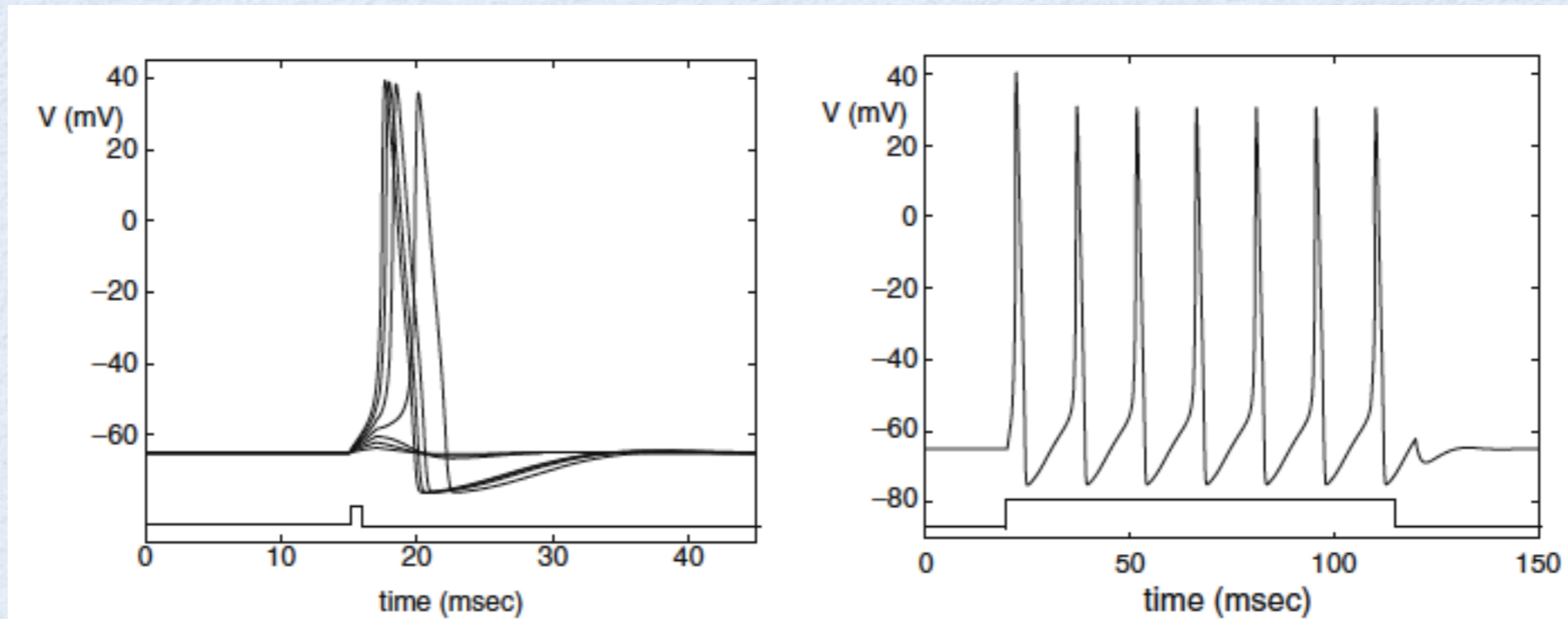


Fig. 1.13 Responses of the Hodgkin–Huxley model to applied currents. *Left* transient responses showing “all-or-none” behavior and *right* sustained periodic response

Hodgkin-Huxley model

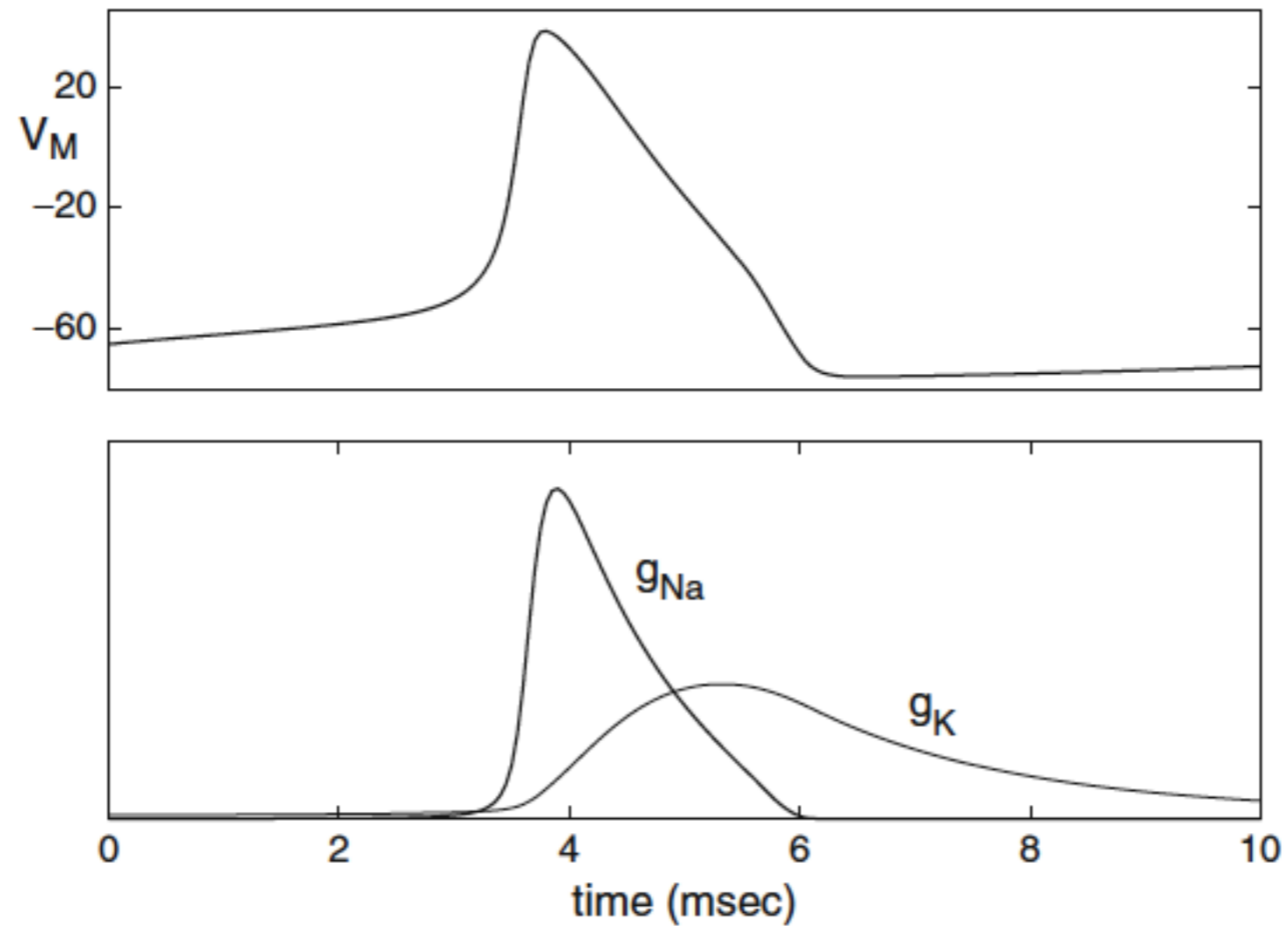
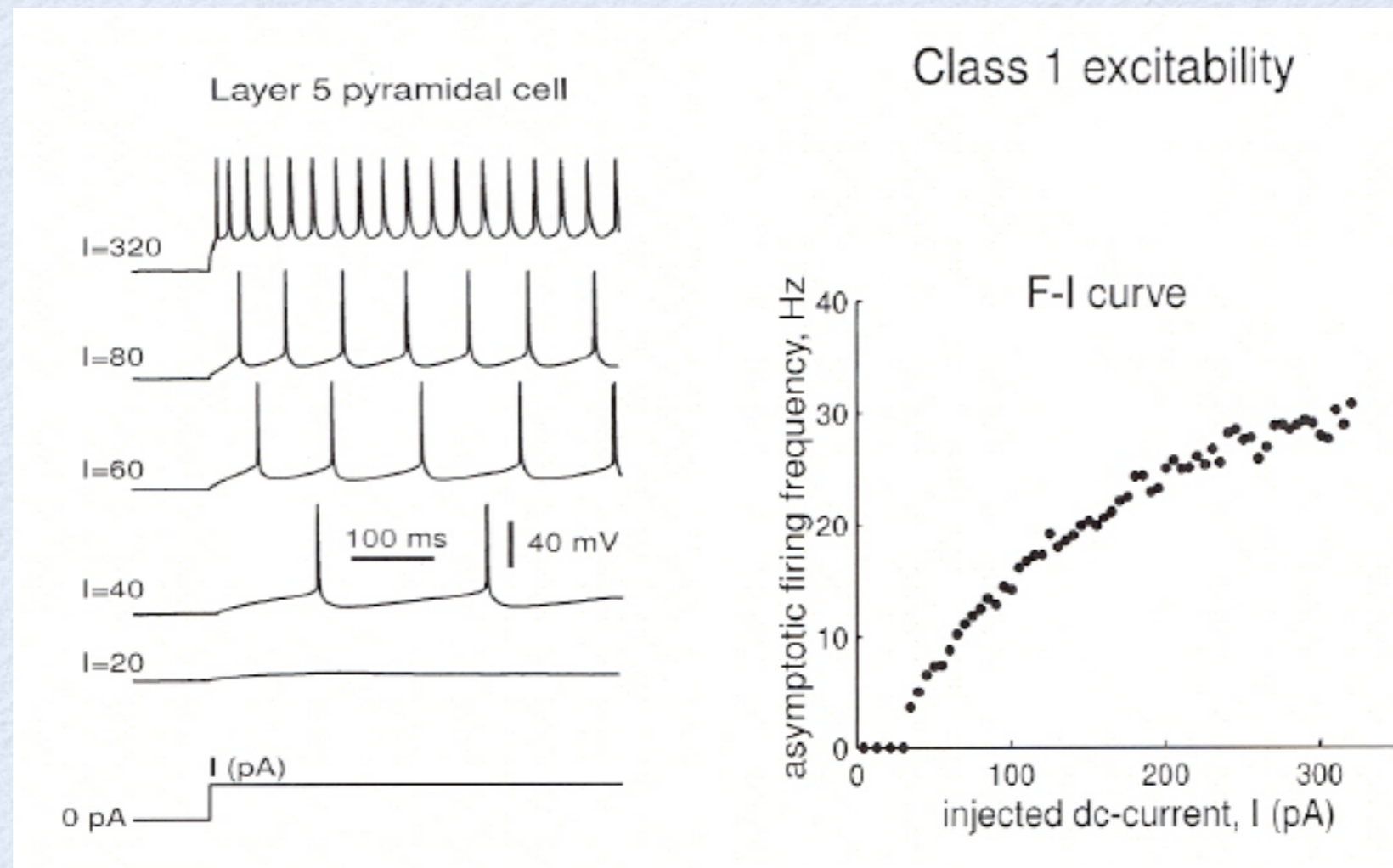
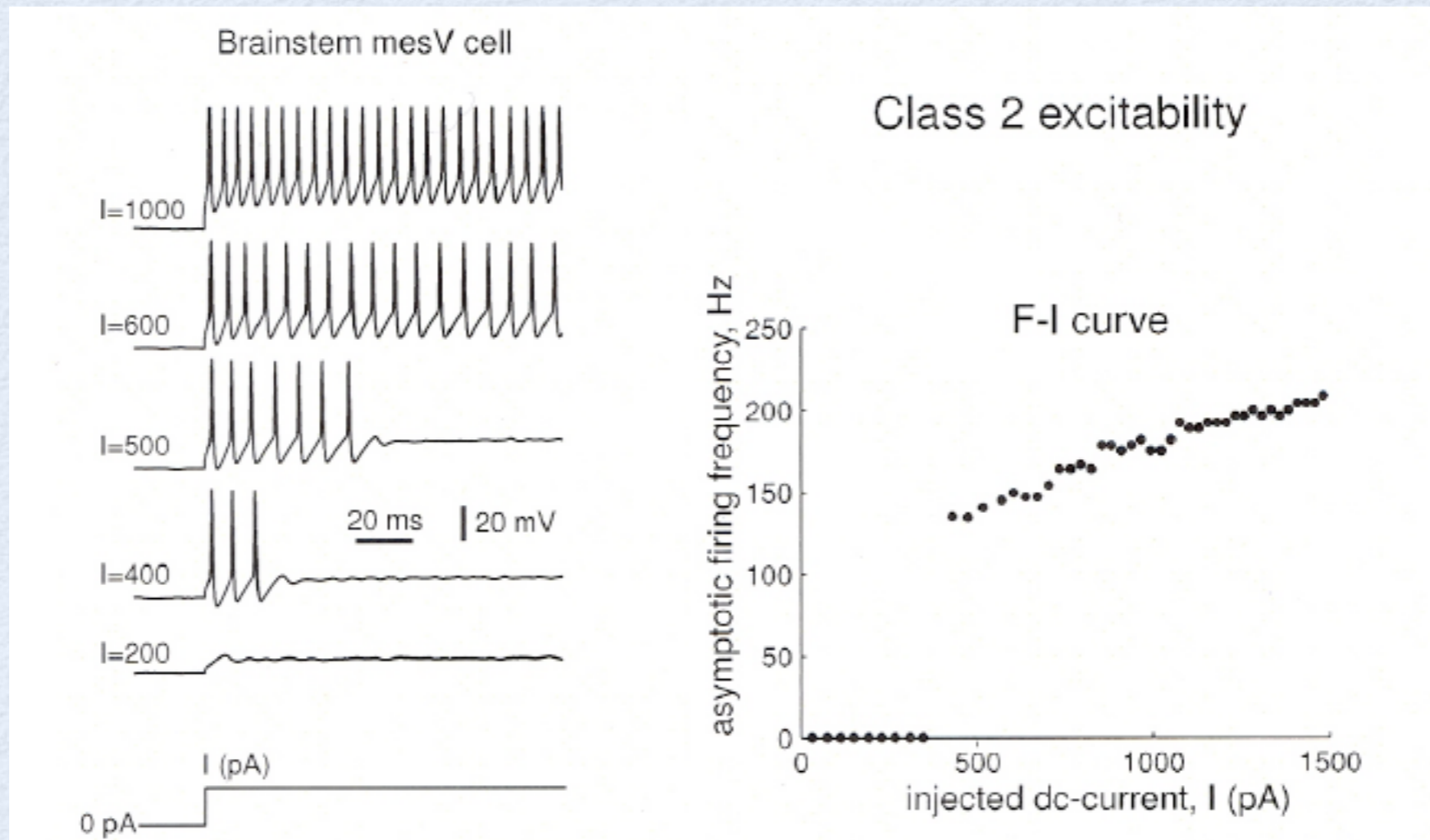


Fig. 1.14 Solution of the Hodgkin–Huxley equations showing an action potential. Also shown are the Na^+ and K^+ conductances

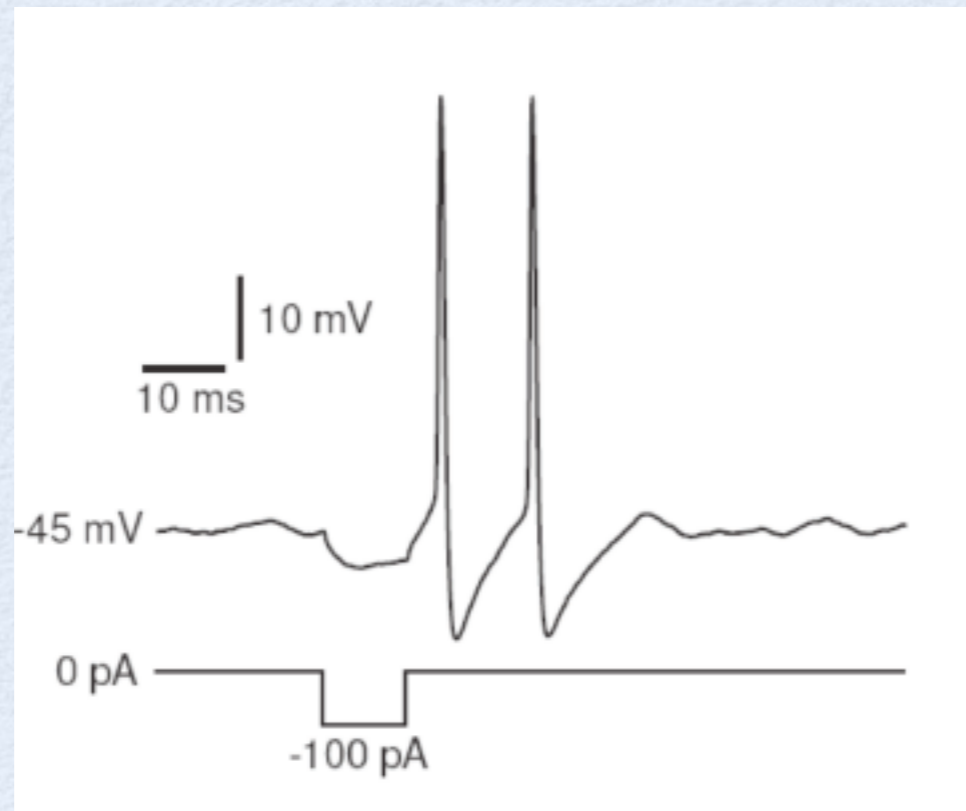
Dynamics of single neurons



Dynamics of single neurons



Dynamics of single neurons



In vitro recording of **rebound spikes** of a rat's brainstem mesencephalic V neuron in response to a brief hyperpolarizing pulse of current.

(Izhikevich)

Dynamics systems

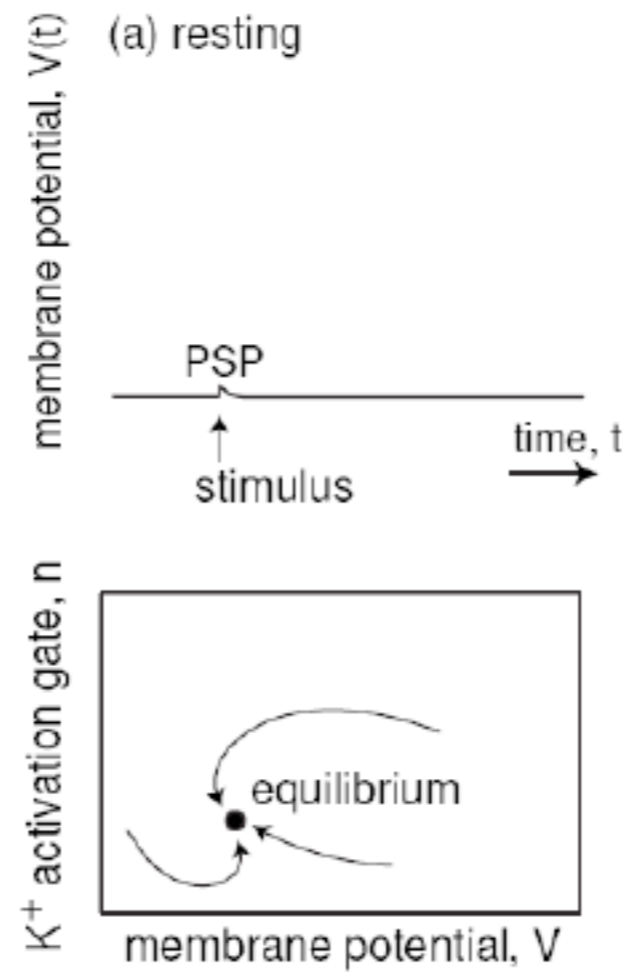
Dynamical System:

- set of variables that describes its state
- law that describes the evolution of the state variables with time

Dynamical systems approach to neuroscience: One can tell things about a system without knowing all the details that govern the system evolution

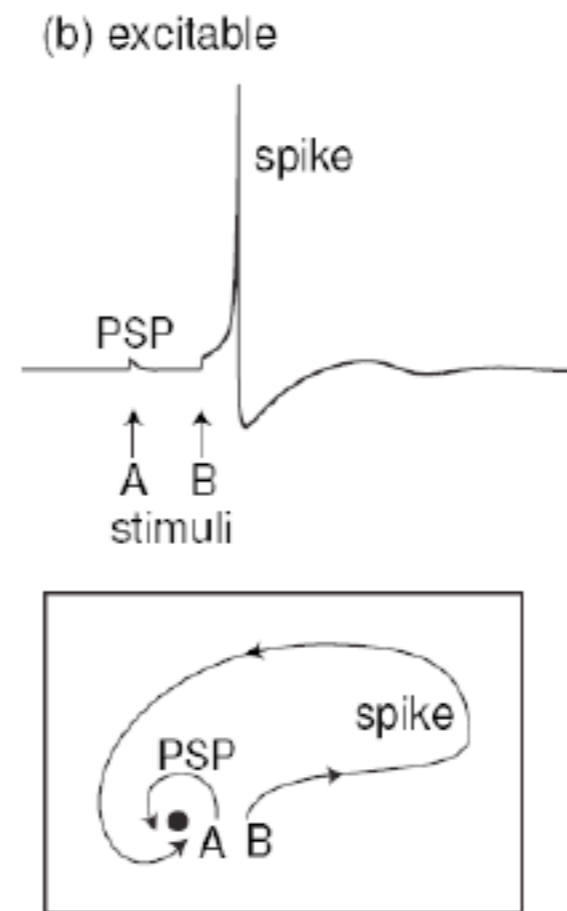
Dynamics systems

□ Stable equilibrium point



Dynamics systems

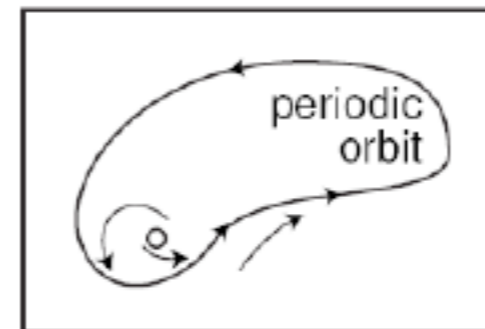
□ Stable equilibrium point



Dynamics systems

□ Stable limit cycle

(c) periodic spiking



Dynamics systems

Important concepts:

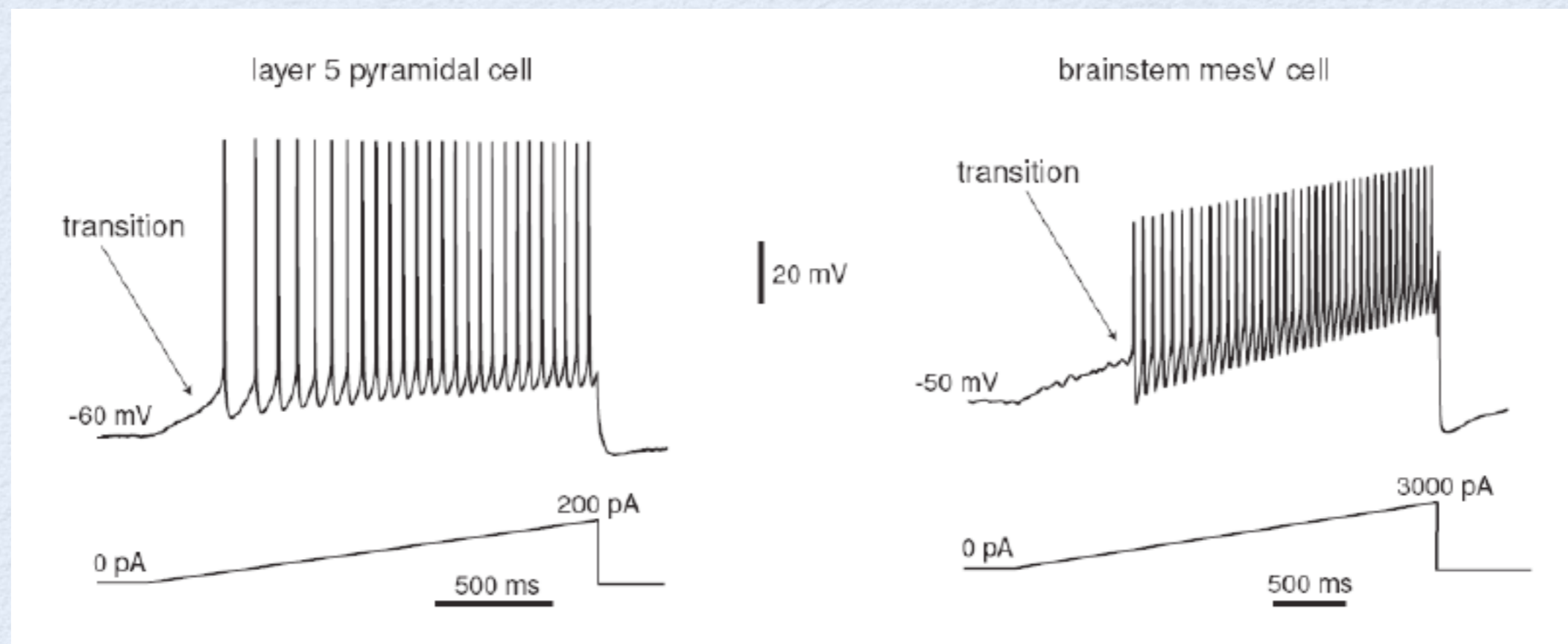
- Phase portrait (phase-plane / phase-space)
- Equilibrium (fixed-) point
- Limit cycle
- Stability
- Threshold
- Attractor
- Attraction domain (basin of attraction)
- Separatrix
- Perturbation
- Bifurcation

Bifurcations

- Qualitative change of the phase portrait of a dynamical system as the magnitude of a control parameter changes
- Example of a control parameter: Applied (injected) current
- Example of a bifurcation: transition from a fixed-point to a limit cycle

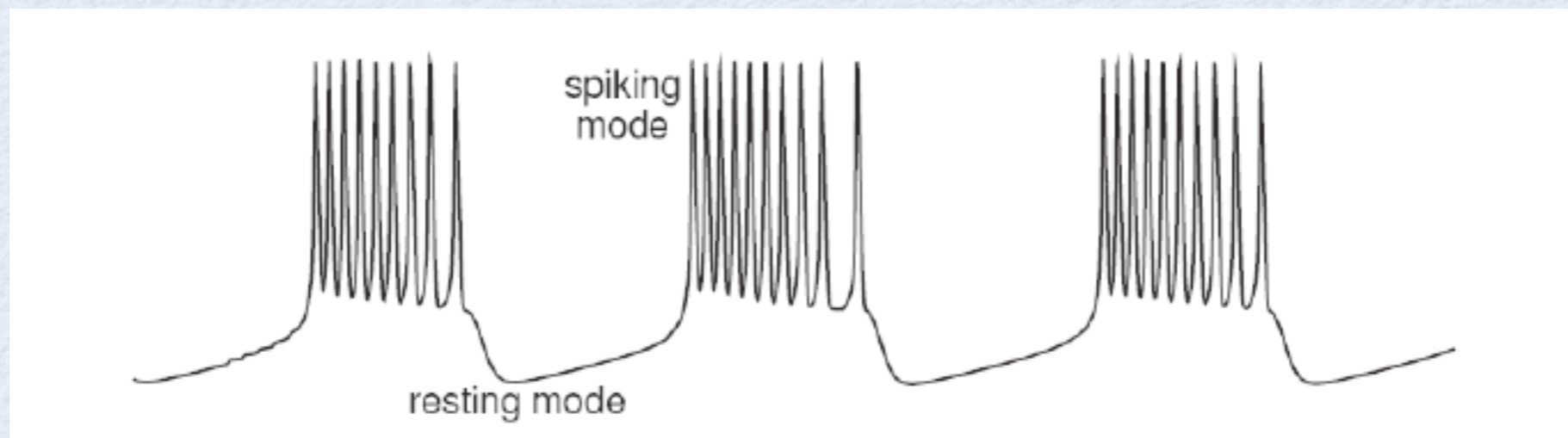
Bifurcations

- As the magnitude of the injected current slowly increases, the neurons bifurcated from resting (equilibrium) to tonic spiking (limit cycle) modes



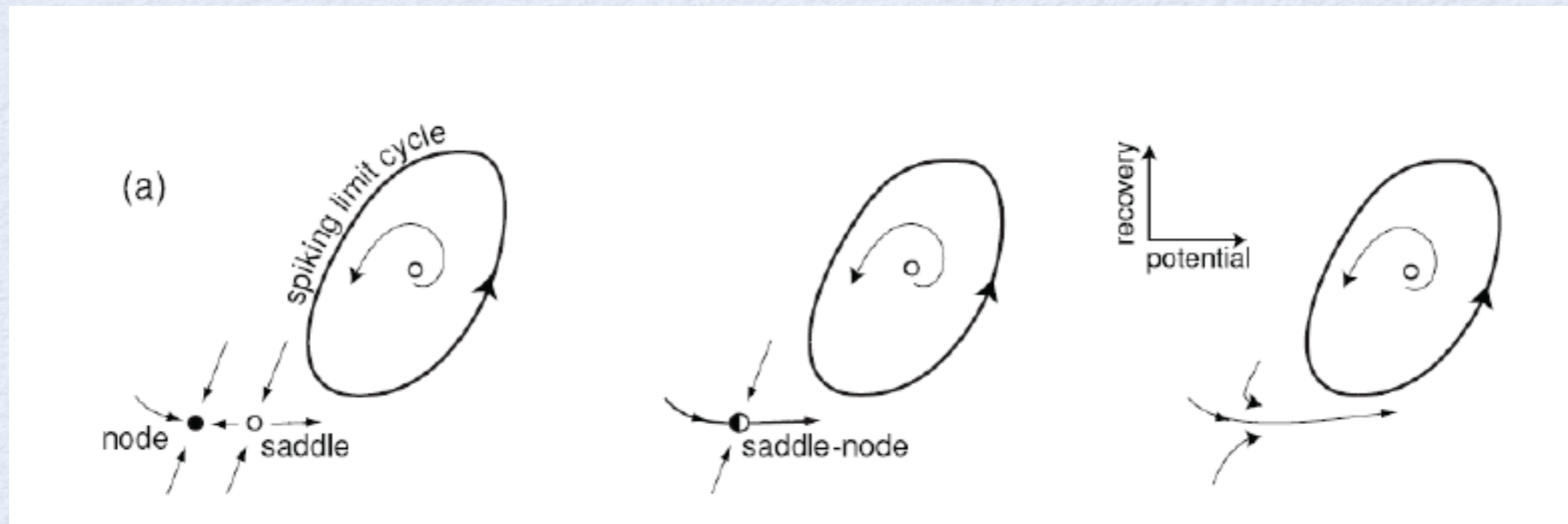
Bifurcations

- Transitions between rhythmic and spiking modes result in bursting behavior



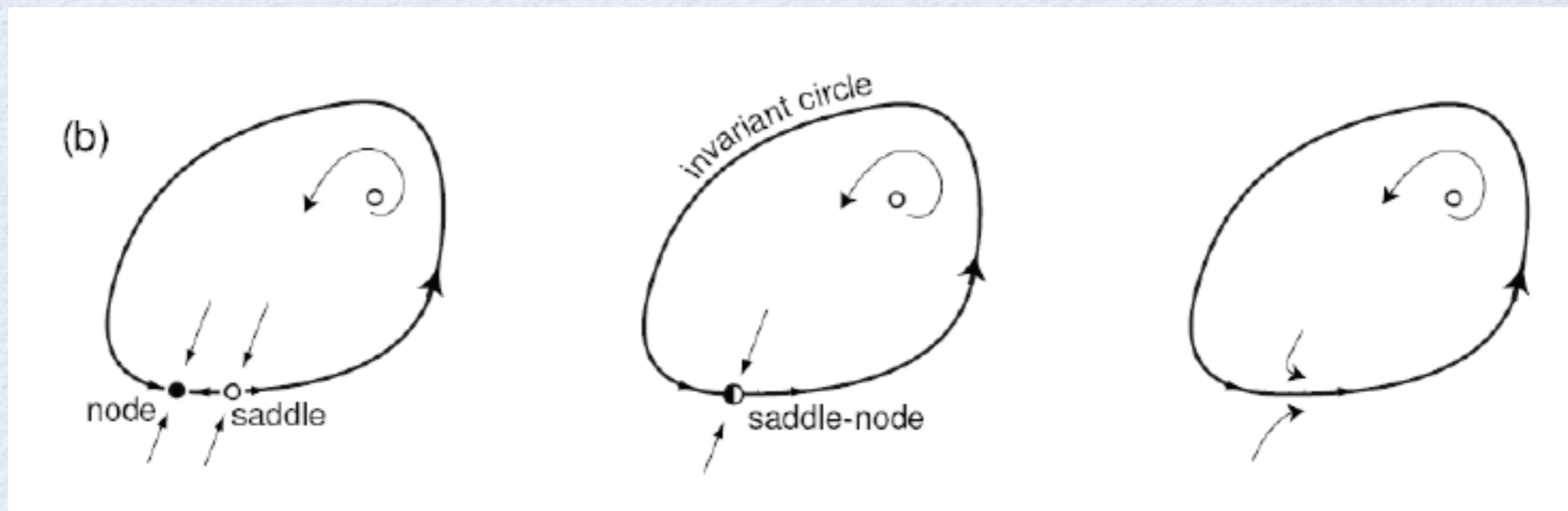
Bifurcations

- Saddle-node bifurcation



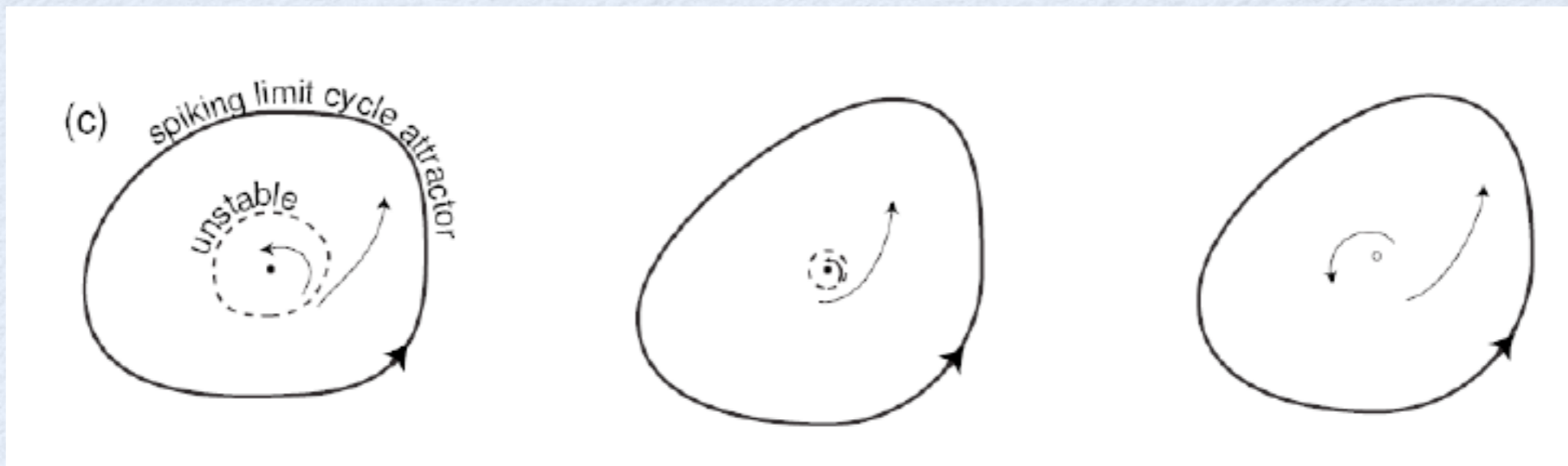
Bifurcations

- Saddle-node on invariant cycle (SNIC) bifurcation



Bifurcations

- Subcritical Hopf (Andronov-Hopf) bifurcation



Bifurcations

- Supercritical Hopf (Andronov-Hopf) bifurcation



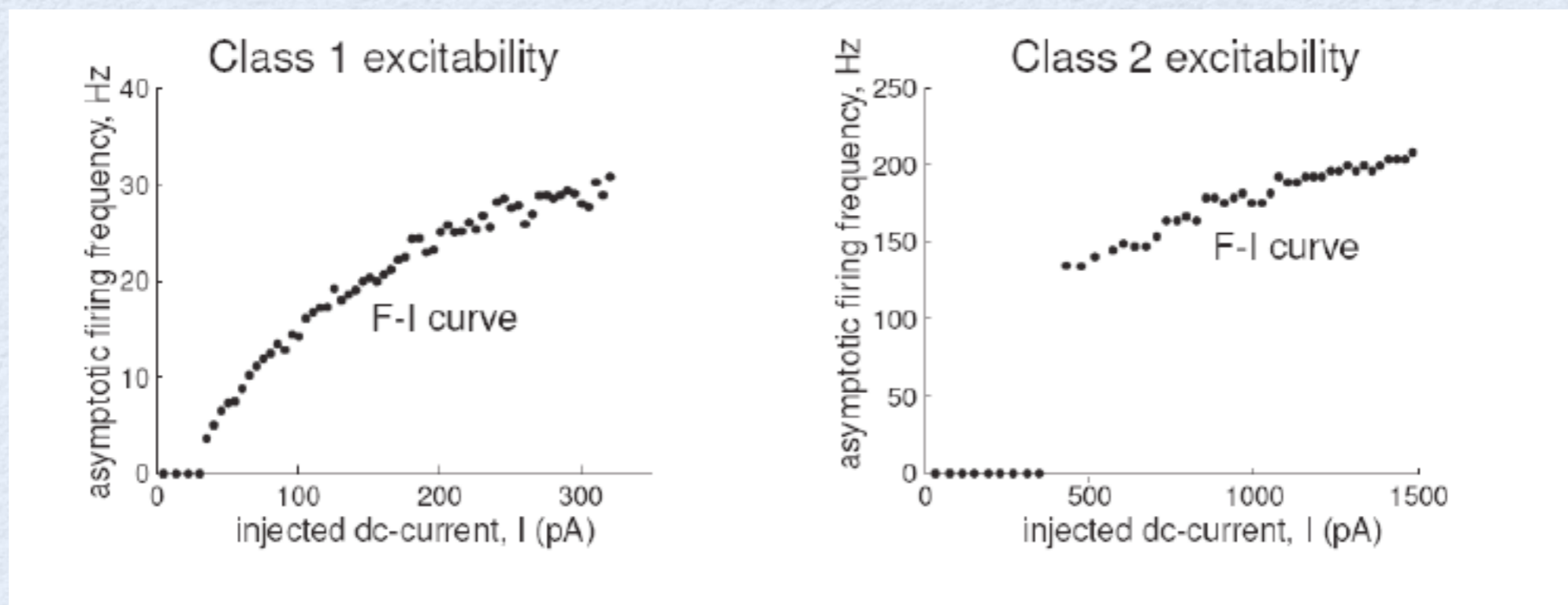
Bifurcations

- Classification of neurons according to the bifurcation of the resting state

		co-existence of resting and spiking states	
		YES (bistable)	NO (monostable)
subthreshold oscillations	NO (integrator)	saddle-node	saddle-node on invariant circle
	YES (resonator)	subcritical Andronov-Hopf	supercritical Andronov-Hopf

Bifurcations

- Frequency-current (F-I) curves of cortical pyramidal neurons and mes V neurons



One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$
$$\dot{n} = \overbrace{(n_\infty(V) - n) / \tau_n(V)}^{I_K},$$
$$\dot{m} = \overbrace{(m_\infty(V) - m) / \tau_m(V)}^{I_{Na}},$$
$$\dot{h} = \overbrace{(h_\infty(V) - h) / \tau_h(V)}^{I_L},$$

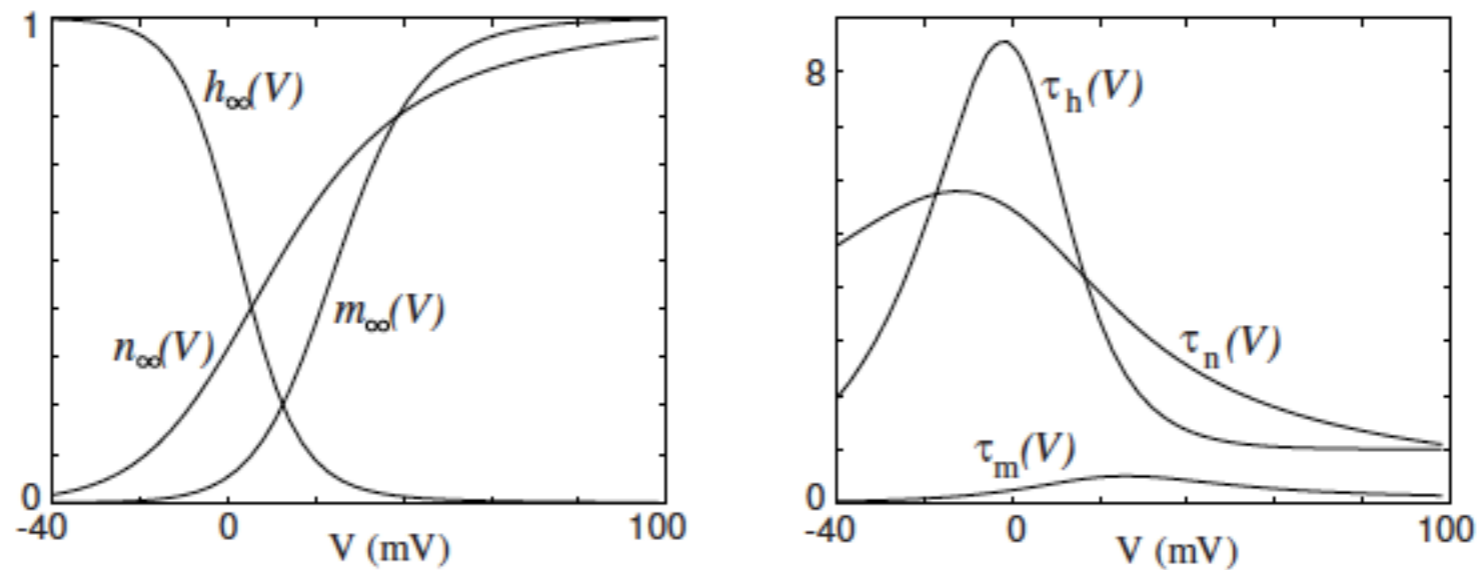


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = (n_\infty(V) - n) / \tau_n(V),$$

$$\dot{m} = (m_\infty(V) - m) / \tau_m(V),$$

$$\dot{h} = (h_\infty(V) - h) / \tau_h(V),$$

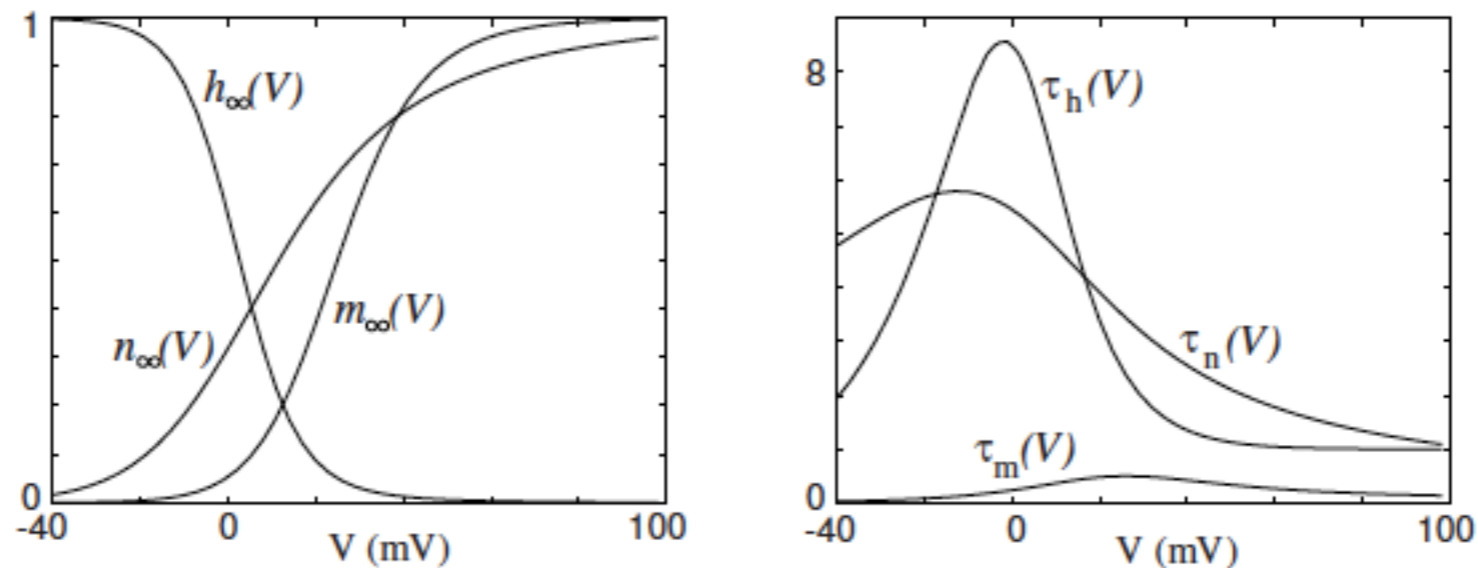


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

- The passive membrane equation

$$C \frac{dV}{dt} = - G_L (V - E_L) + I_{app}$$

One-dimensional neural model

$$\begin{aligned}
 C\dot{V} &= I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L} \\
 \dot{n} &= (n_\infty(V) - n) / \tau_n(V), \\
 \dot{m} &= (m_\infty(V) - m) / \tau_m(V), \\
 \dot{h} &= (h_\infty(V) - h) / \tau_h(V),
 \end{aligned}$$

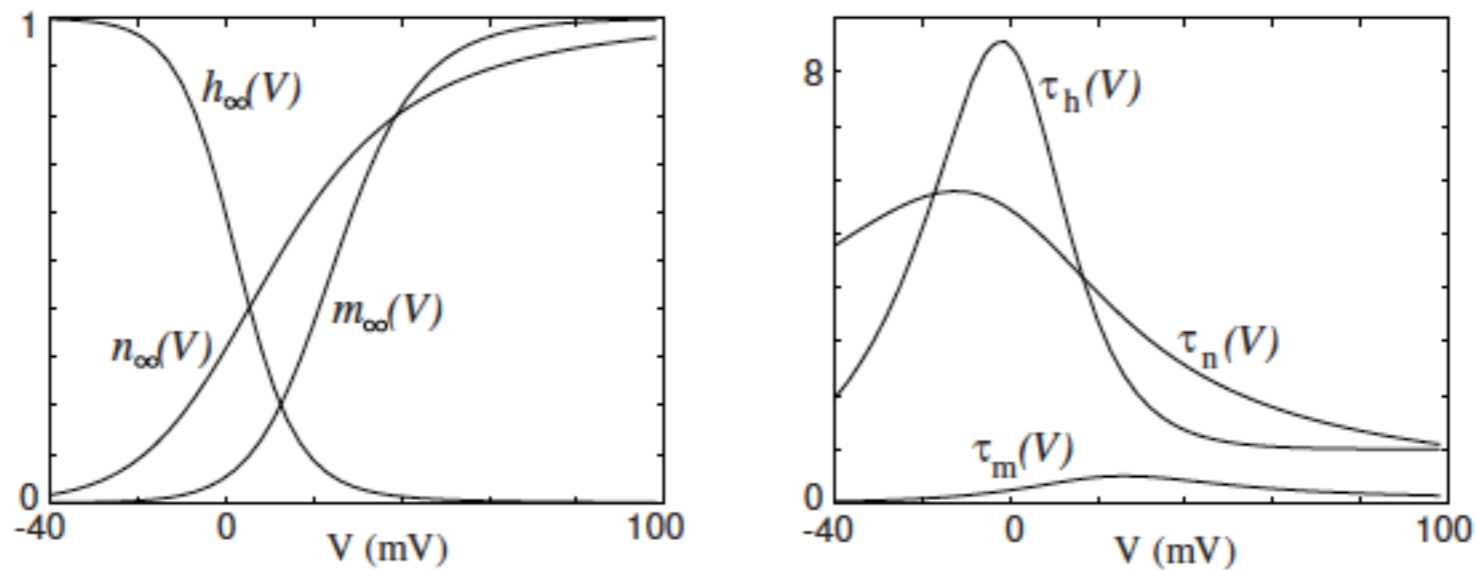


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = \overbrace{(n_\infty(V) - n) / \tau_n(V)}^{\text{activation}} ,$$

$$\dot{m} = (m_\infty(V) - m) / \tau_m(V) ,$$

$$\dot{h} = \overbrace{(h_\infty(V) - h) / \tau_h(V)}^{\text{inactivation}} ,$$

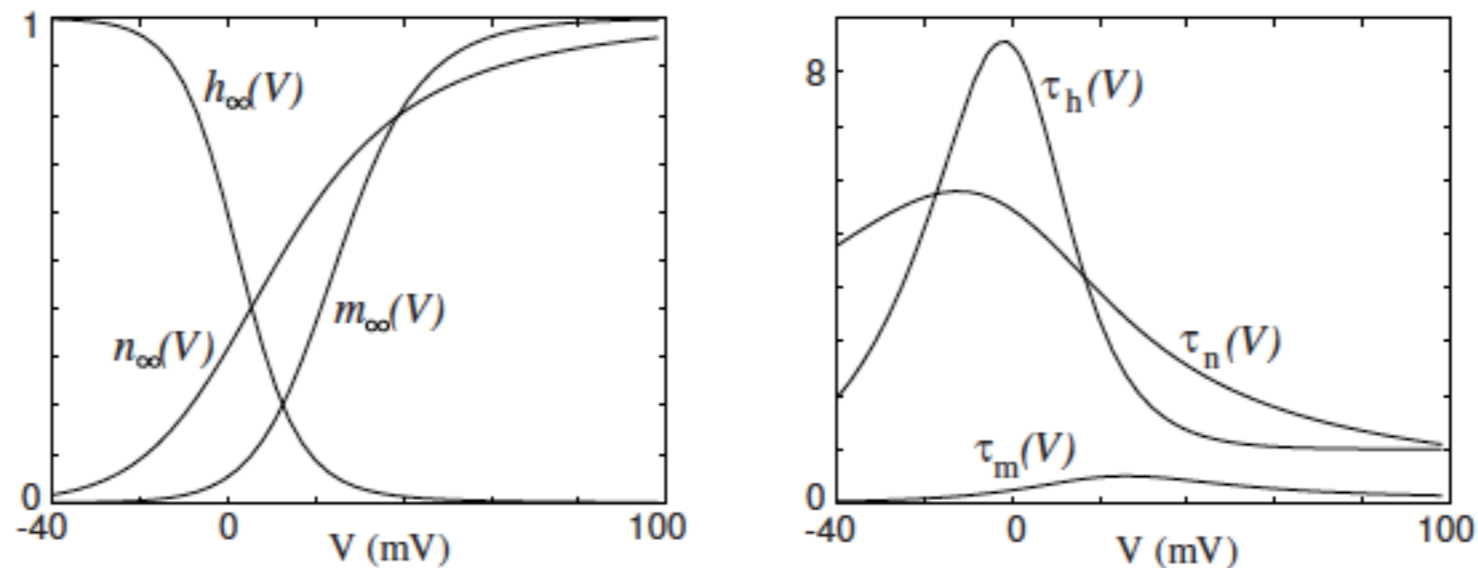


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = (n_\infty(V) - n) / \tau_n(V),$$

$$\dot{m} = (m_\infty(V) - m) / \tau_m(V),$$

$$\dot{h} = (h_\infty(V) - h) / \tau_h(V),$$

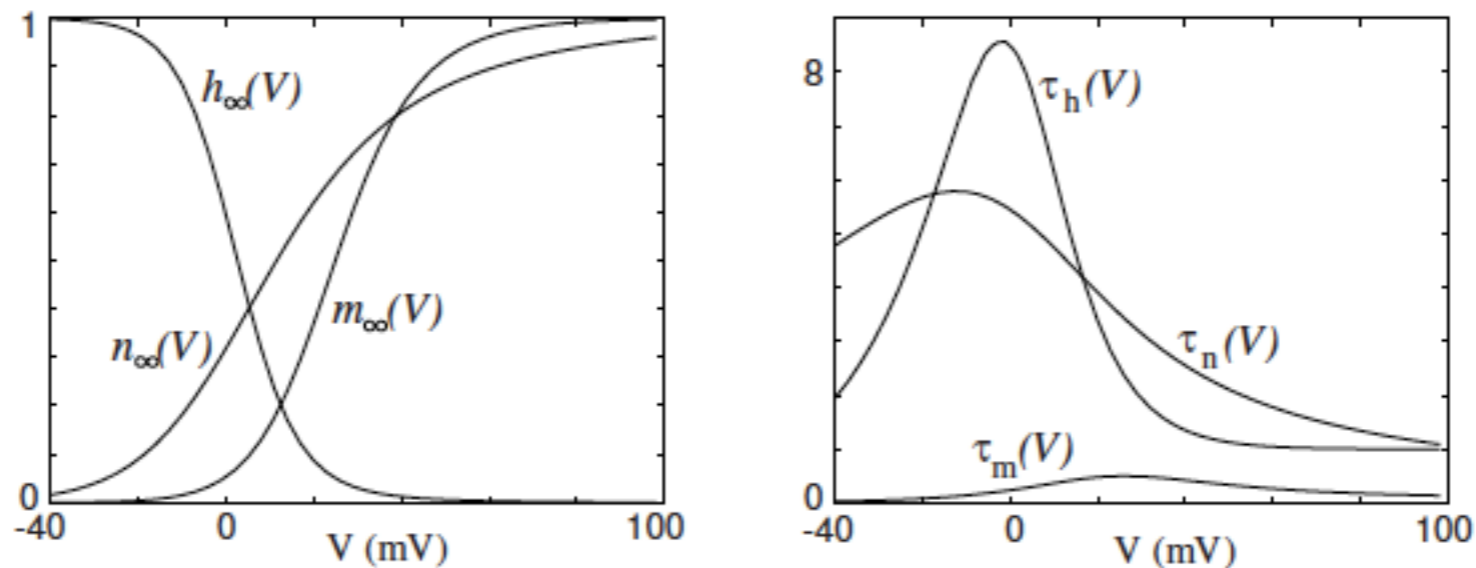
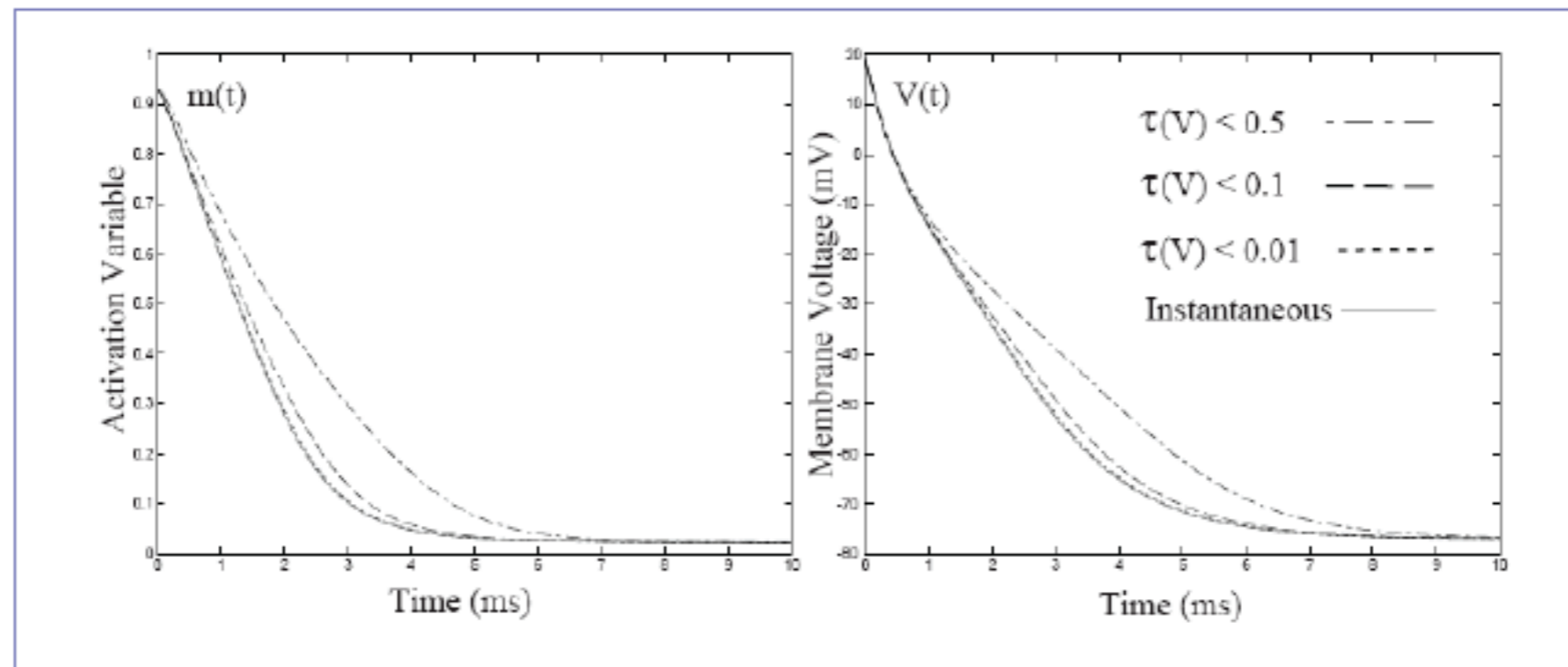


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C \frac{dV}{dt} = - G_{Na} m_{\infty}^3(V) (V - E_{Na}) - G_L (V - E_L) + I_{app}$$

$$\frac{dm}{dt} = \frac{m_{\infty}(V) - m}{\tau(V)}$$



One-dimensional neural model

- Persistent sodium model

$$C \frac{dV}{dt} = - G_{Na} m_{\infty}^3 (V) (V - E_{Na}) - G_L (V - E_L) + I_{app}$$

One-dimensional neural model

- Persistent sodium model

$$C \frac{dV}{dt} = - G_{Na} p_{\infty}^3(V) (V - E_{Na}) - G_L (V - E_L) + I_{app}$$

Two-dimensional neural model

$$\begin{aligned}
 C\dot{V} &= I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 \cancel{h} (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L} \\
 \dot{n} &= (n_\infty(V) - n) / \tau_n(V), \\
 \dot{m} &= (m_\infty(V) - m) / \tau_m(V), \\
 \dot{h} &= \cancel{(h_\infty(V) - h) / \tau_h(V)},
 \end{aligned}$$

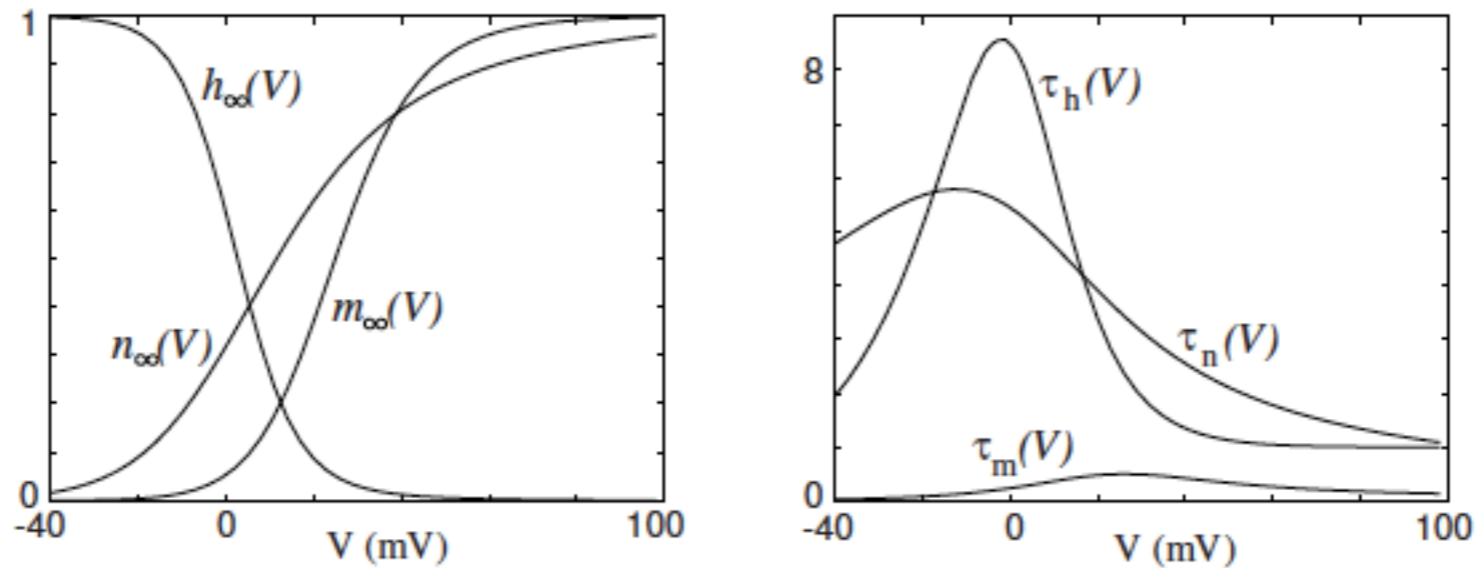


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Two-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 \cancel{h} (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = (n_\infty(V) - n) / \tau_n(V),$$

$$\dot{m} = (m_\infty(V) - m) / \tau_m(V),$$
~~$$\dot{h} = (h_\infty(V) - h) / \tau_h(V),$$~~

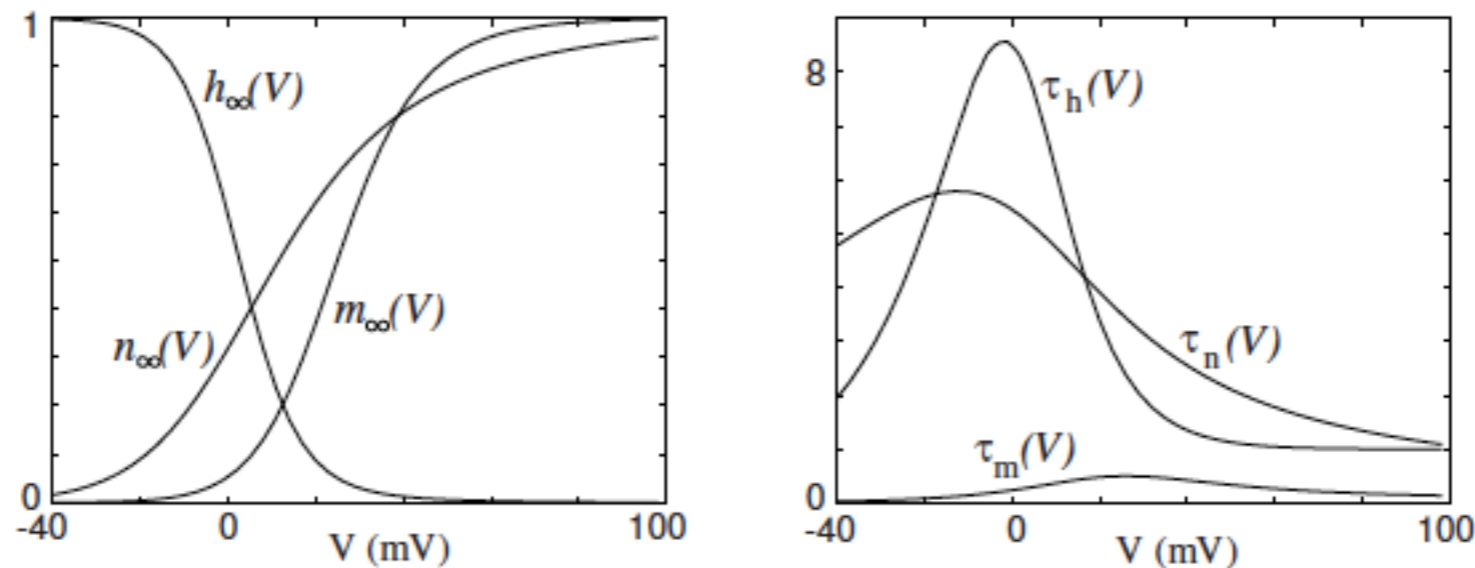


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Two-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L}$$

$$\dot{n} = (n_\infty(V) - n) / \tau_n(V),$$

$$\dot{m} = (m_\infty(V) - m) / \tau_m(V),$$
~~$$\dot{h} = (h_\infty(V) - h) / \tau_h(V),$$~~

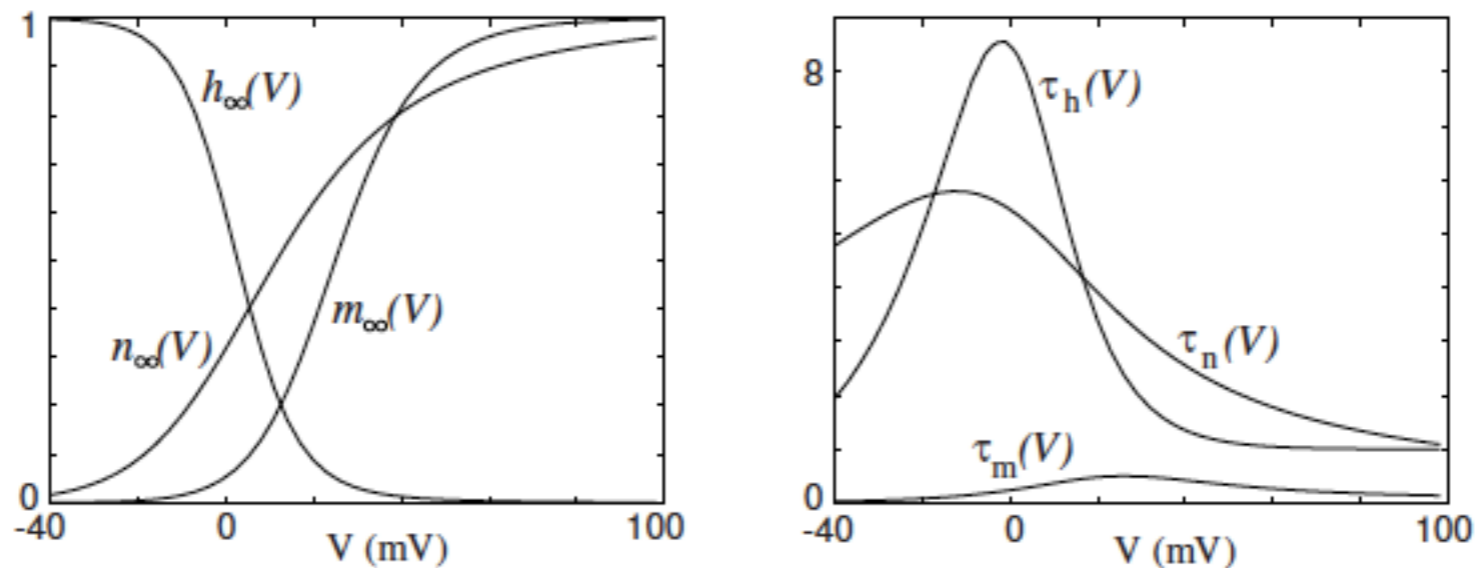


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

Two-dimensional neural model

- Persistent sodium & potassium model

$$C \frac{dV}{dt} = - G_{Na} m_{\infty}^3(V) (V - E_{Na}) - G_K n^4 (V - E_K) - G_L (V - E_L) + I_{app}$$

$$\frac{dn}{dt} = \frac{n_{\infty}(V) - n}{\tau_n(V)}$$

Two-dimensional neural model

- Persistent sodium & potassium model

$$C \frac{dV}{dt} = - G_{Na} p_{\infty}^3(V) (V - E_{Na}) - G_K n^4 (V - E_K) - G_L (V - E_L) + I_{app}$$

$$\frac{dn}{dt} = \frac{n_{\infty}(V) - n}{\tau_n(V)}$$

Two-dimensional neural model

- Morris-Lecar model

$$C \frac{dV}{dt} = - G_{Ca} m_{\infty}(V) (V - E_{Ca}) - G_K w (V - E_K) - G_L (V - E_L) + I_{app}$$

$$\frac{dw}{dt} = \frac{w_{\infty}(V) - w}{\tau_w(V)}$$

Two-dimensional neural model

- Morris-Lecar model

$$m_{\infty}(V) = 0.5 \left(1 + \tanh \frac{V + 1}{15} \right)$$

$$w_{\infty}(V) = 0.5 \left(1 + \tanh \frac{V}{30} \right)$$

$$\tau_w(V) = \frac{5}{\cosh(V / 60)}$$

Two-dimensional neural model

- FitzHugh-Nagumo (FHN) model

$$\frac{dV}{dt} = V - \frac{V^3}{3} - W + I$$

$$\frac{dW}{dt} = \phi (V + a - bW)$$

a, b, ϕ : dimensionless & positive

$\phi \ll 1$: inverse of a time constant

Two-dimensional neural model

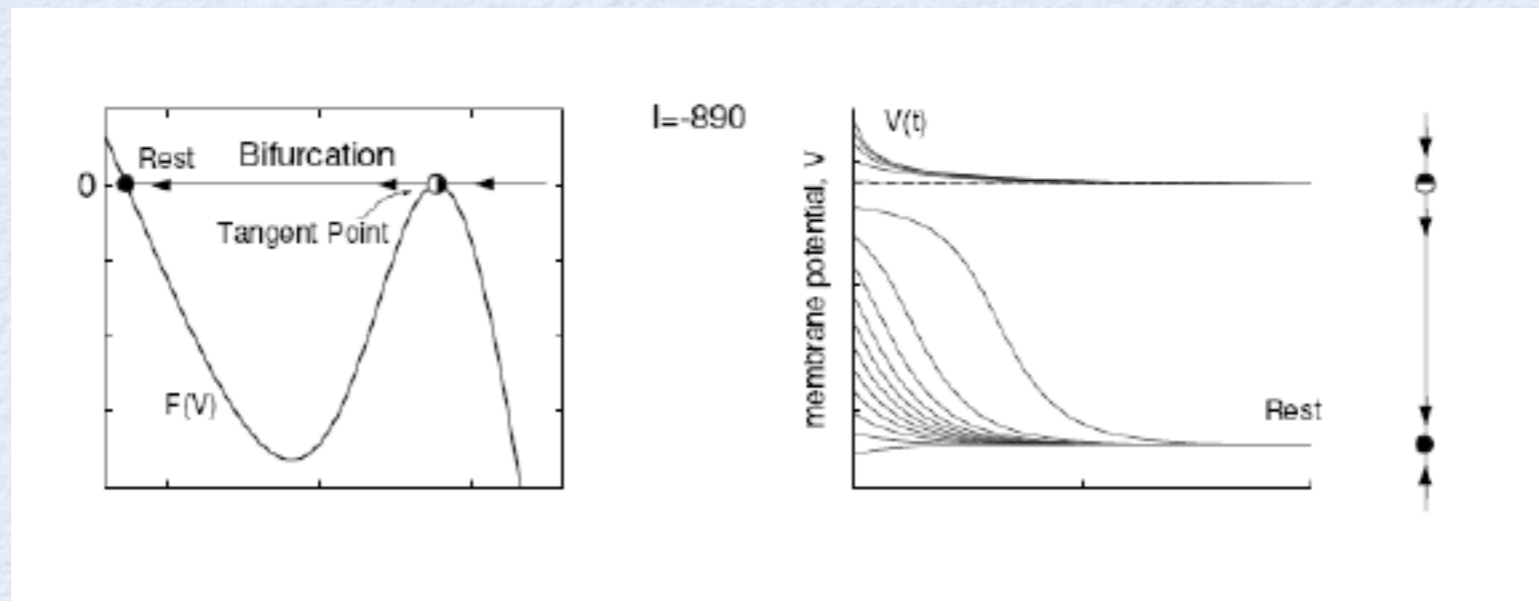
- Quadratic integrate-and-fire model

$$\frac{dV}{dt} = I + V^2 \quad \text{if } V \geq V_{peak}, \text{ then } V \leftarrow V_{reset}$$

$$\text{Equilibria: } V_{rest} = -\sqrt{I} \quad \text{and} \quad V_{thresh} = +\sqrt{I}$$

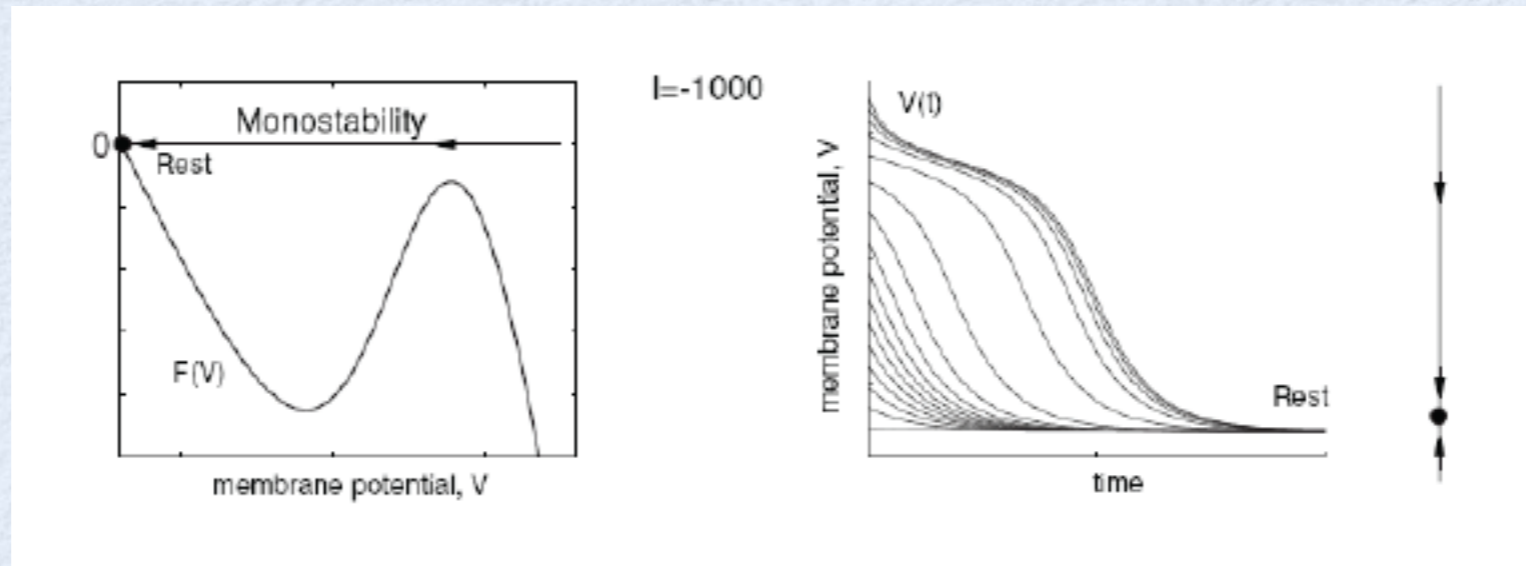
Two-dimensional neural model

- Quadratic integrate-and-fire model



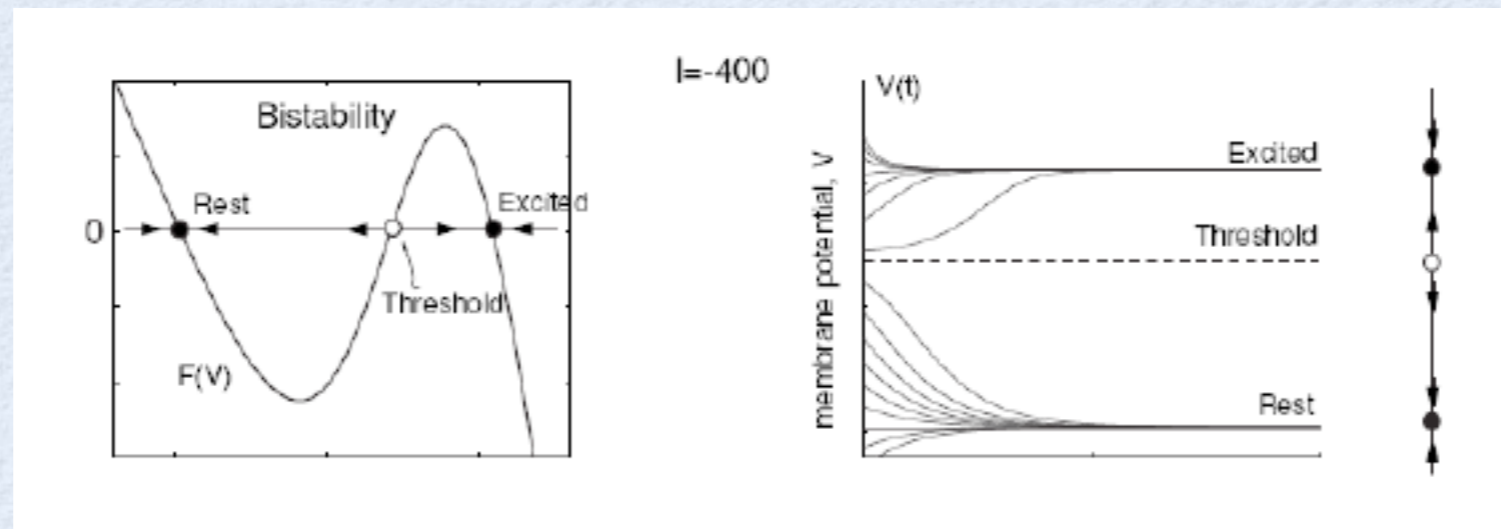
Two-dimensional neural model

- Quadratic integrate-and-fire model



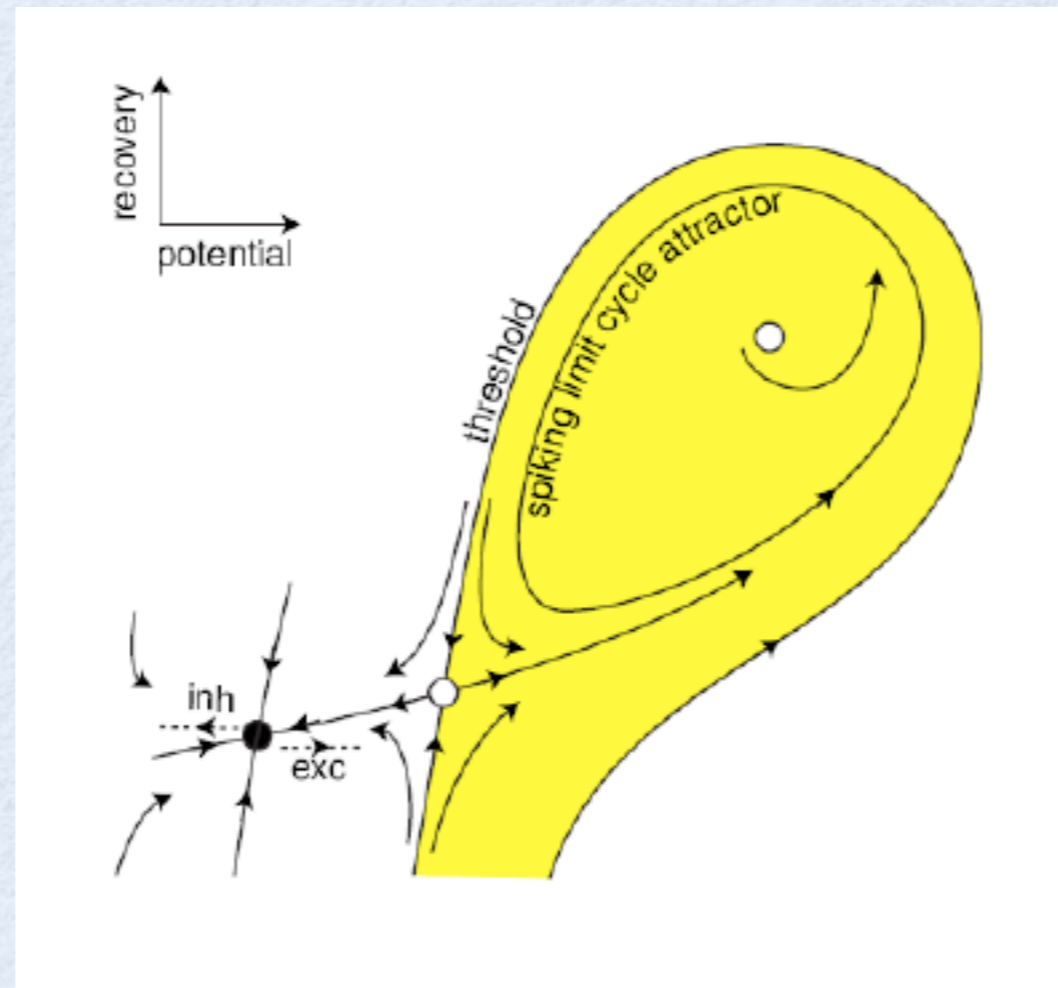
Two-dimensional neural model

- Quadratic integrate-and-fire model



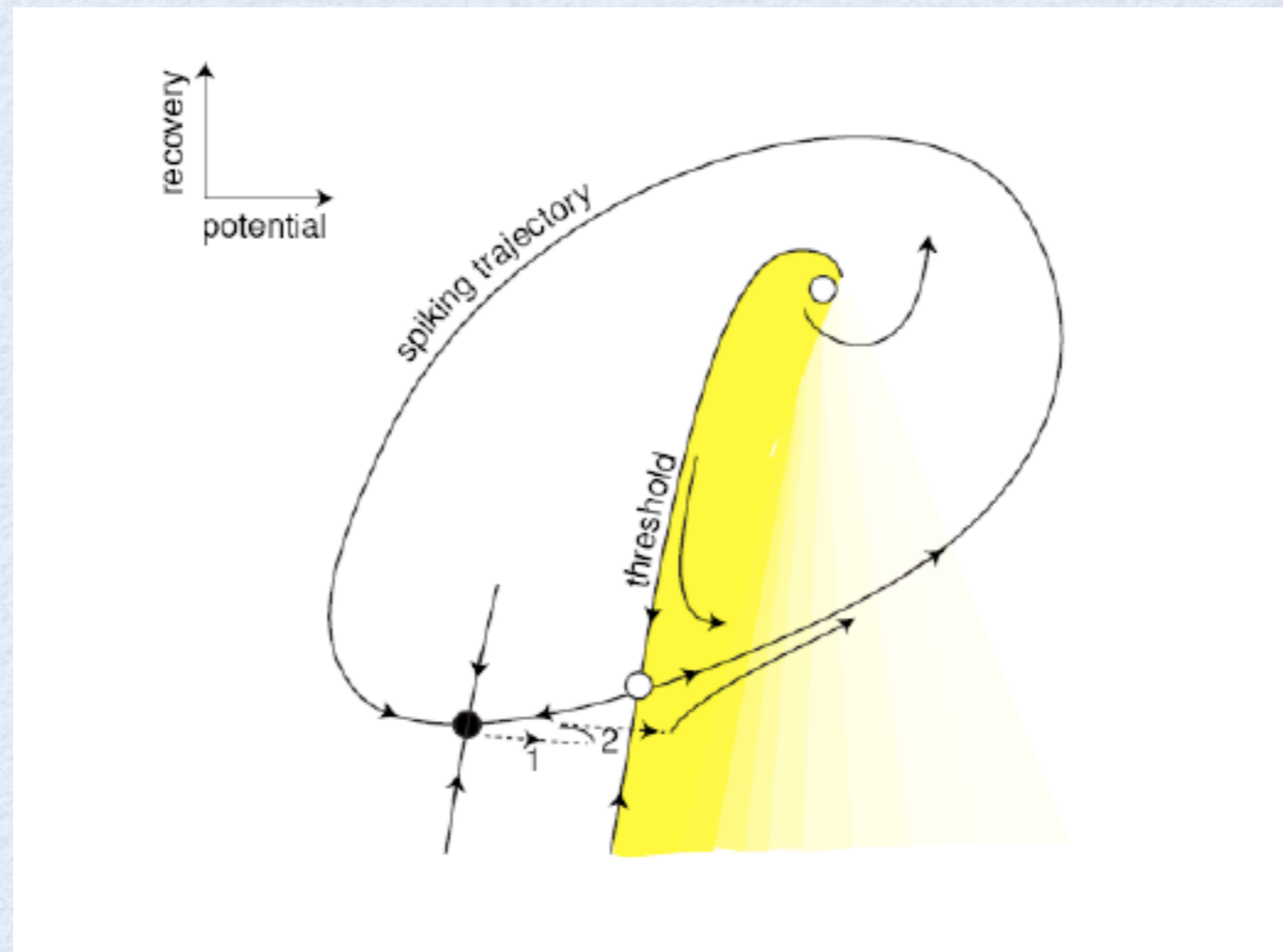
Bifurcations

- Saddle-node



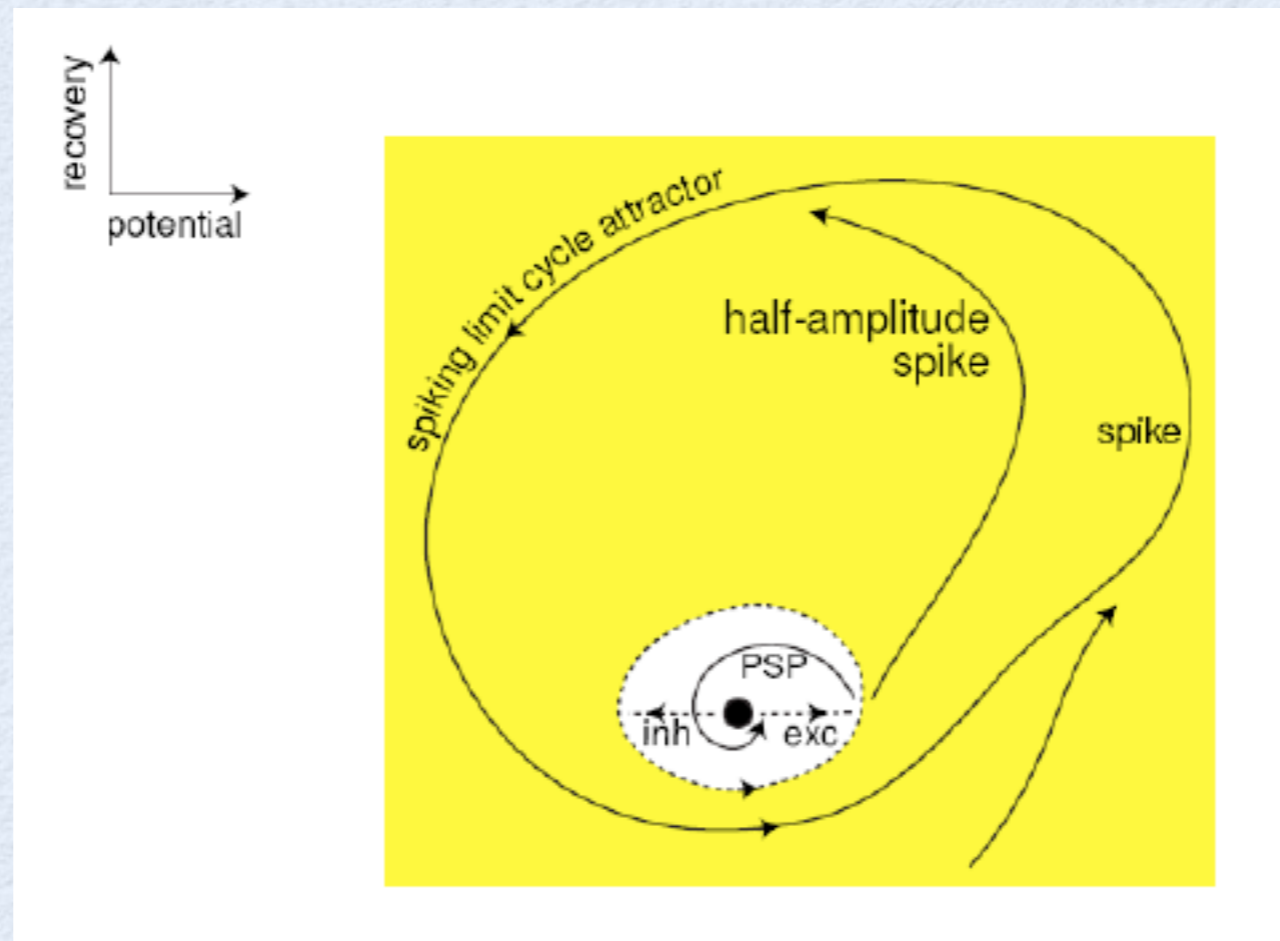
Bifurcations

- Saddle-node on an invariant circle (SNIC)



Bifurcations

- Subcritical Hopf (Andronov-Hopf)



Bifurcations

- Supercritical Hopf (Andronov-Hopf)

