Introduction to Computational Neuroscience

Biol 698
Math 635
Biol 498
Math 430

Bibliography:

"Mathematical Foundations of Neuroscience", by G. B. Ermentrout & D. H. Terman - Springer (2010), 1st edition. ISBN 978-0-387-87707-5

* "Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting", by Eugene M. Izhikevich. The MIT Press, 2007. ISBN 0-262-09043-8

Figures were taken from some of these books

Overview

One-dimensional models

- Reduction of the Hodgkin-Huxley model to a one-dimensional model (review)
- One-dimensional models
- Equilibria and stability
- Bistability and hysteresis
- Topological equivalence
- Bifurcations

One-dimensional neural model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}} \\
\dot{n} &= (n_\infty(V) - n)/\tau_n(V)\,, \\
\dot{m} &= (m_\infty(V) - m)/\tau_m(V)\,, \\
\dot{h} &= (h_\infty(V) - h)/\tau_h(V)\,,
\end{aligned}
$$

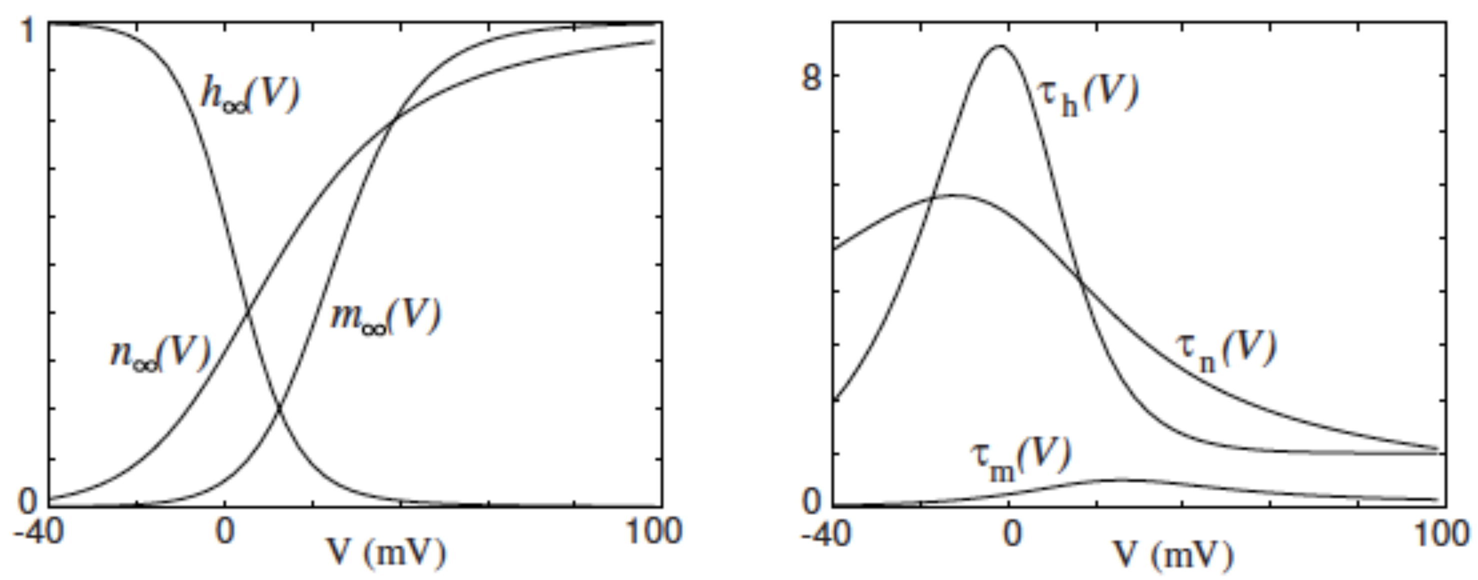

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}} (V - E_{\mathrm{L}})}^{I_{\mathrm{L}}}$$

$$\begin{aligned}
\dot{n} &= (n_\infty(V) - n)/\tau_n(V)\,, \\
\dot{m} &= (m_\infty(V) - m)/\tau_m(V)\,, \\
\dot{h} &= (h_\infty(V) - h)/\tau_h(V)\,,
\end{aligned}$$

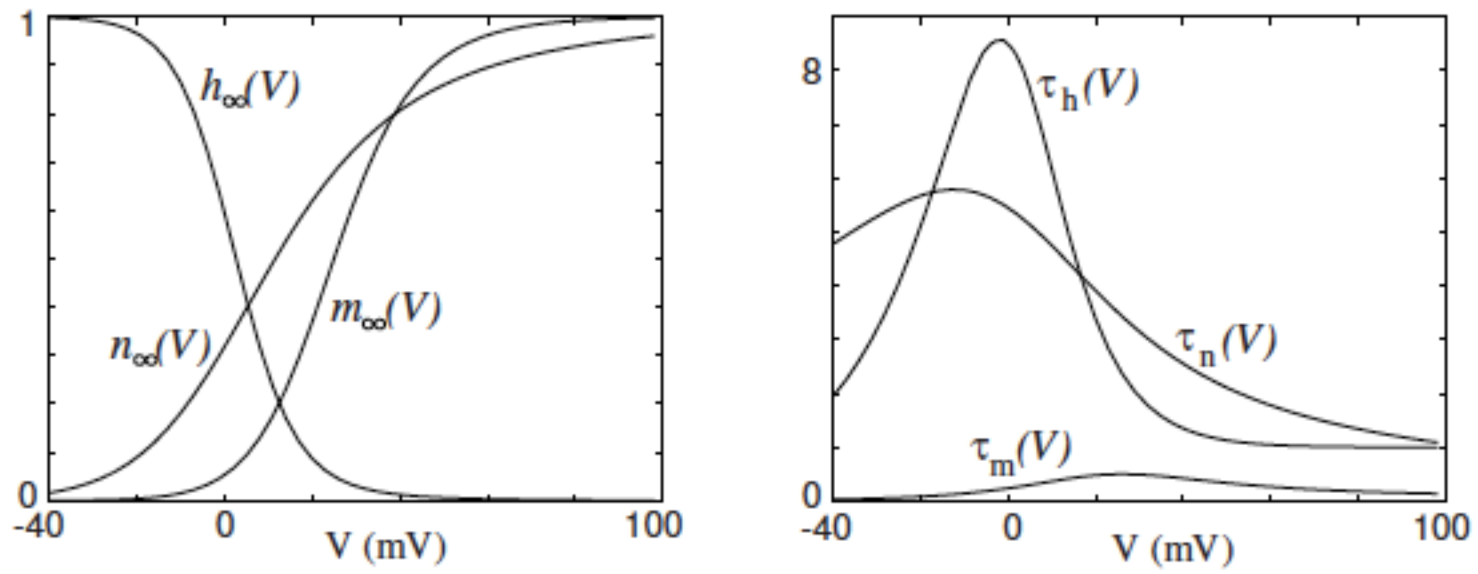

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

- The passive membrane equation

$$C\frac{dV}{dt} = -G_L\,(V - E_L) + I_{app}$$

One-dimensional neural model

$$
\begin{aligned}
C\dot{V} &= I - \overbrace{\bar{g}_K n^4 (V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_L (V - E_L)}^{I_L} \\
\dot{n} &= (n_\infty(V) - n)/\tau_n(V)\,, \\
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\dot{h} &= (h_\infty(V) - h)/\tau_h(V)\,,
\end{aligned}
$$

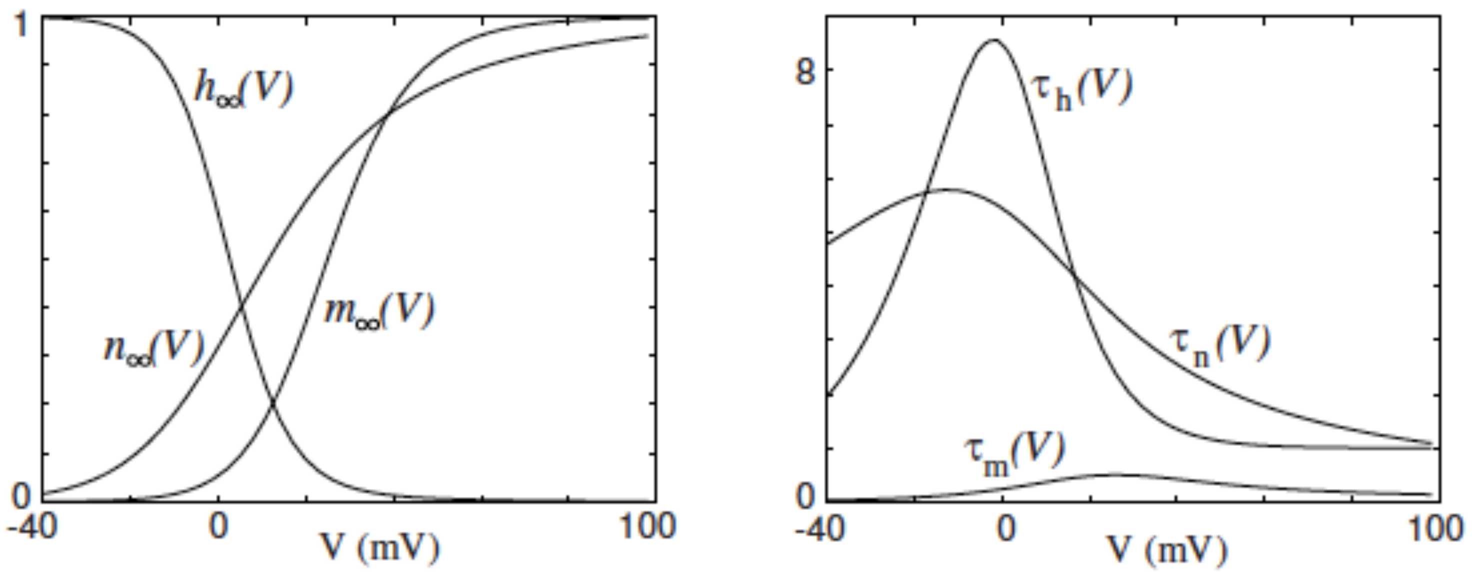

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_{\mathrm{K}} n^4 (V - E_{\mathrm{K}})}^{I_{\mathrm{K}}} - \overbrace{\bar{g}_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}})}^{I_{\mathrm{Na}}} - \overbrace{g_{\mathrm{L}}(V - E_{\mathrm{L}})}^{I_{\mathrm{L}}}$$

$$\dot{n} = (n_\infty(V) - n)/\tau_n(V)\,,$$

$$\dot{m} = (m_\infty(V) - m)/\tau_m(V)\,,$$

$$\dot{h} = (h_\infty(V) - h)/\tau_h(V)\,,$$

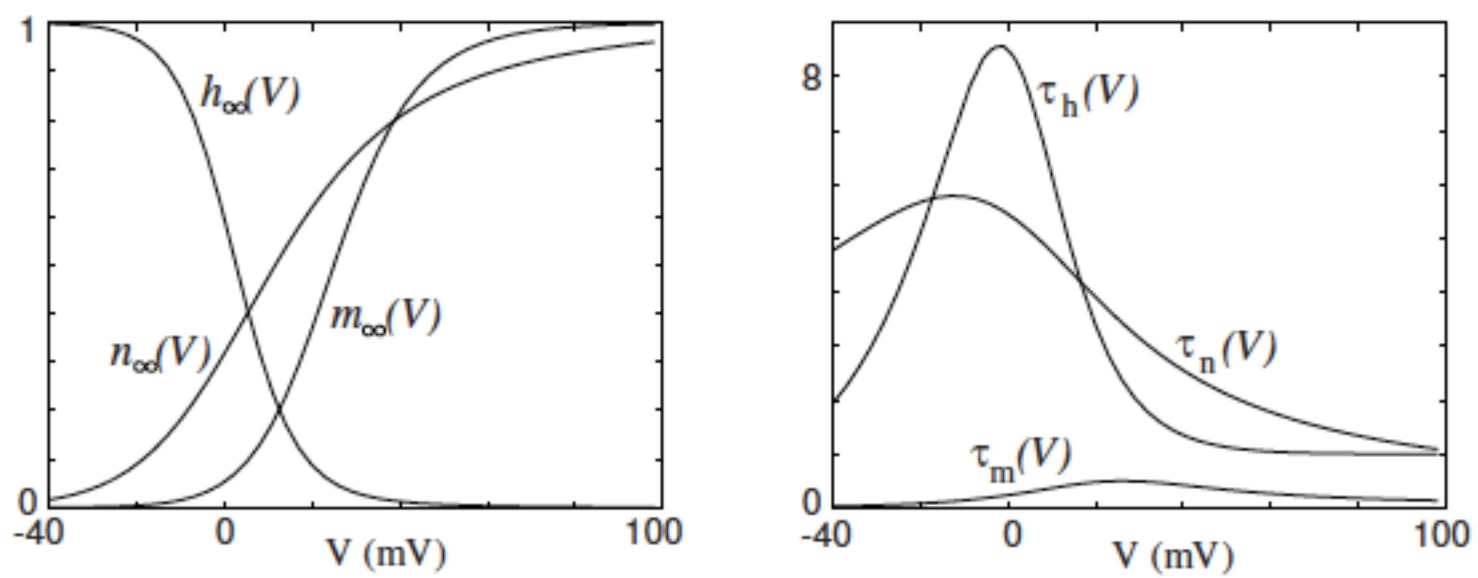

Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\dot{V} = I - \overbrace{\bar{g}_{K} n^4 (V - E_{K})}^{I_{K}} - \overbrace{\bar{g}_{Na} m^3 h (V - E_{Na})}^{I_{Na}} - \overbrace{g_{L}(V - E_{L})}^{I_{L}}$$

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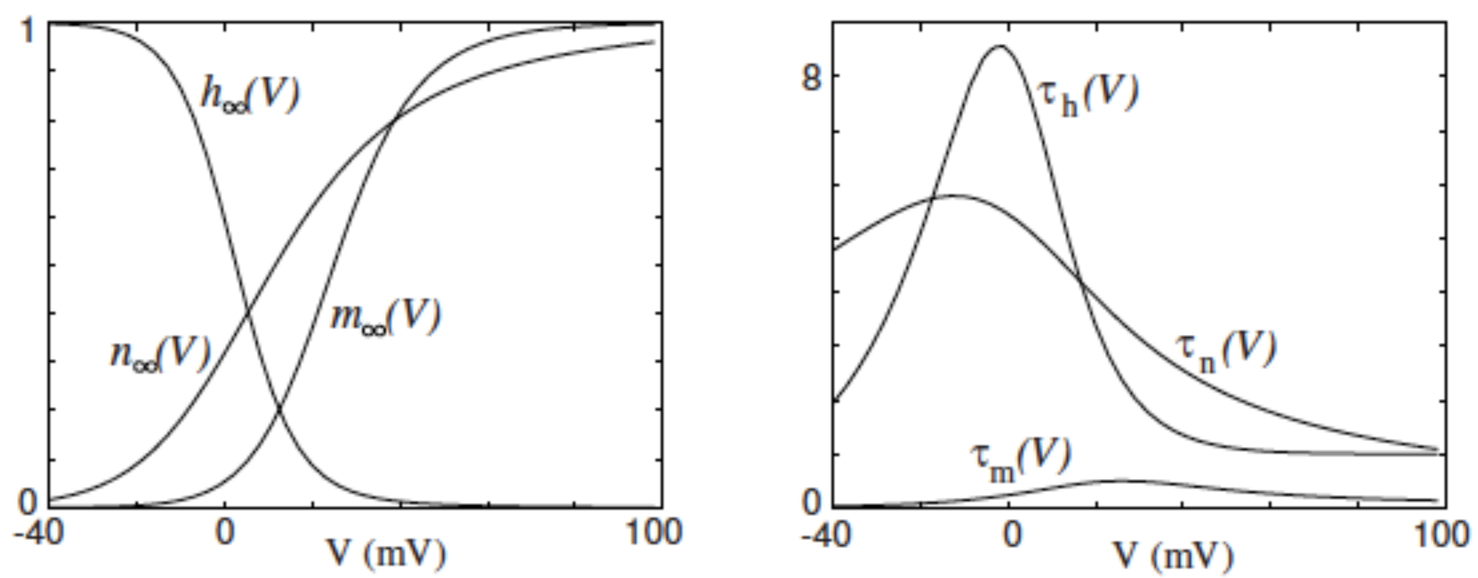
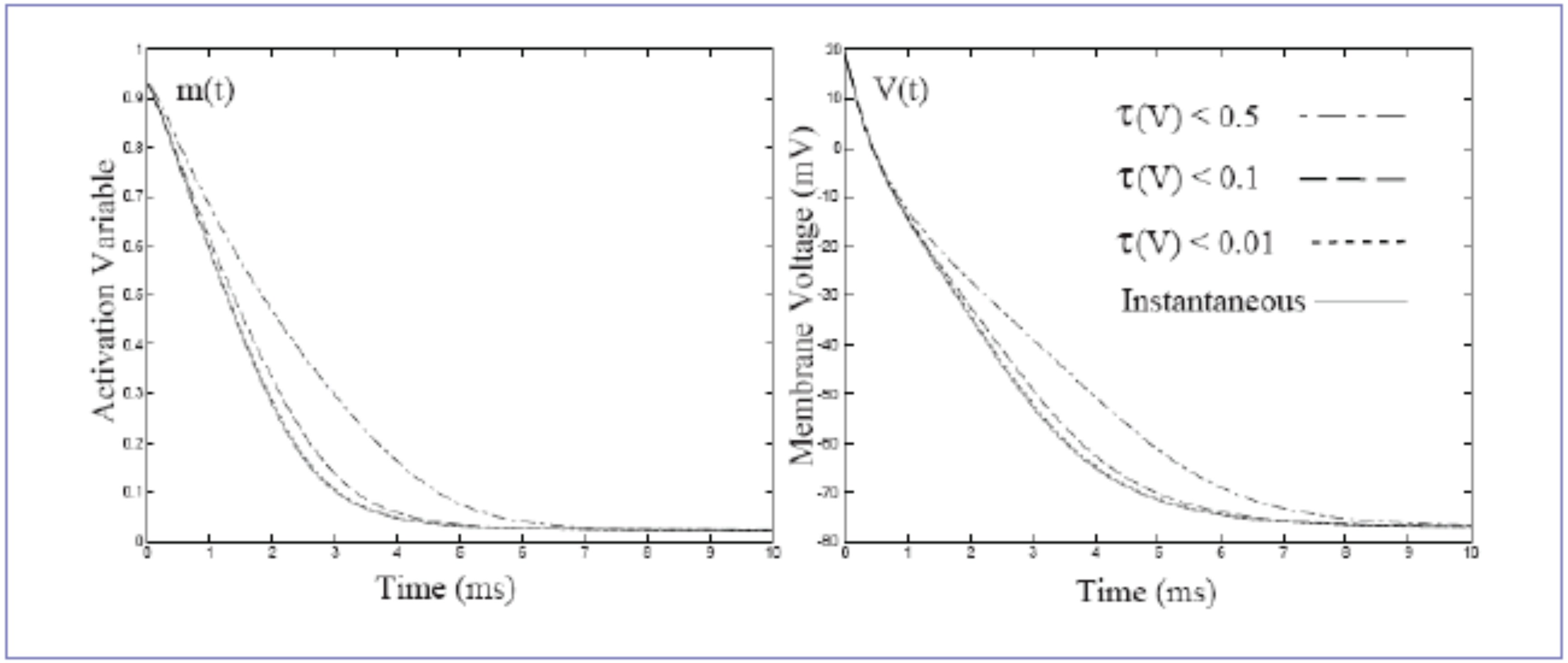


Figure 2.13: Steady-state (in)activation functions (left) and voltage-dependent time constants (right) in the Hodgkin-Huxley model.

One-dimensional neural model

$$C\frac{dV}{dt} = -G_{Na} m_\infty^3(V)(V - E_{Na}) - G_L(V - E_L) + I_{app}$$

$$\frac{dm}{dt} = \frac{m_\infty(V) - m}{\tau(V)}$$



One-dimensional neural model

- Persistent sodium model

$$C\frac{dV}{dt} = -G_{Na}m_{\infty}^{3}(V)\,(V - E_{Na}) - G_{L}\,(V - E_{L}) + I_{app}$$

One-dimensional neural model

- Persistent sodium model

$$C\frac{dV}{dt} = -G_{Na}p_{\infty}^{3}(V)(V - E_{Na}) - G_L(V - E_L) + I_{app}$$

One-dimensional neural model

- Models with one gating variable

$$C\frac{dV}{dt} = -G_X x_\infty(V)(V - E_X) - G_L(V - E_L) + I_{app}$$

One-dimensional neural model

- Models with one gating variable

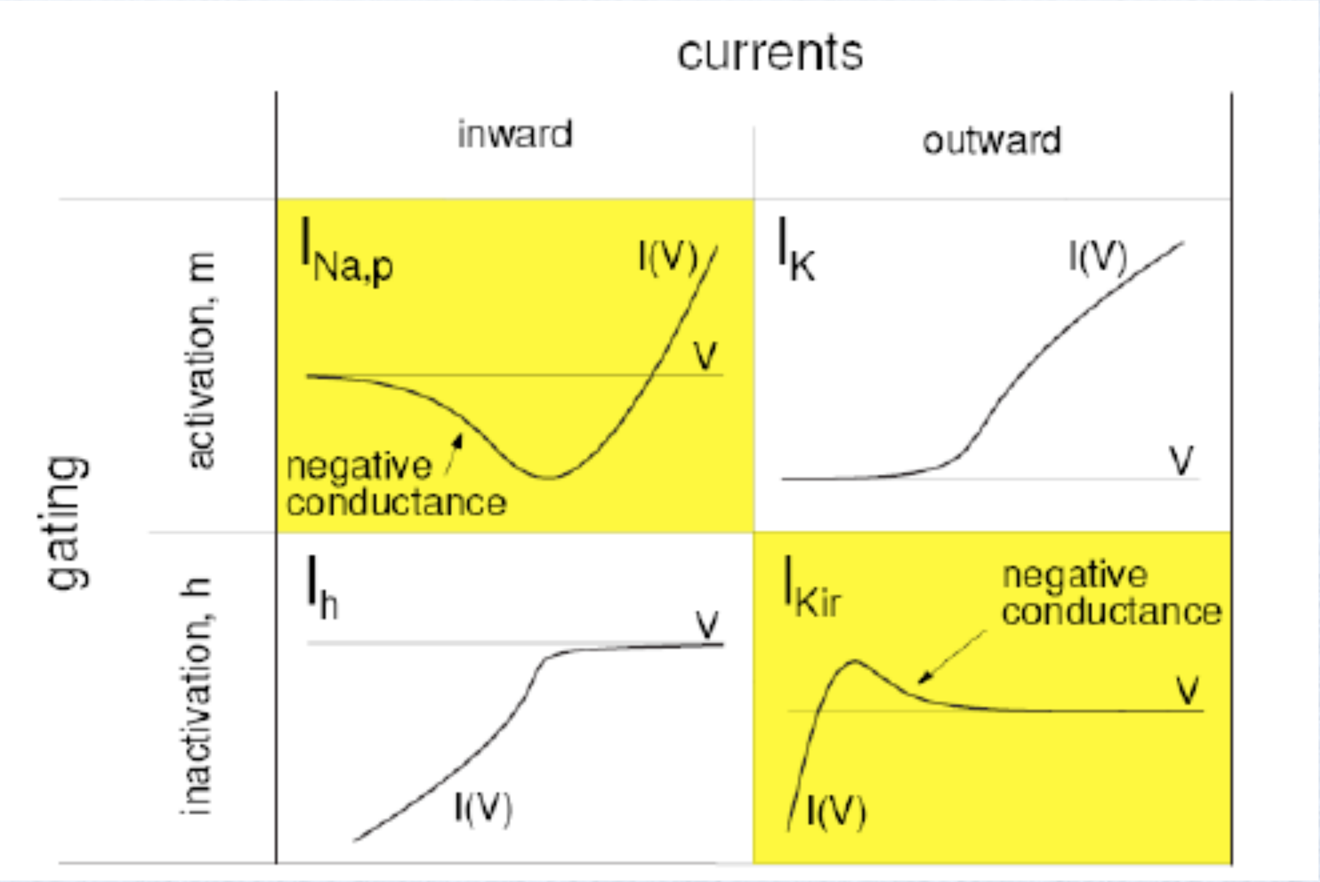
$$C\frac{dV}{dt} = -G_X x_\infty(V)(V - E_X) - G_L(V - E_L) + I_{app}$$

Gating		Current: inward	Current: outward
	activation	$I_{Na,p}$	I_K
	inactivation	I_h	I_{Kir}

One-dimensional neural model

$$C\frac{dV}{dt} = -\ G_X x_\infty(V)\ (V - E_X)\ -\ G_L\ (V - E_L)\ + I_{app}$$

- Negative conductance in the I-V relation creates positive feedback between V and x
- Amplifying role in neural dynamics
- Amplifying currents



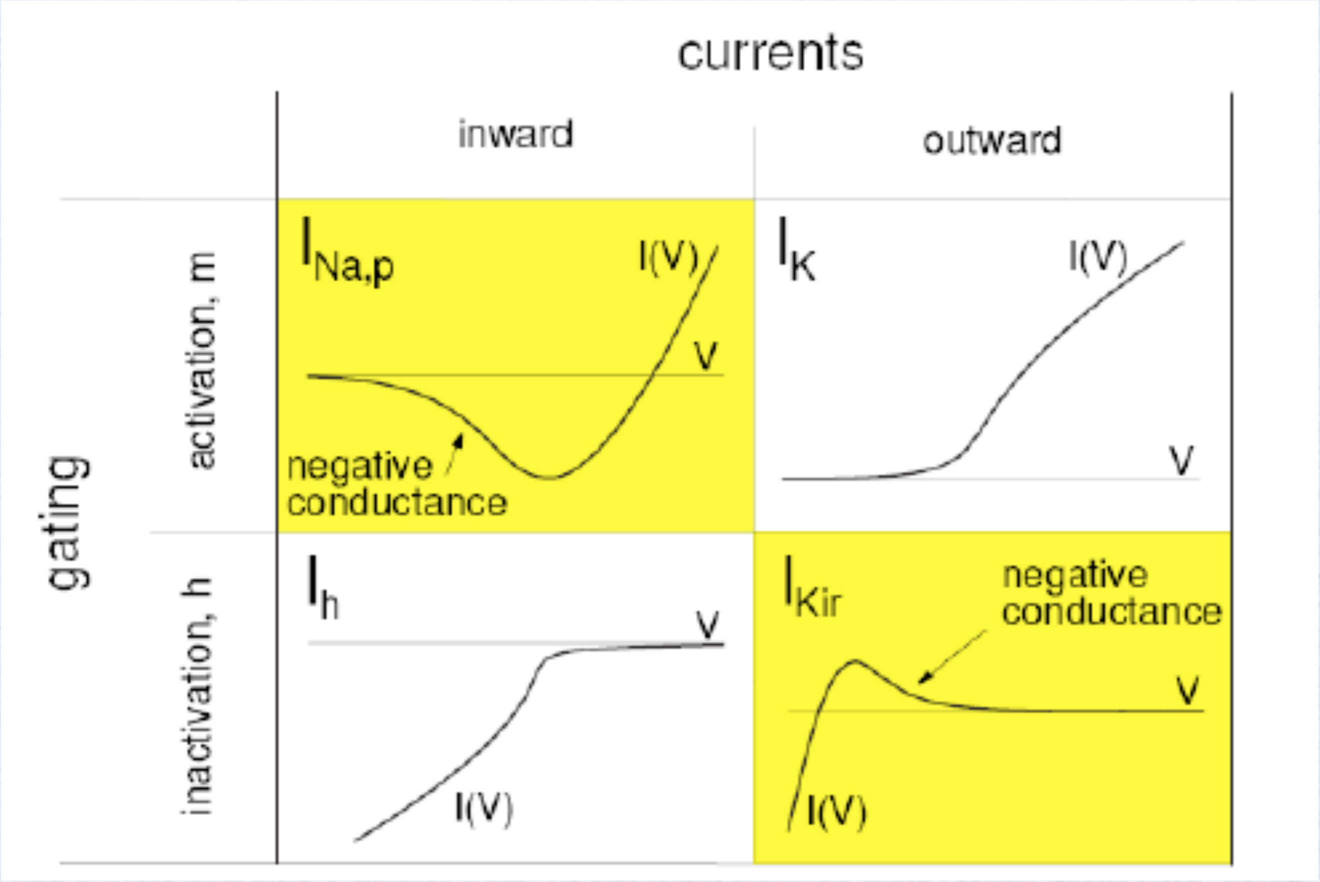
One-dimensional neural model

$$C\frac{dV}{dt} = - G_X x_\infty(V)\,(V - E_X) - G_L\,(V - E_L) + I_{app}$$

- The currents in the other group have negative feedback between V and x

- Damped oscillations of the membrane potential

- Resonant currents



One-dimensional neural model

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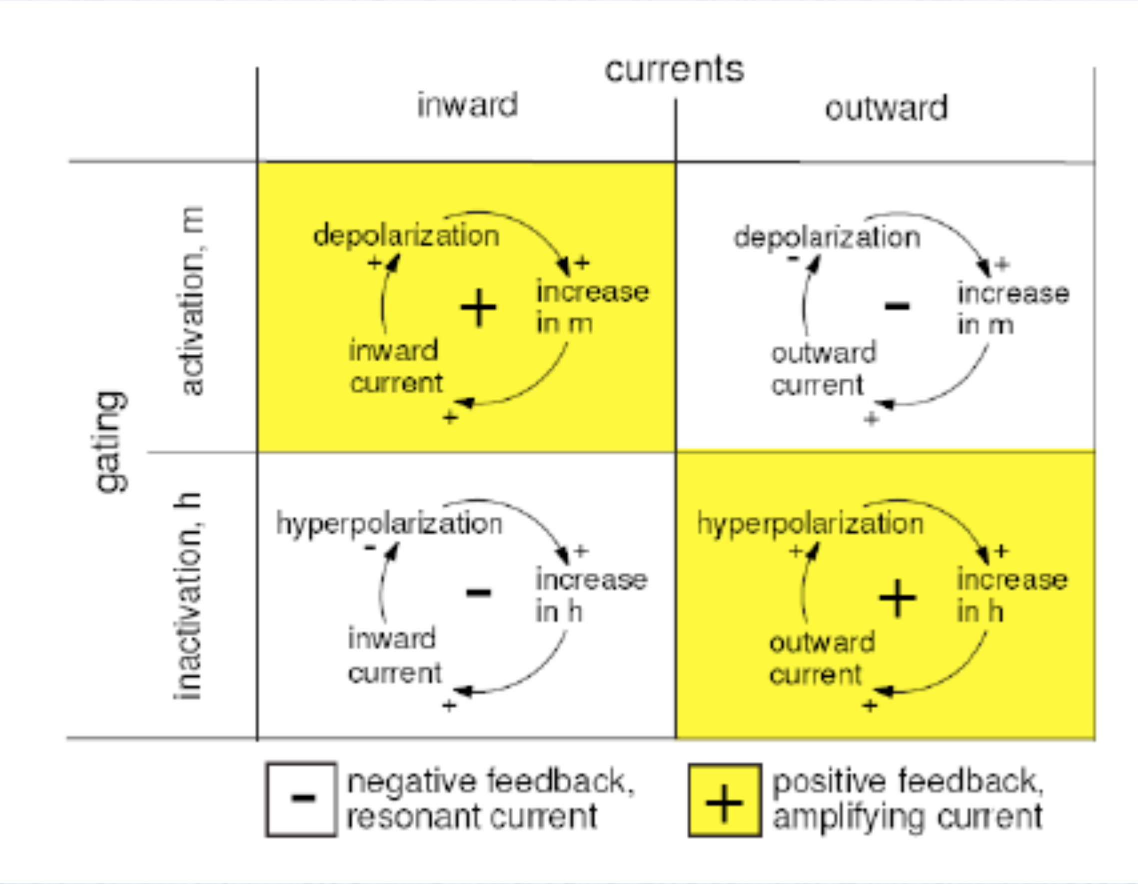

One-dimensional neural model

$$C\frac{dV}{dt} = -G_X x_\infty(V)(V - E_X) - G_L(V - E_L) + I_{app}$$

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- Damped oscillations of the membrane potential
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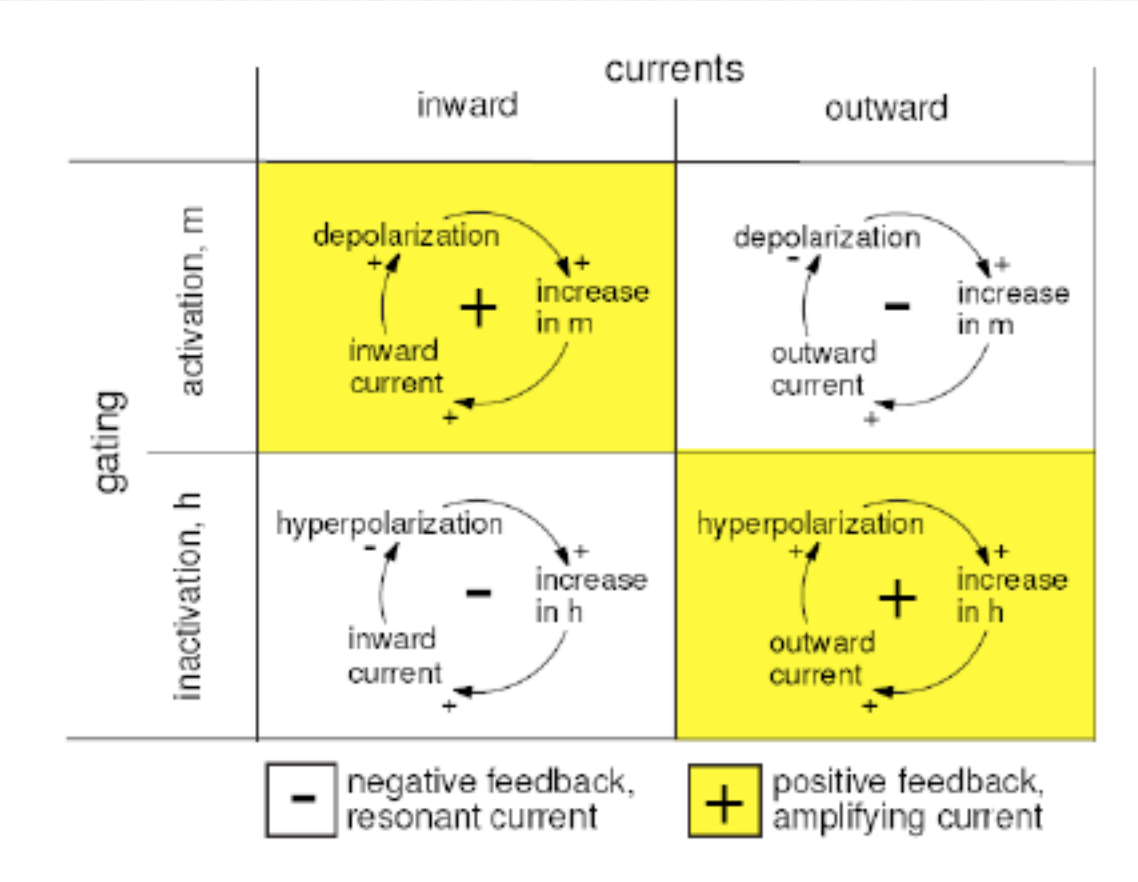


One-dimensional neural model

- $I_{Na,p}$ model

$$C\frac{dV}{dt} = - G_{Na}p_\infty^3(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I$$

$$p_\infty(V) = \frac{1}{1 + e^{\frac{V_{1/2}-V}{V_{sl}}}}$$

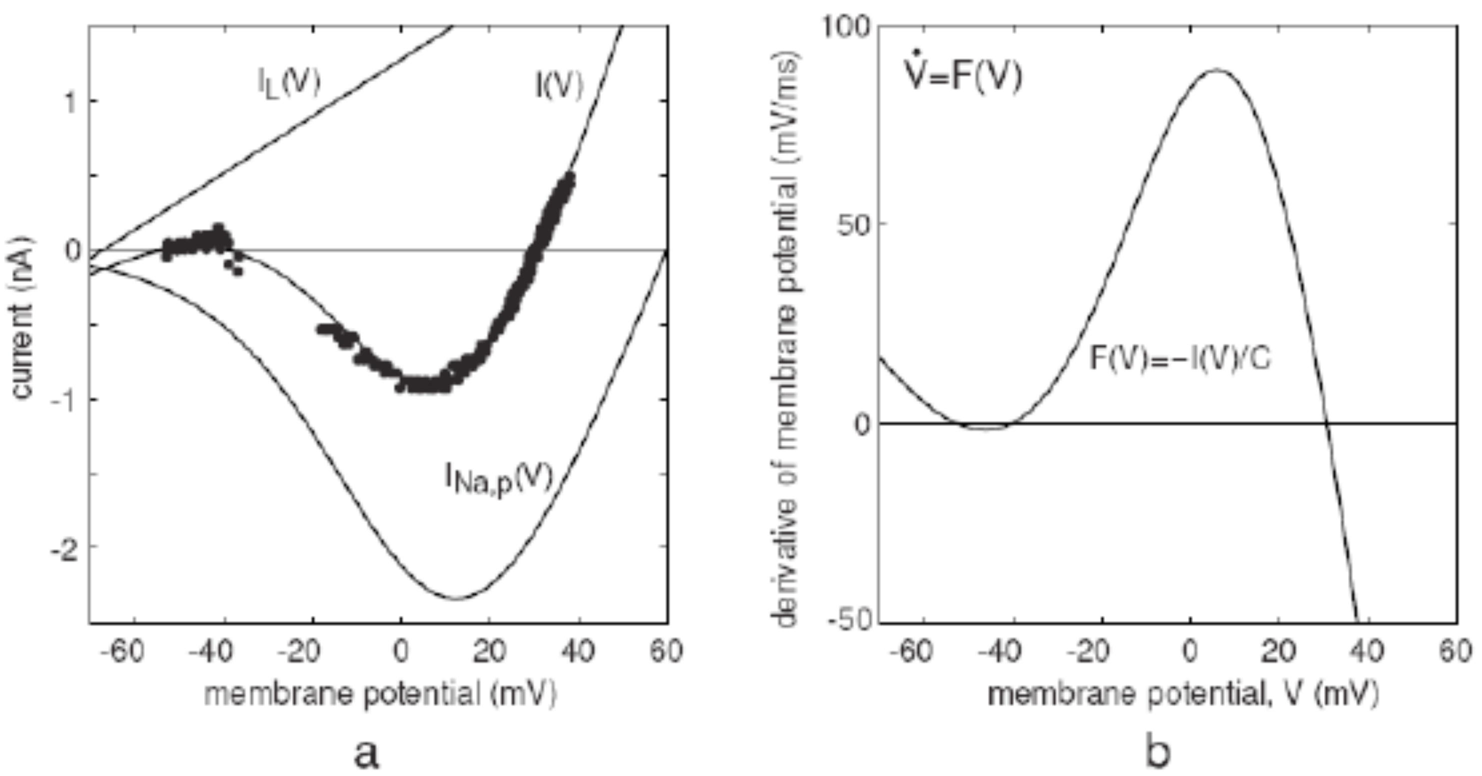
$C = 10\mu F$	$I = 0\ pA$	$G_L = 19\ mS$	$E_L = -67\ mV$
$G_{Na} = 74\ mS$	$V_{1/2} = 1.5mV$	$V_{sl} = 16mV$	$E_{Na} = 60\ mV$

- Experimental parameter values obtained using whole-patch clamp recordings of a layer V pyramidal neuron in the visual cortex of a rat at room temperature.

One-dimensional neural model

- $I_{Na,p}$ model

$$C\frac{dV}{dt} = -G_{Na}\, p_{\infty}(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I$$



One-dimensional neural model

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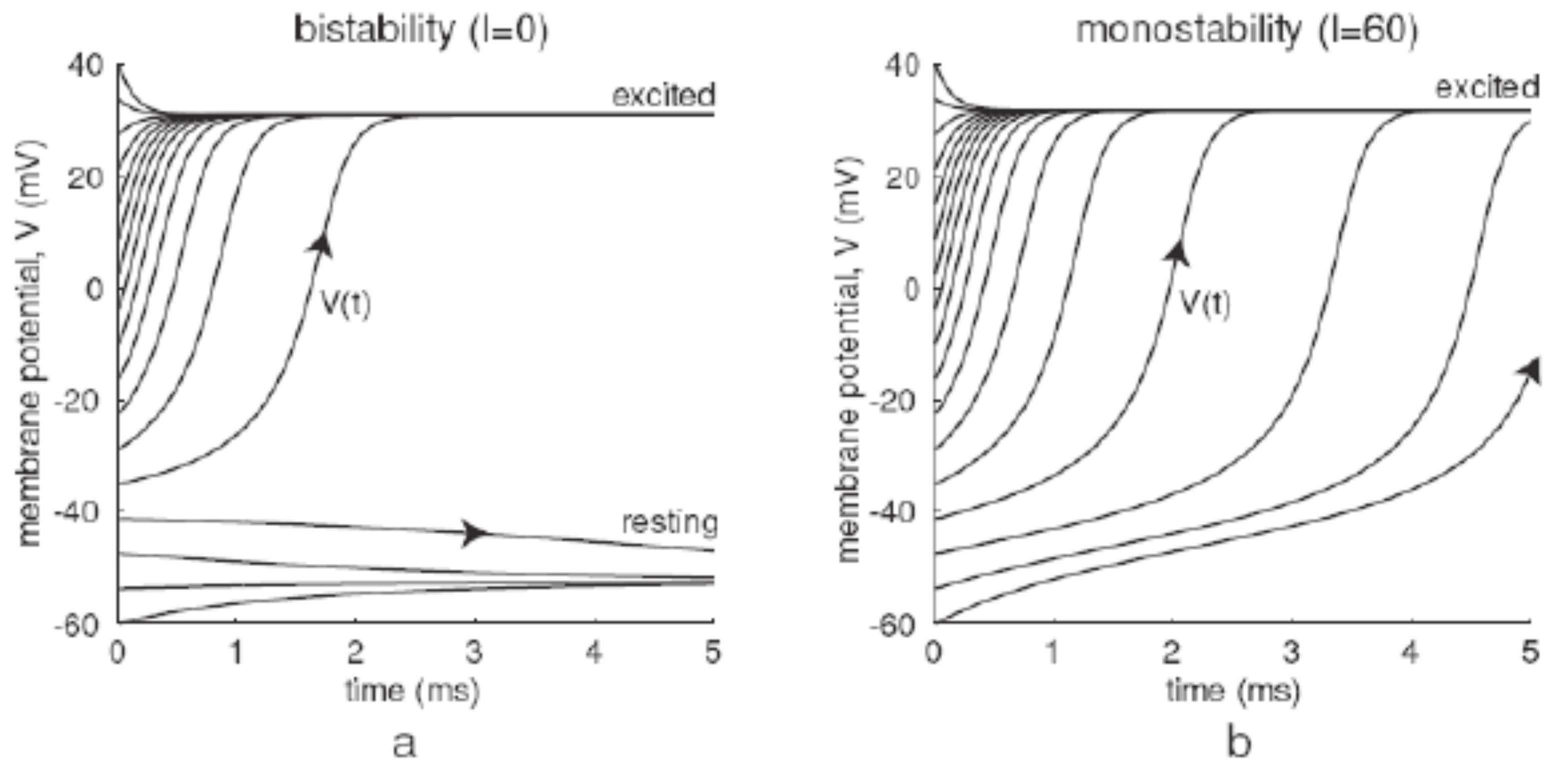
$$C\frac{dV}{dt} = -G_{Na}\, p_{\infty}(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I$$



Typical voltage trajectories of the $I_{\mathrm{Na,p}}$-model

One-dimensional neural model

- Equilibria

$$C\frac{dV}{dt} = -\,G_{Na}\,p_{\infty}(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I$$

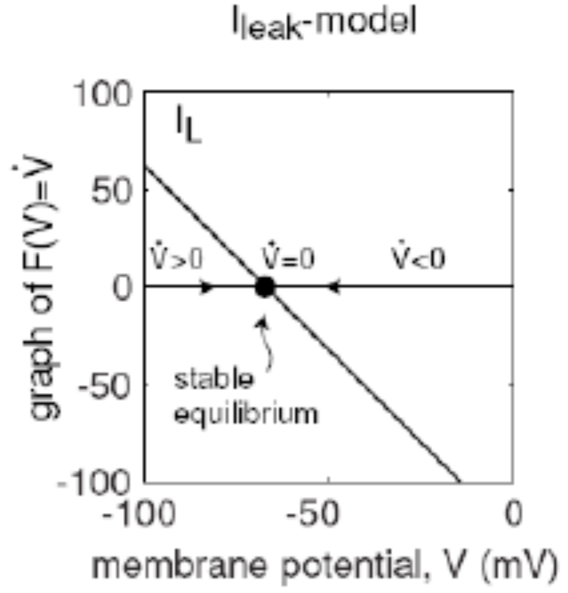
$$\frac{dV}{dt} = 0 \quad \Rightarrow \quad V = V_{eq}$$

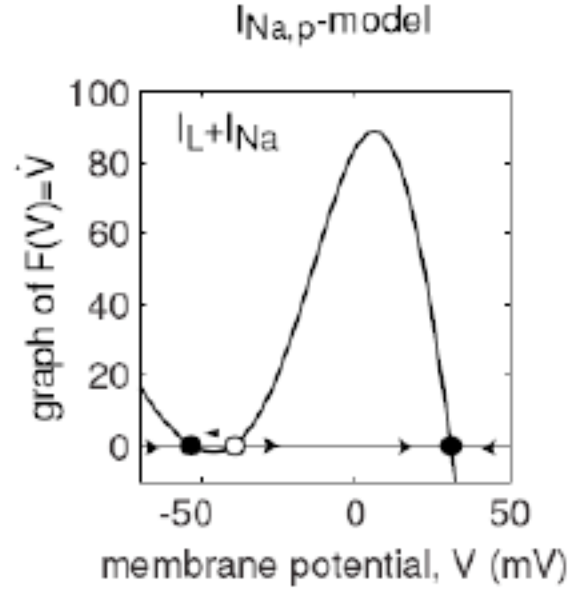
$$\frac{dV}{dt} = F(V) \qquad \rightarrow \qquad F(V_{eq}) = 0$$

One-dimensional neural model

- Equilibria

$$C\frac{dV}{dt} = -G_{Na}\, p_\infty(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I$$

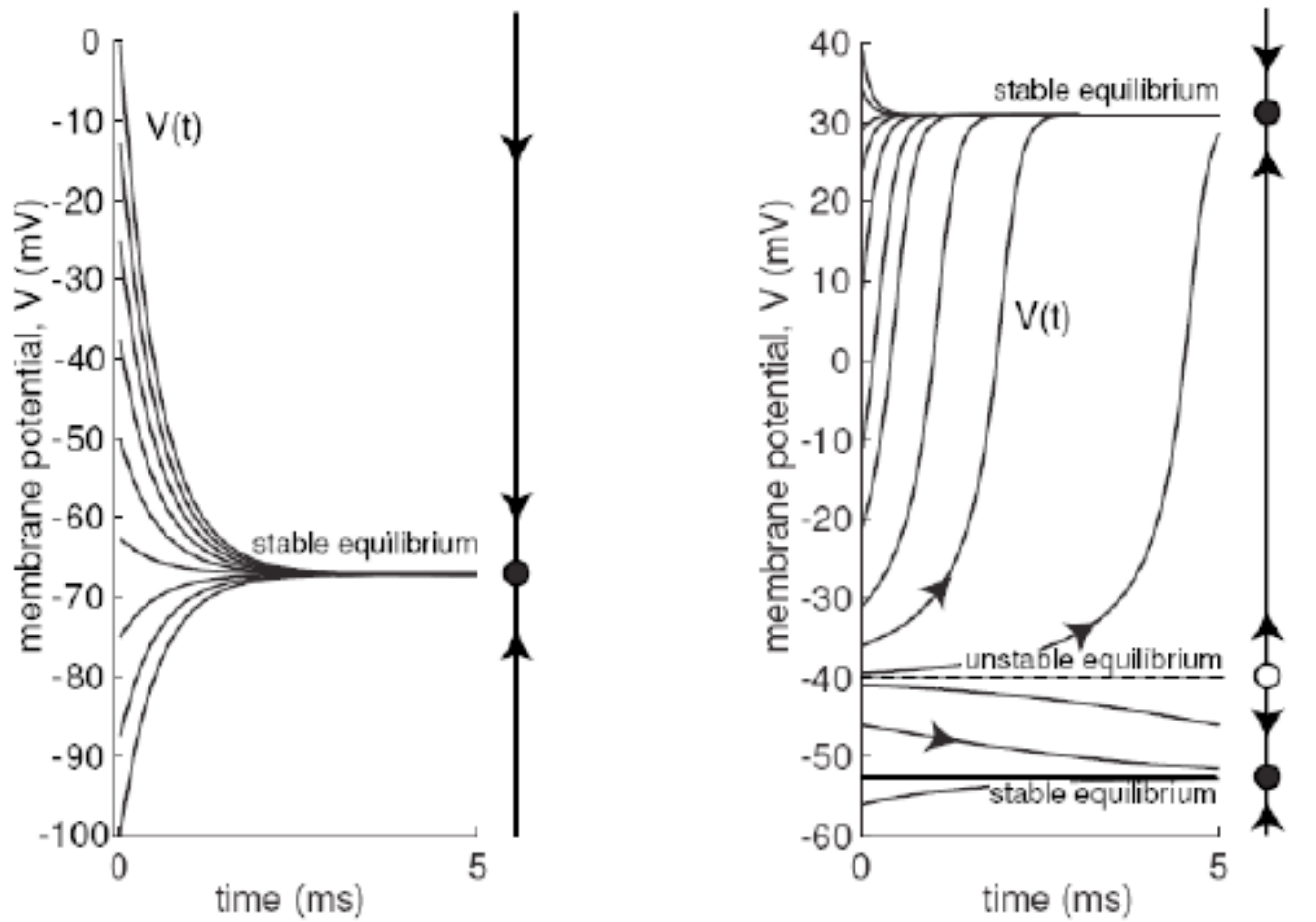




One-dimensional neural model

- Equilibria

$$C\frac{dV}{dt} = -\,G_{Na}\,p_{\infty}(V)\,(V - E_{Na}) - G_L\,(V - E_L) + I$$

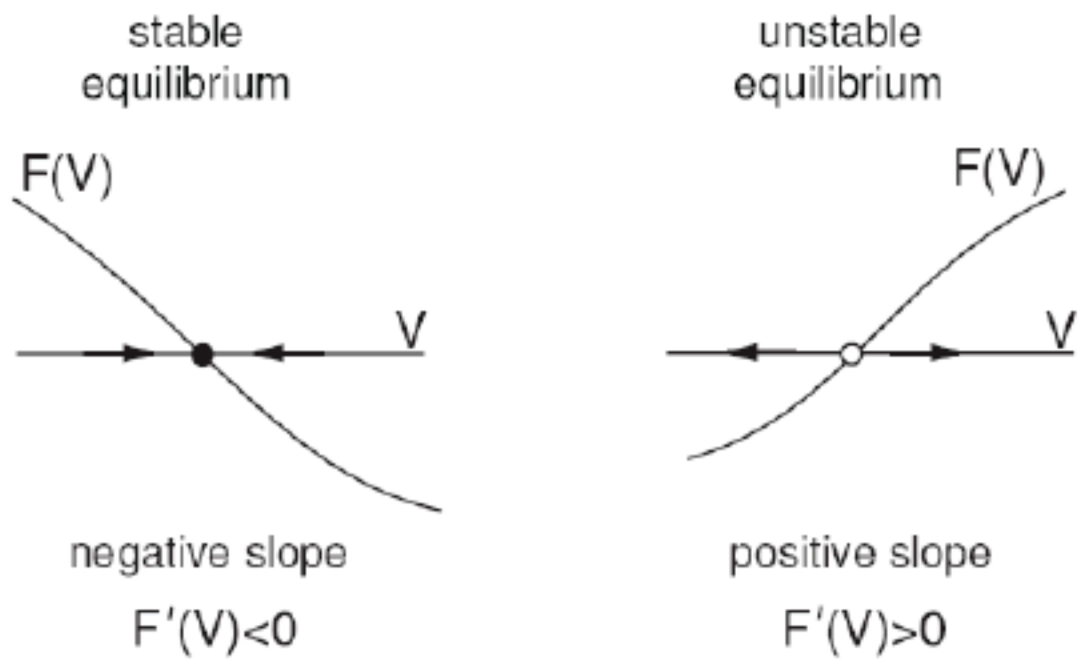


One-dimensional neural model

- Stability

$$\frac{dV}{dt} = F(V) \qquad V(0) = V_0$$

$$\lambda = F'(V_{eq}) \text{ (eigenvalue)} \qquad \mathrm{F}(\mathrm{V}_{eq}) = 0$$



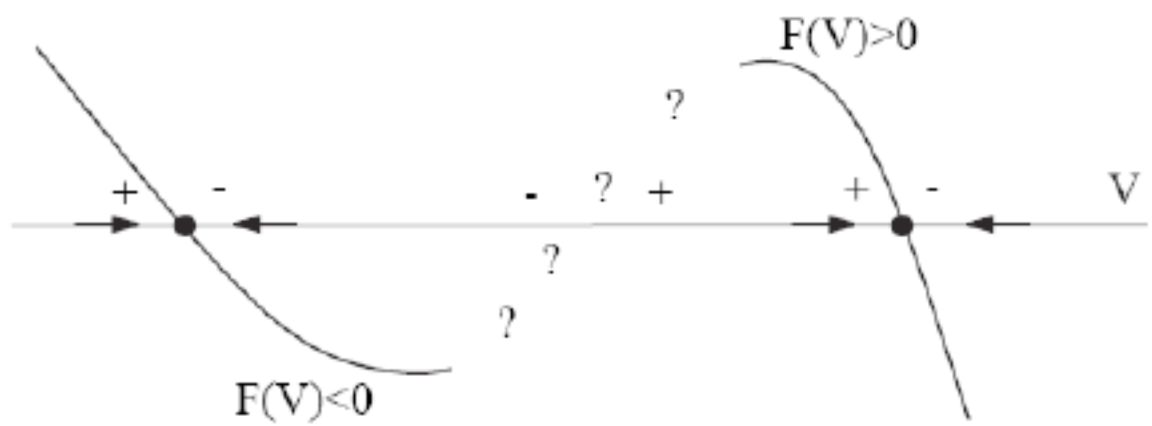
Stability of an equilibrium is determined by the signs of F(V) around it

One-dimensional neural model

- Stability

$$\frac{dV}{dt} = F(V) \qquad V(0) = V_0$$

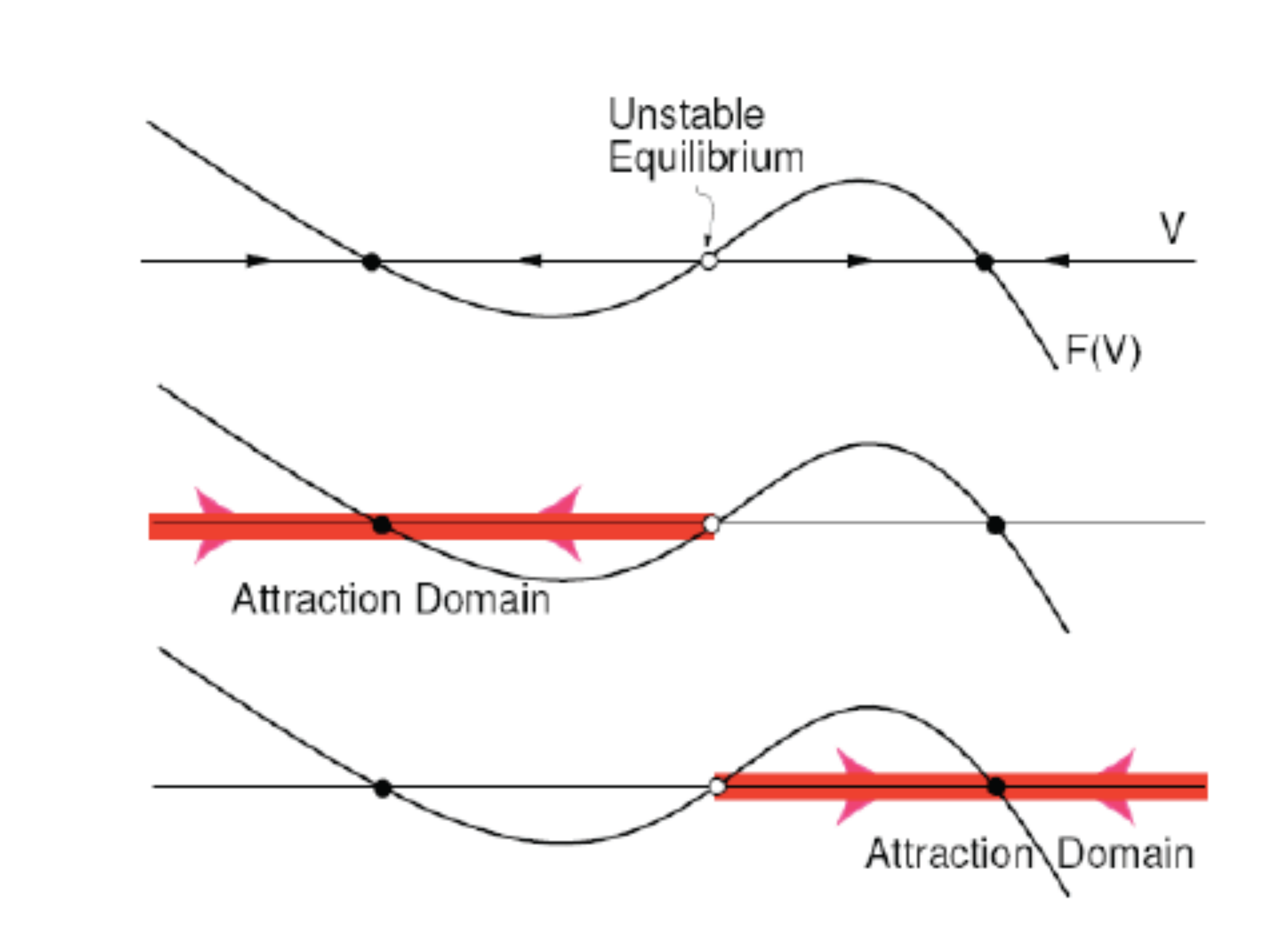
$$\lambda = F'(V_{eq}) \text{ (eigenvalue)} \qquad F(V_{eq}) = 0$$



Stability of an equilibrium is determined by the signs of F(V) around it

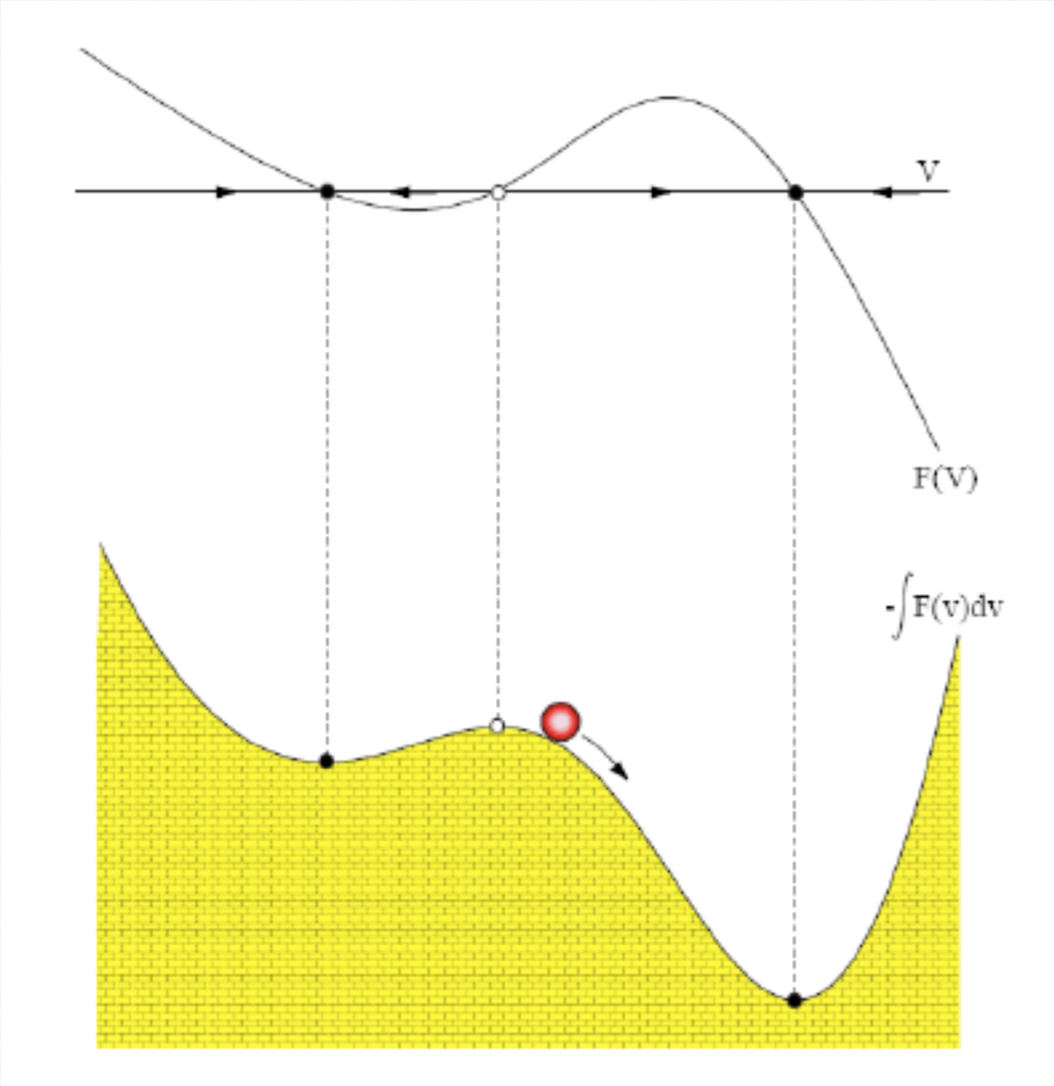
One-dimensional neural model

- Threshold and action potential



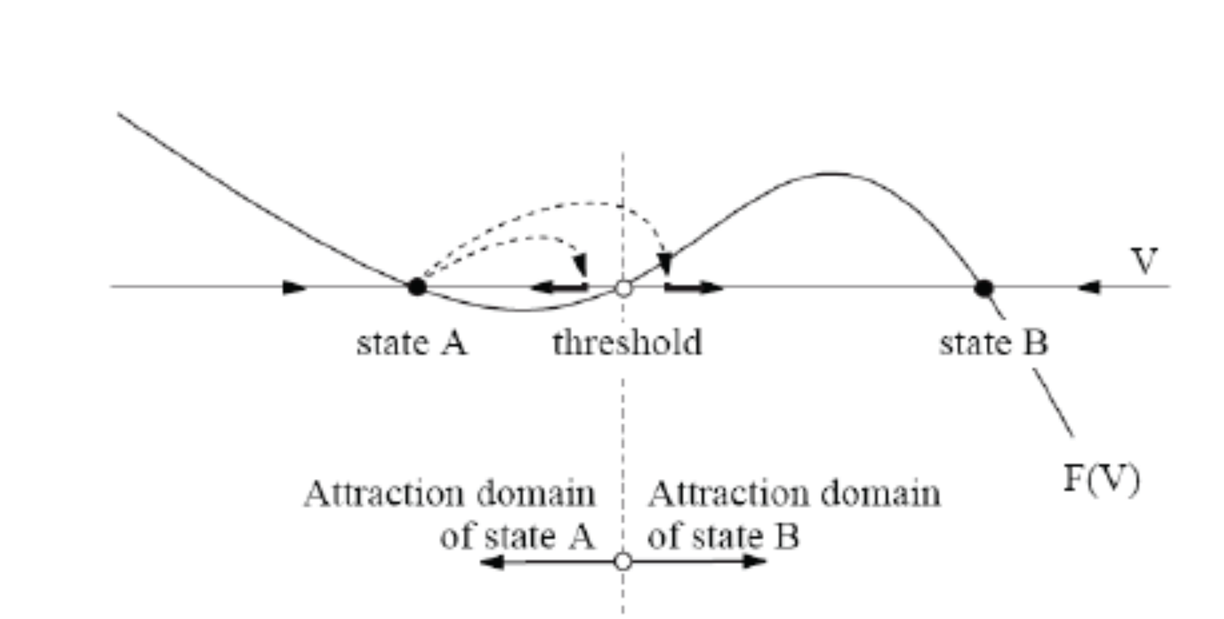
One-dimensional neural model

- Threshold and action potential - mechanistic (energy) interpretation



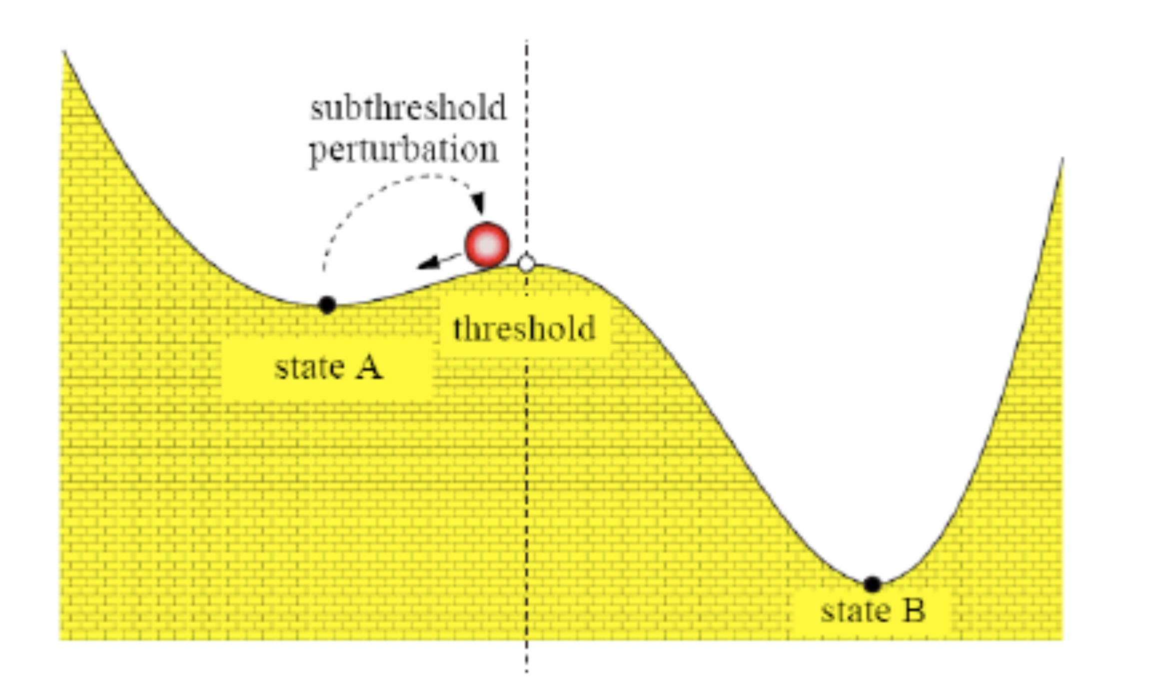
One-dimensional neural model

- Threshold and action potential - mechanistic (energy) interpretation



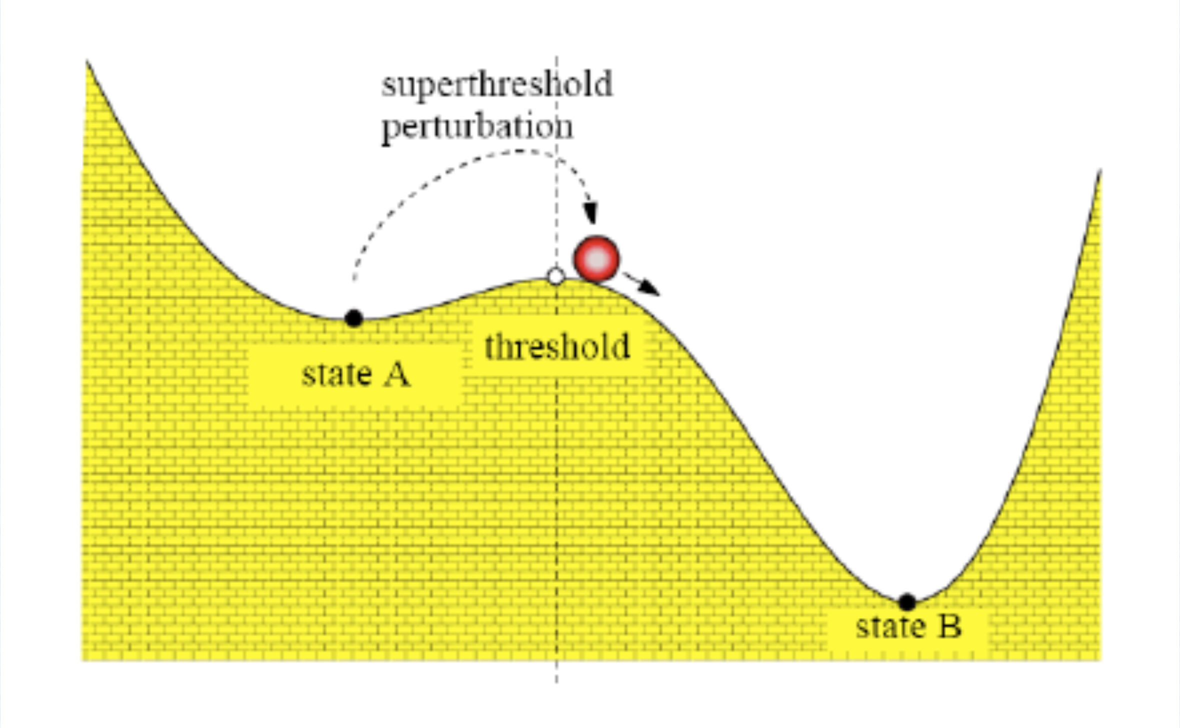
One-dimensional neural model

- Threshold and action potential - mechanistic (energy) interpretation



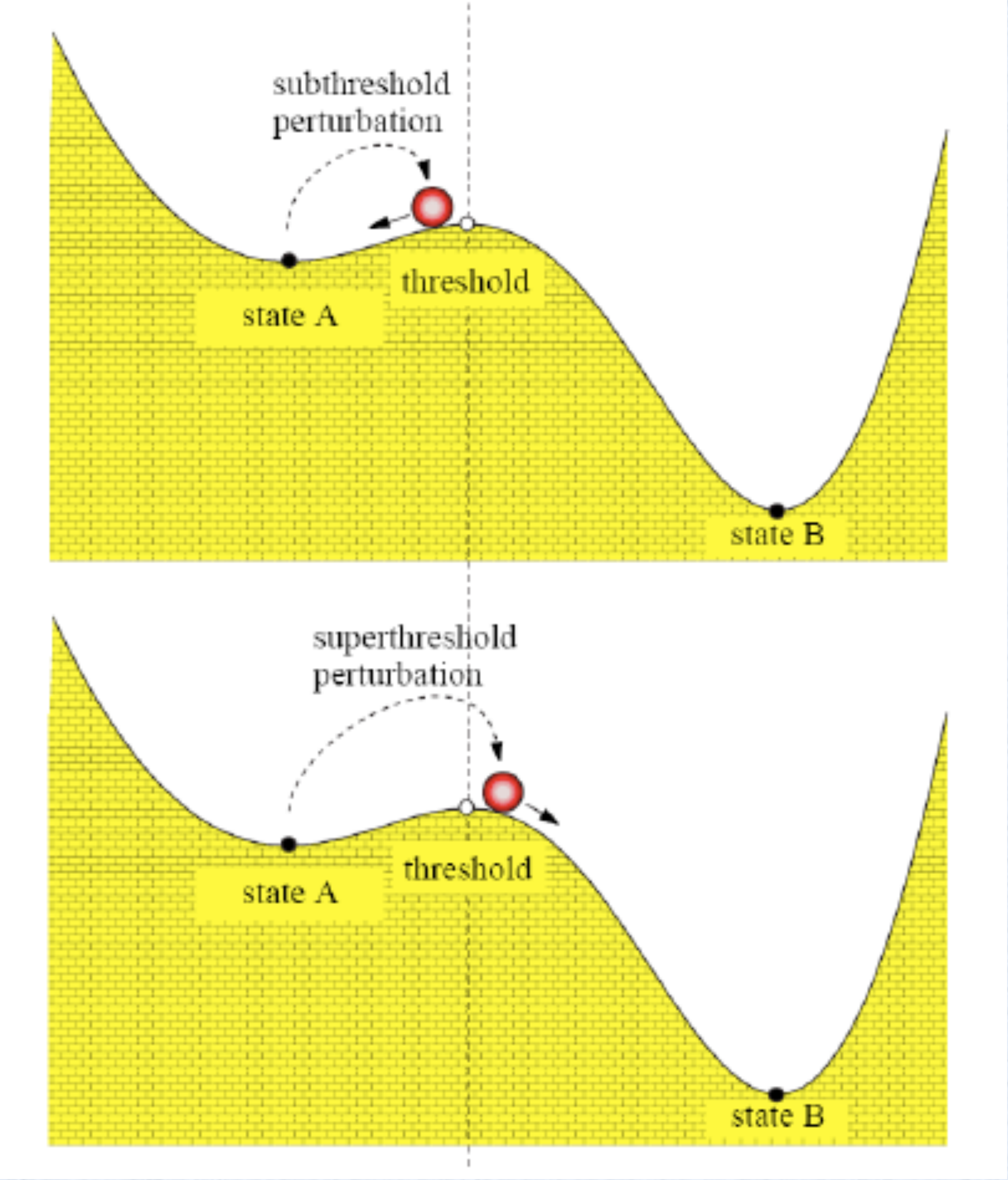
One-dimensional neural model

- Threshold and action potential - mechanistic (energy) interpretation



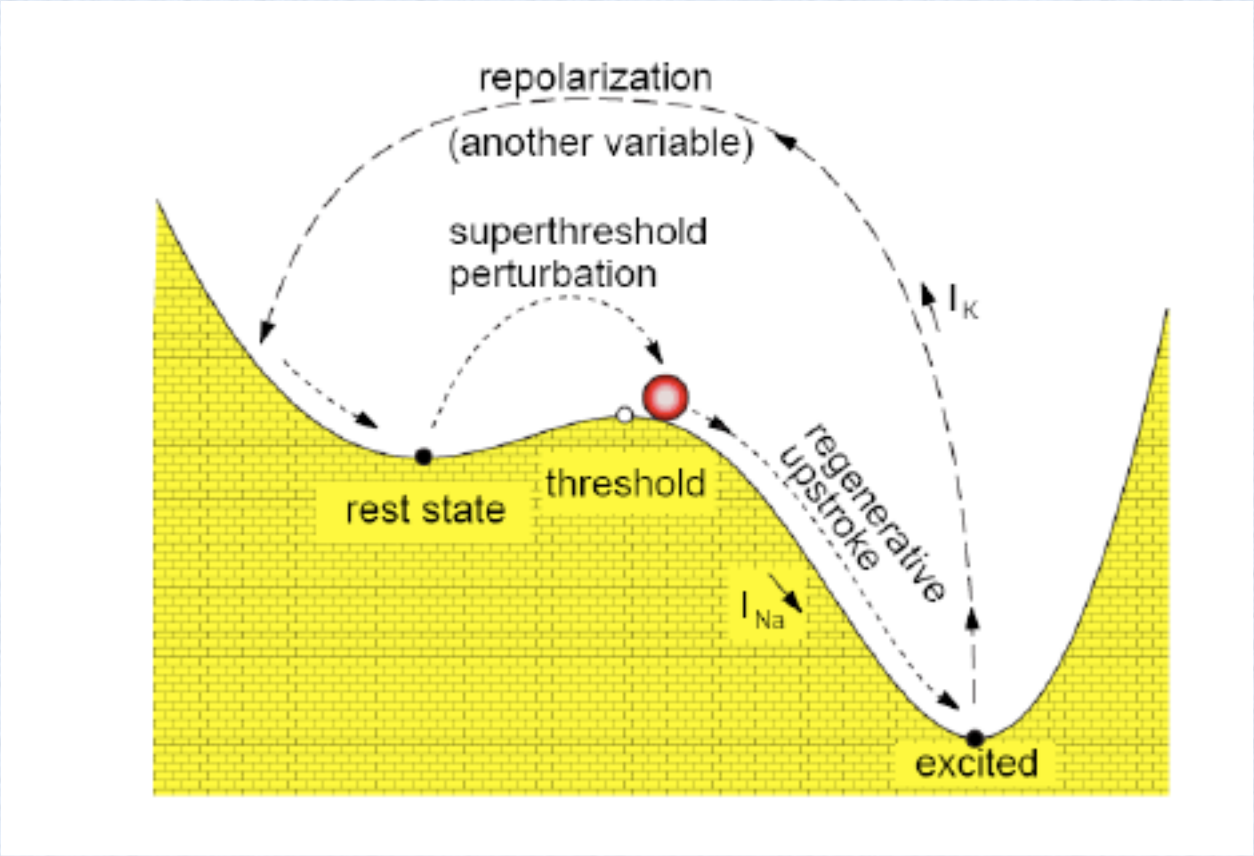
One-dimensional neural model

- Threshold and action potential - mechanistic (energy) interpretation



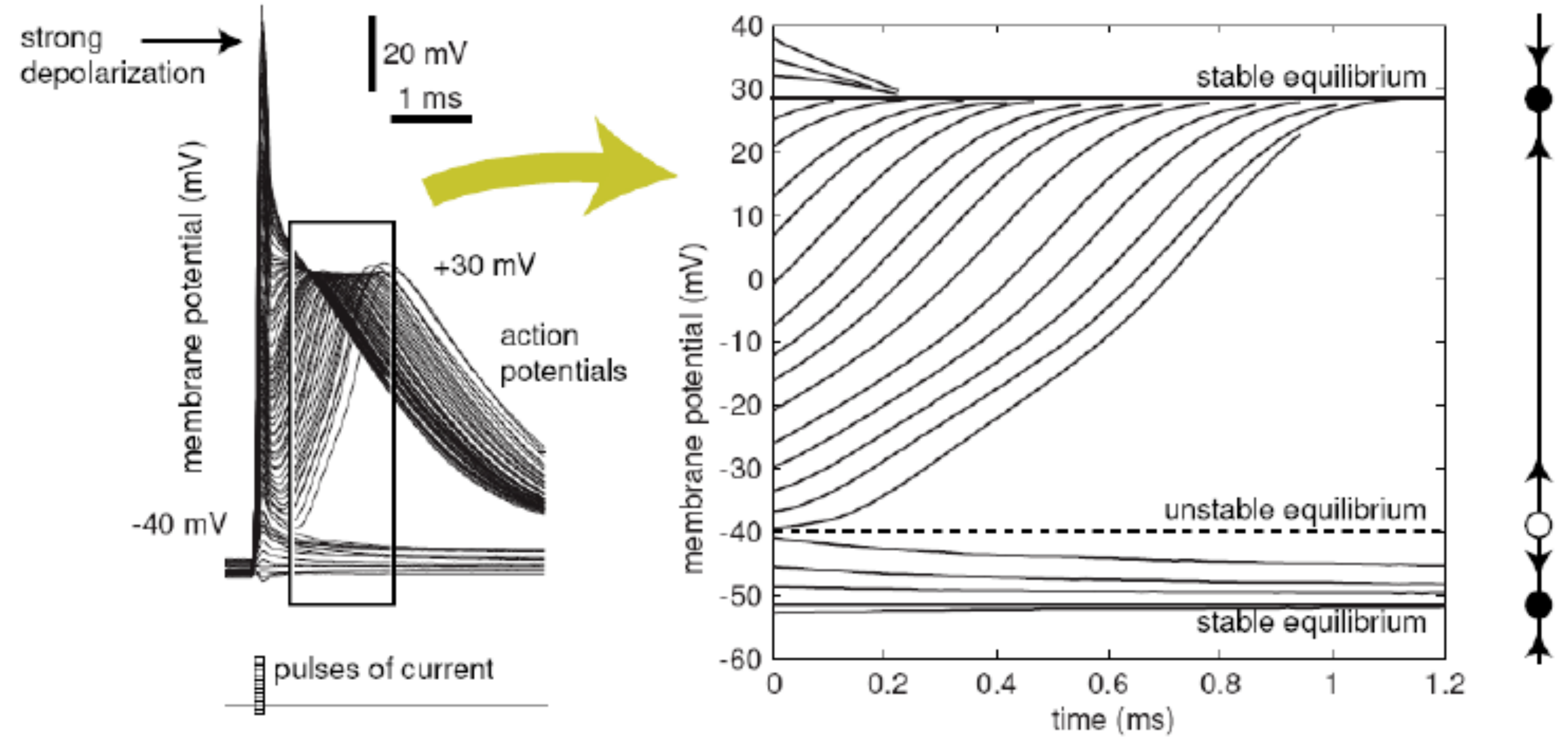
One-dimensional neural model

- Threshold and action potential - mechanistic (energy) interpretation



One-dimensional neural model

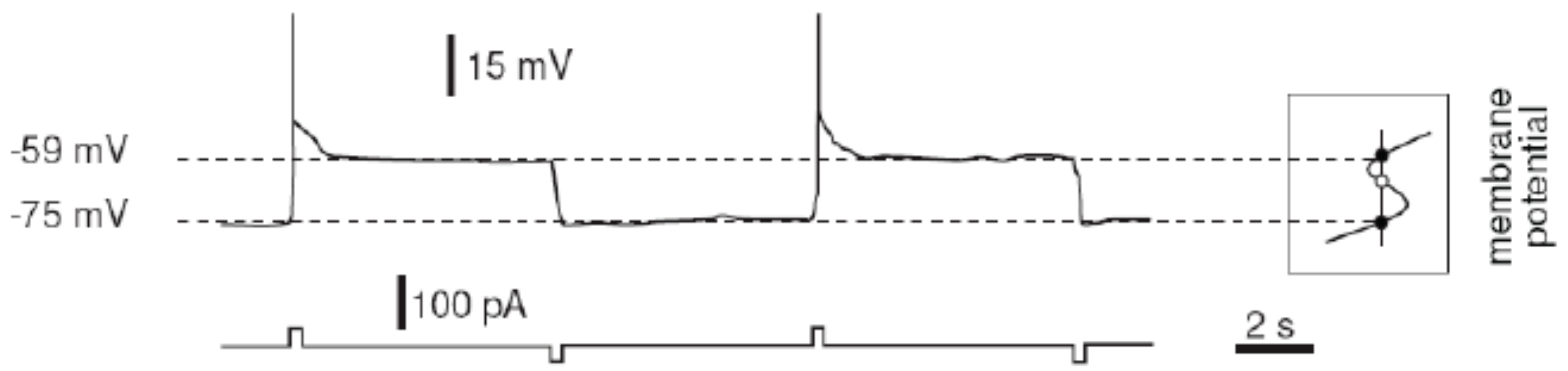
- Threshold and action potential



❑ Upstroke dynamics of layer 5 pyramidal neuron *in vitro*

One-dimensional neural model

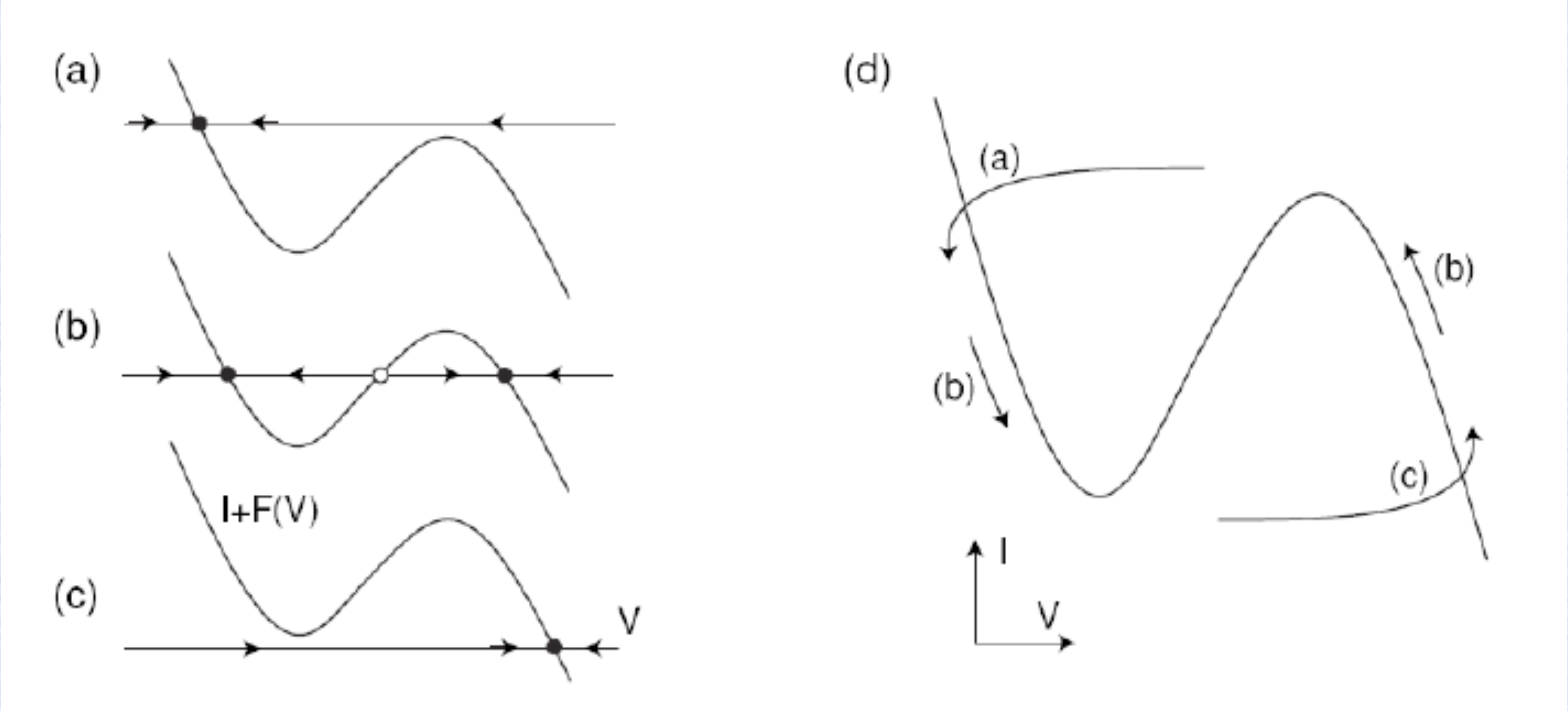
- Bistability and hysteresis



Cat TC neuron in the presence of ZD7288

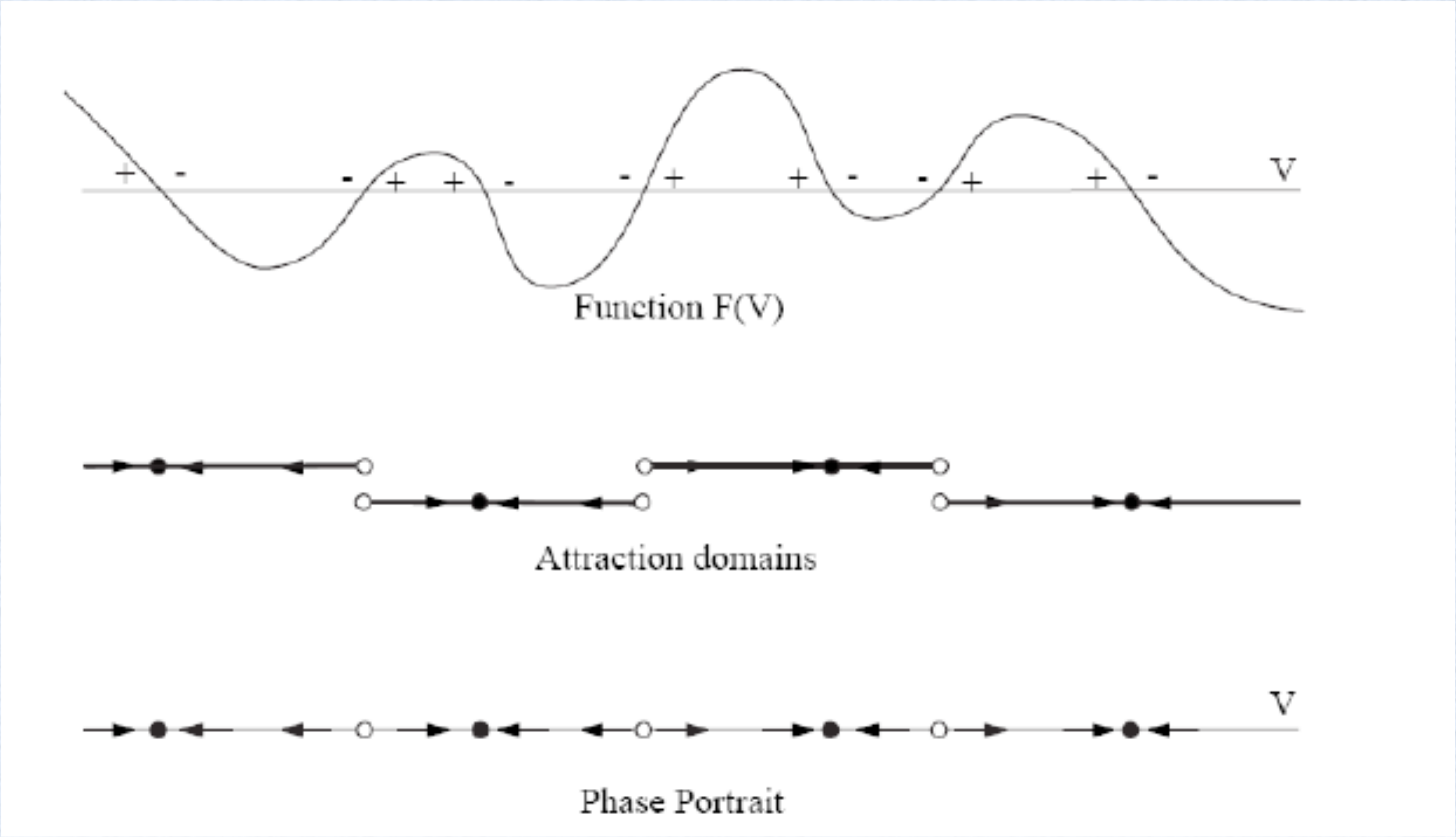
One-dimensional neural model

- Bistability and hysteresis as I changes



One-dimensional neural model

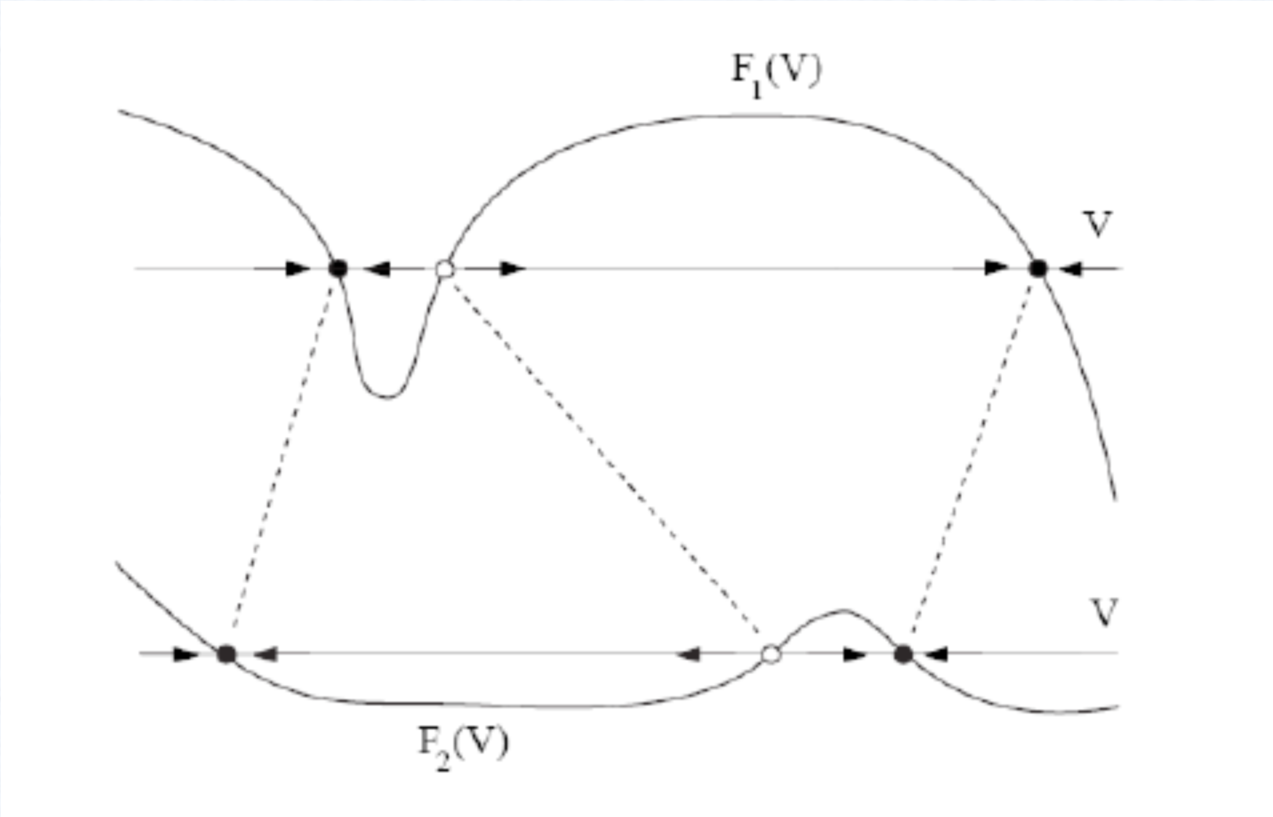
- Multistability



Phase portrait of a one-dimensional system dV / dt = F(V)

One-dimensional neural model

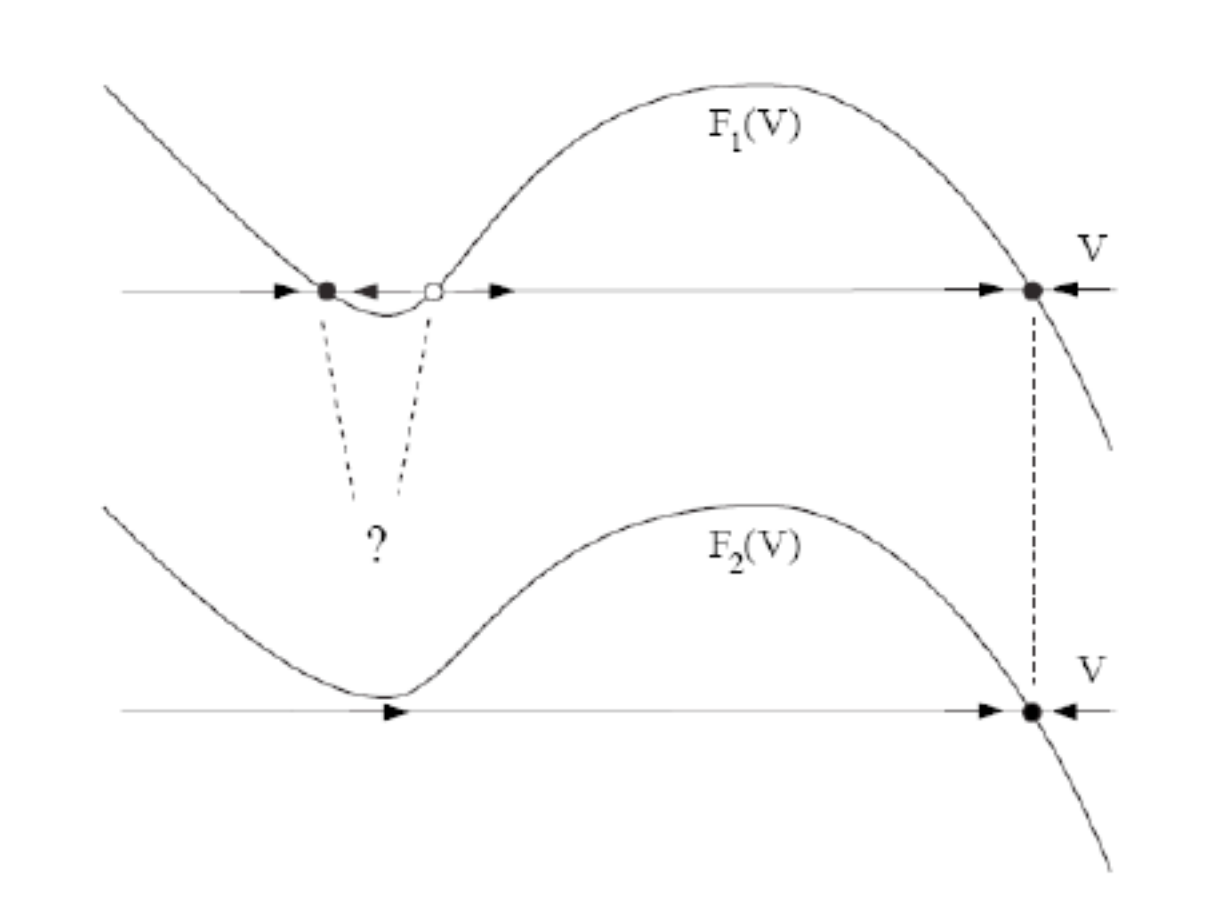
- Topological equivalence



Phase portrait of two seemingly different a one-dimensional systems $dV_1 / dt = F(V_1)$ and $dV_2 / dt = F(V_2)$

One-dimensional neural model

- Topological equivalence

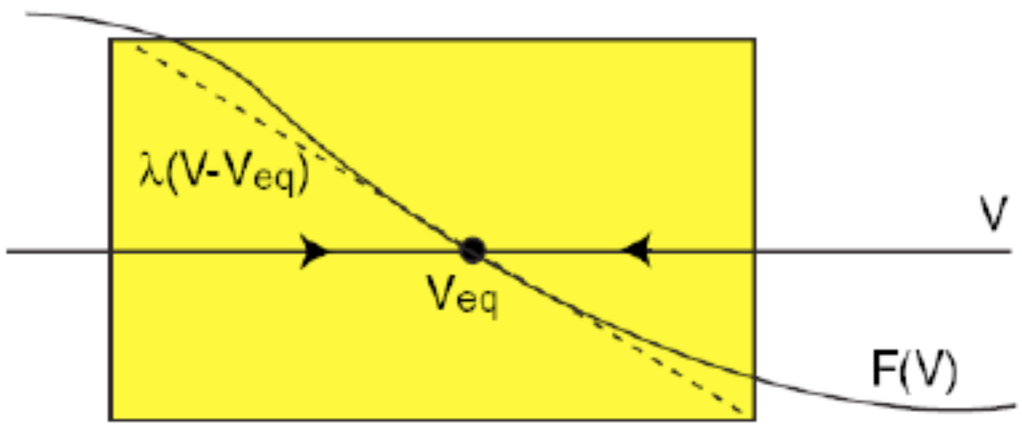


Phase portrait of two seemingly different a one-dimensional systems $dV_1 / dt = F(V_1)$ and $dV_2 / dt = F(V_2)$

One-dimensional neural model

- Local equivalence

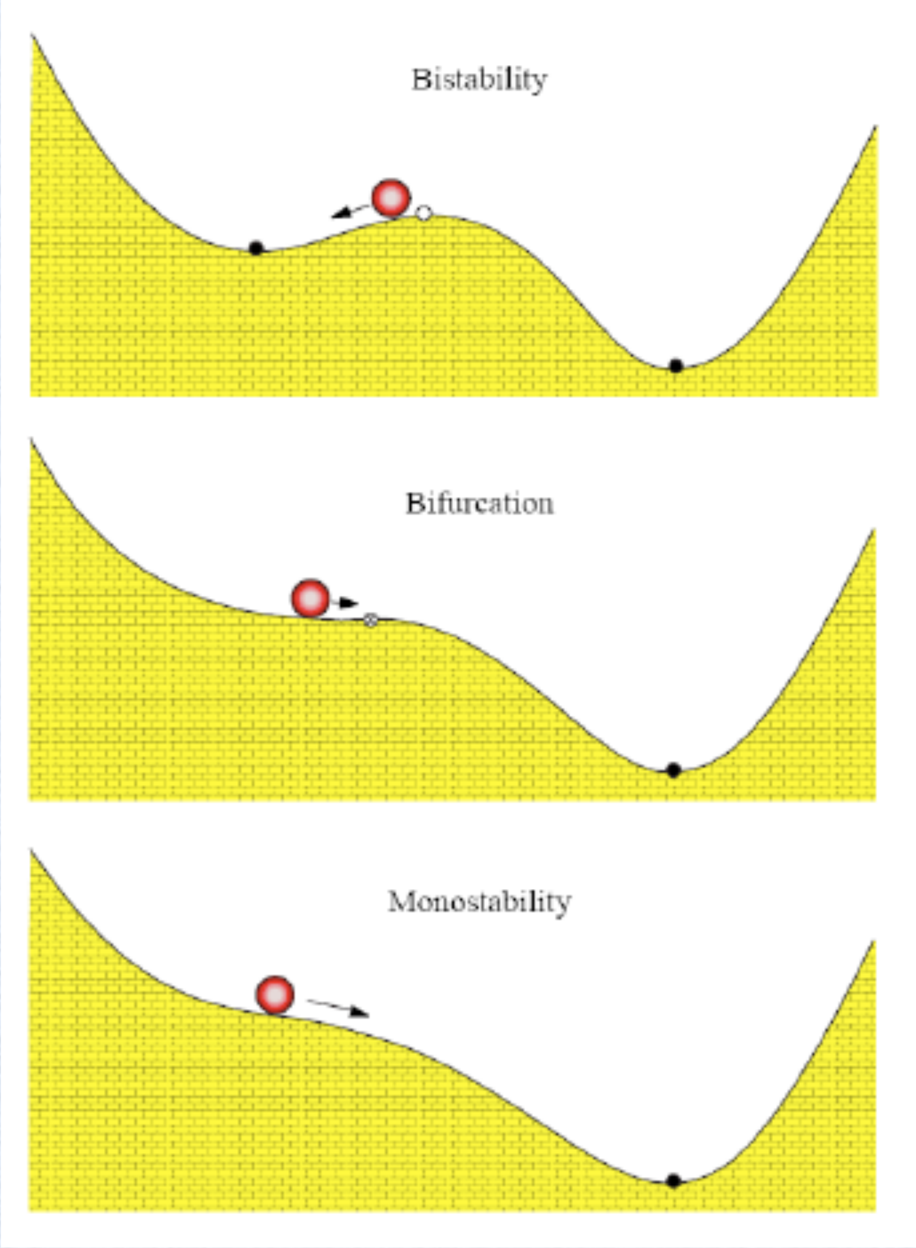
$$\frac{dV}{dt} = F(V) \qquad \lambda = F'(V_{eq}) \neq 0 \qquad \| \qquad \frac{dV}{dt} = \lambda\,(V - V_{eq})$$



Hartman-Grobman theorem: The two systems are topologically equivalent in the local (shaded) neighborhood of the hyperbolic equilibrium

One-dimensional neural model

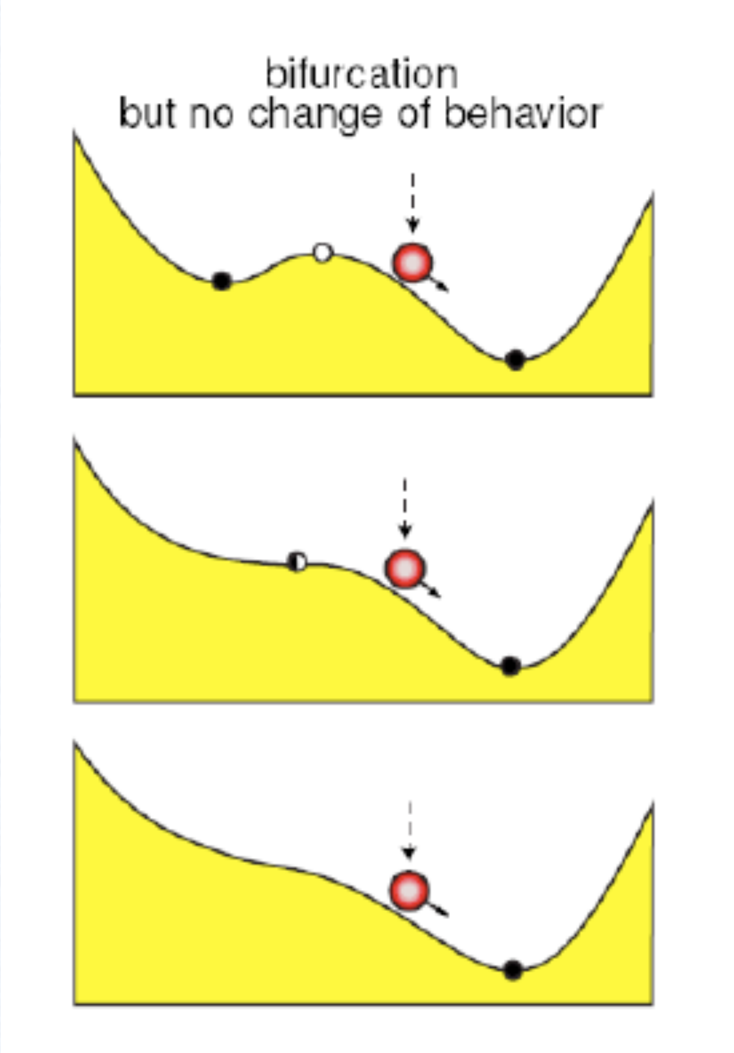
- Bifurcations



Mechanistic illustration of a bifurcation as a change of the landscape

One-dimensional neural model

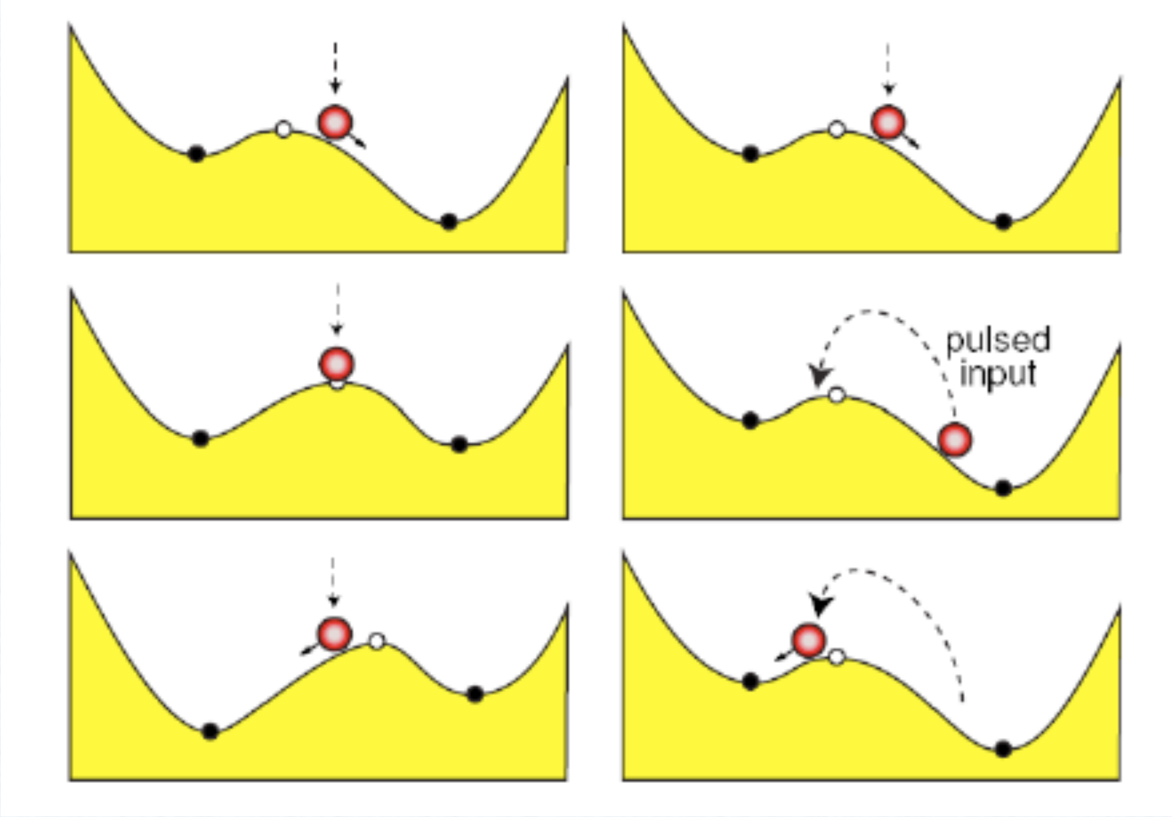
- Bifurcations



No change of behavior

One-dimensional neural model

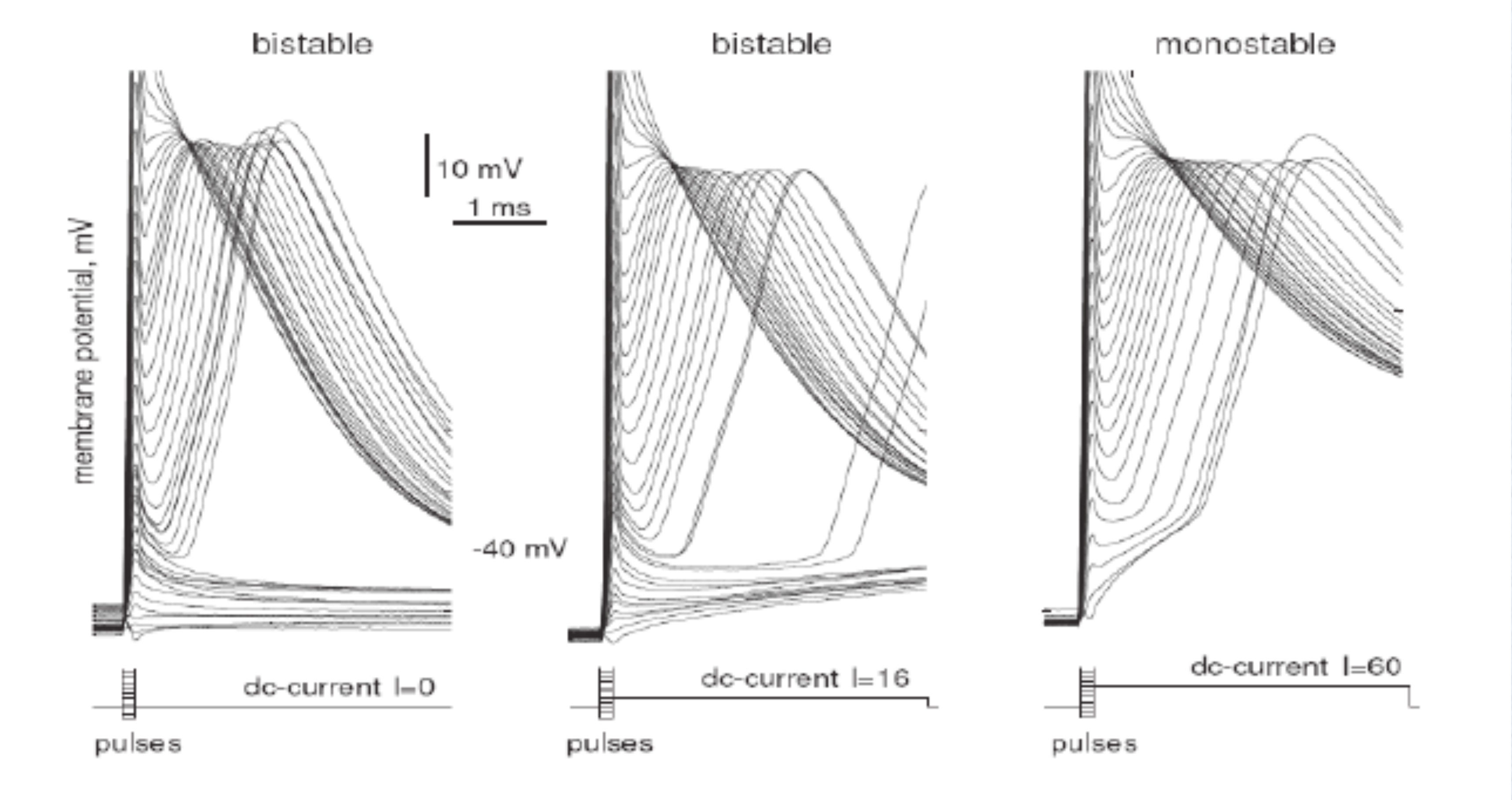
- Bifurcations



Change of behavior

One-dimensional neural model

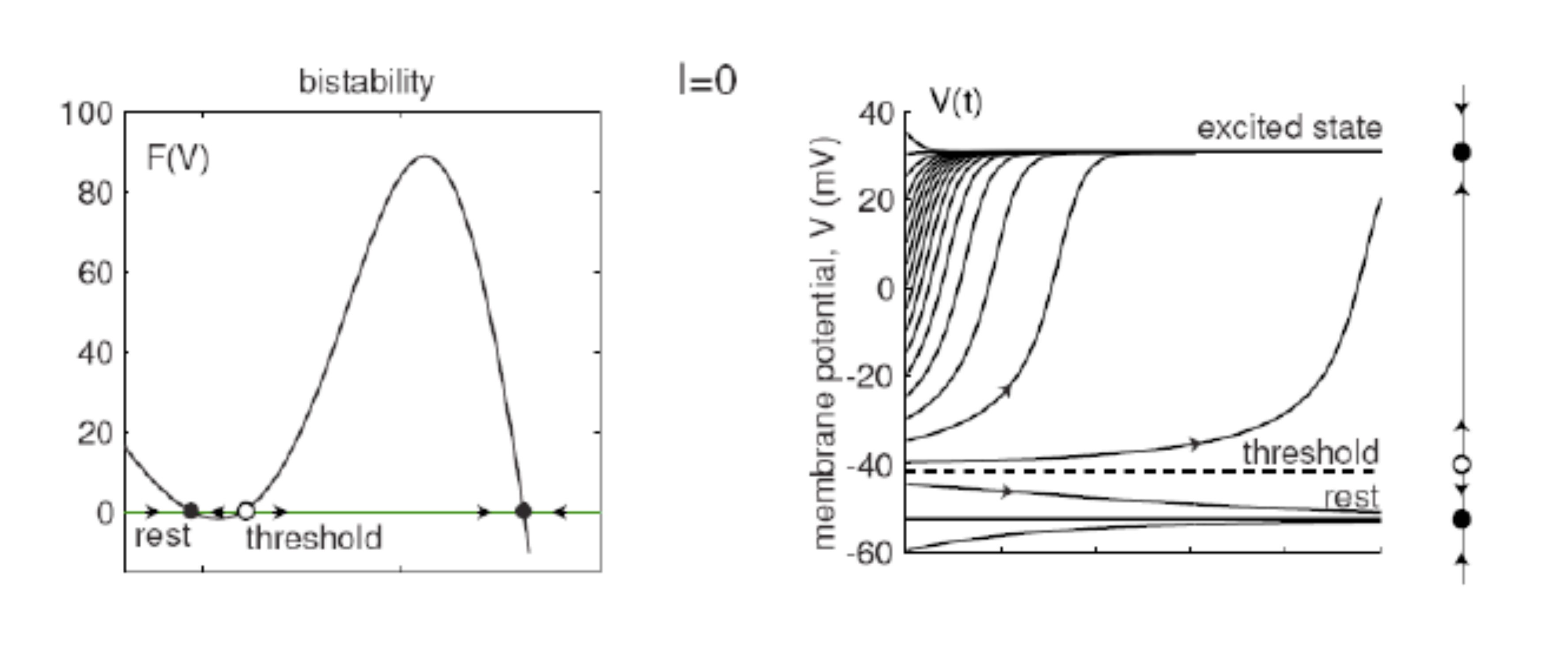
- Bifurcations



Qualitative change of the upstroke dynamics of layer 5 pyramidal neuron from rat visual cortex

One-dimensional neural model

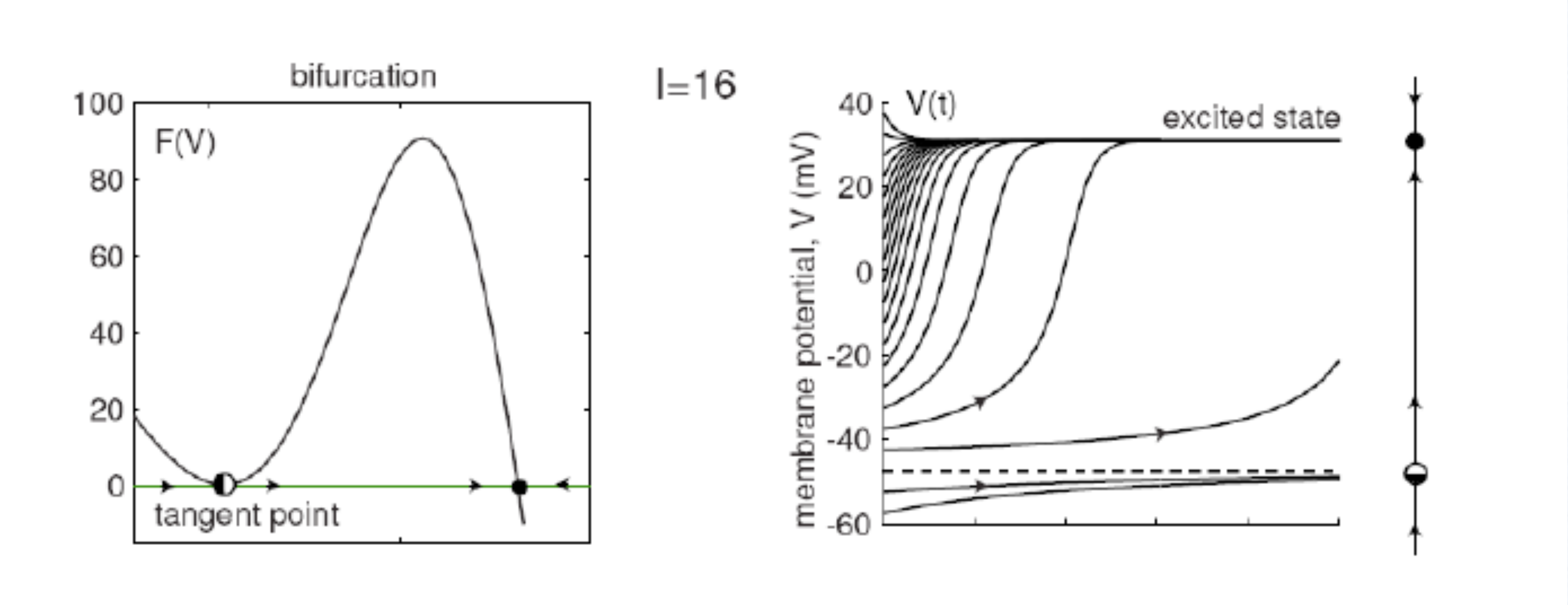
- Bifurcations in the $I_{Na,p}$ model



The resting state and the threshold state coalesce and disappear when the parameter I increases

One-dimensional neural model

- Bifurcations in the $I_{Na,p}$ model



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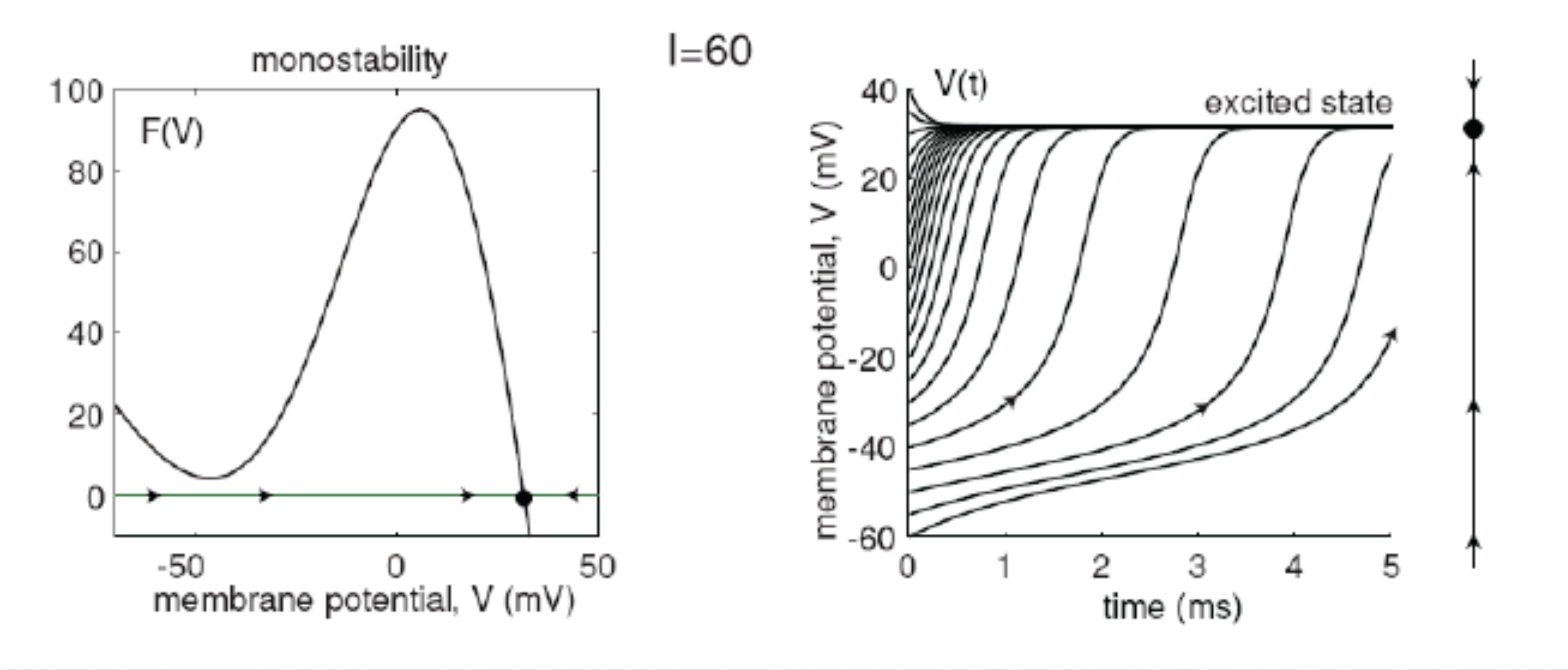
One-dimensional neural model

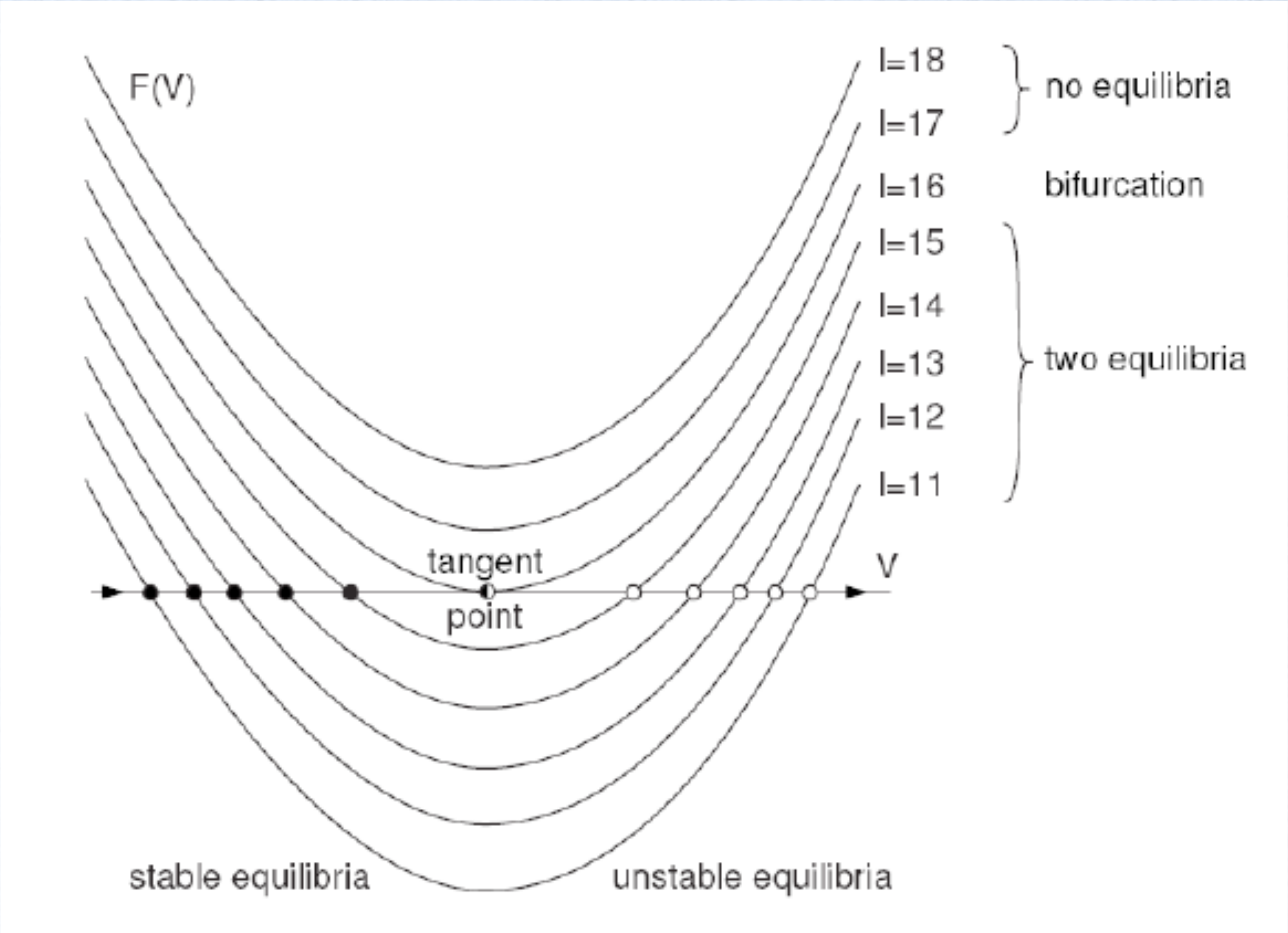
- Bifurcations in the $I_{Na,p}$ model



The resting state and the threshold state coalesce and disappear when the parameter I increases

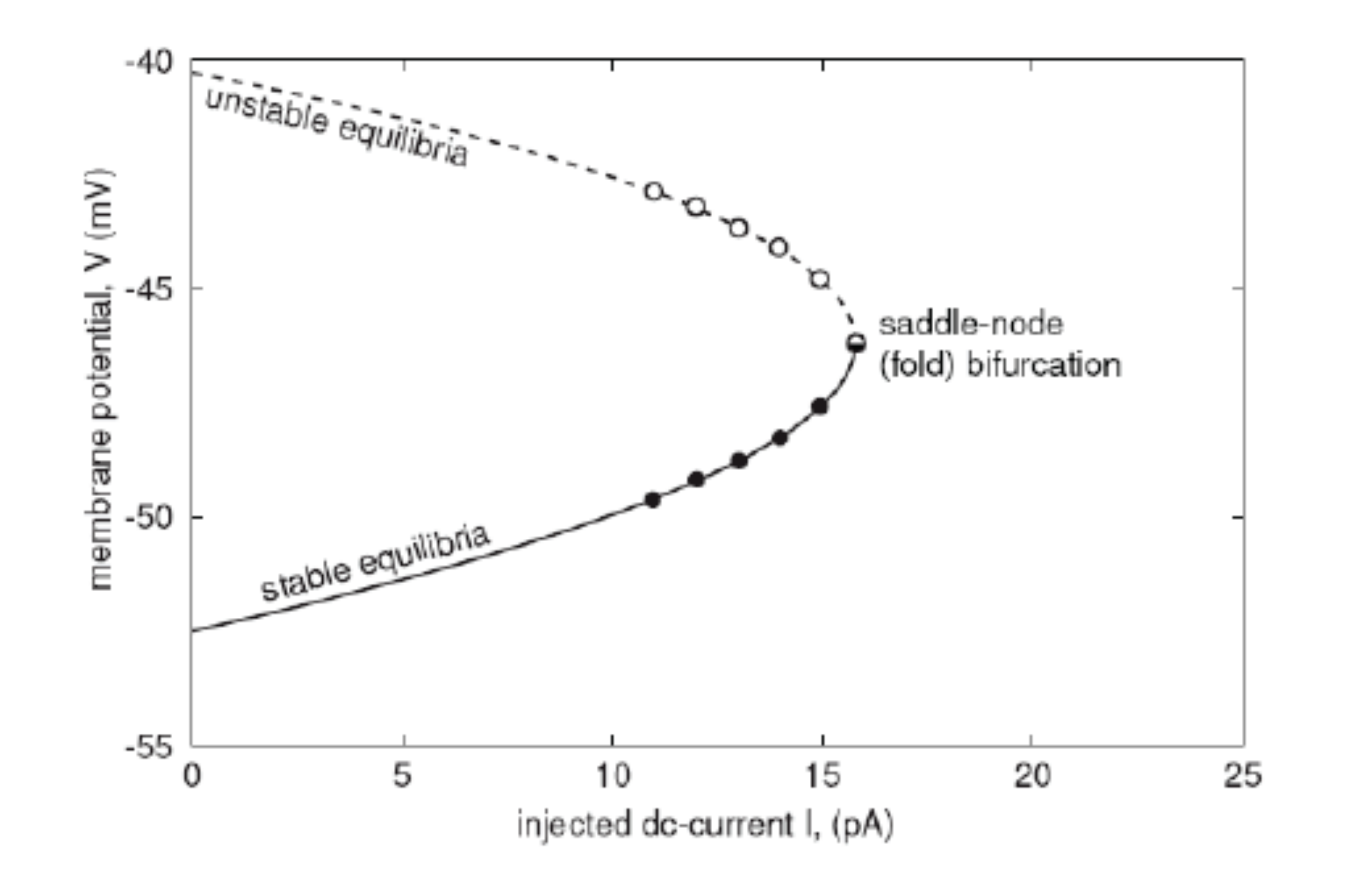
One-dimensional neural model

- Bifurcations



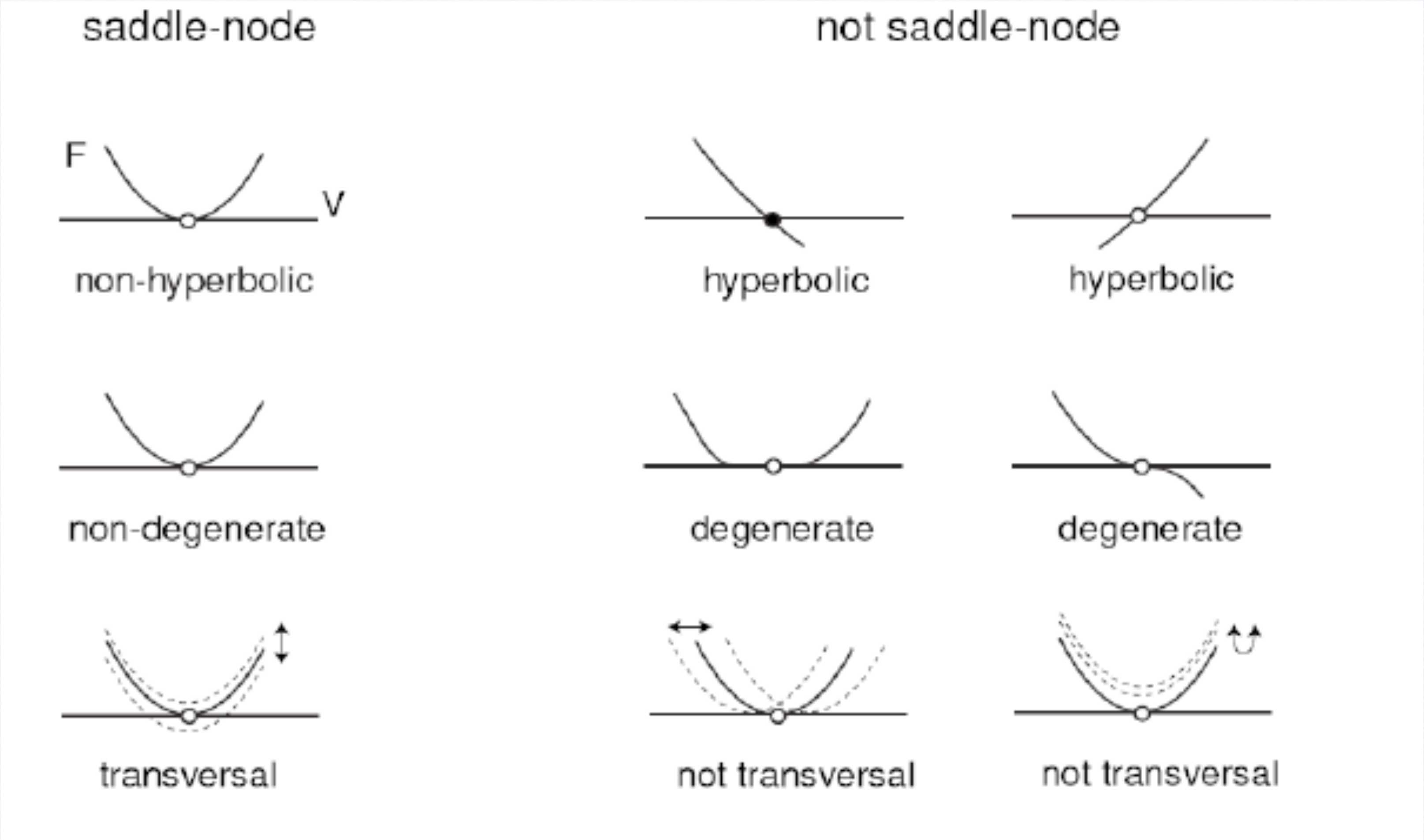
One-dimensional neural model

- Bifurcation diagram



One-dimensional neural model

- Bifurcations: conditions defining saddle-node bifurcations



Arrows denote the direction of displacement of the function F(V,I) as the bifurcation parameter I changes

One-dimensional neural model

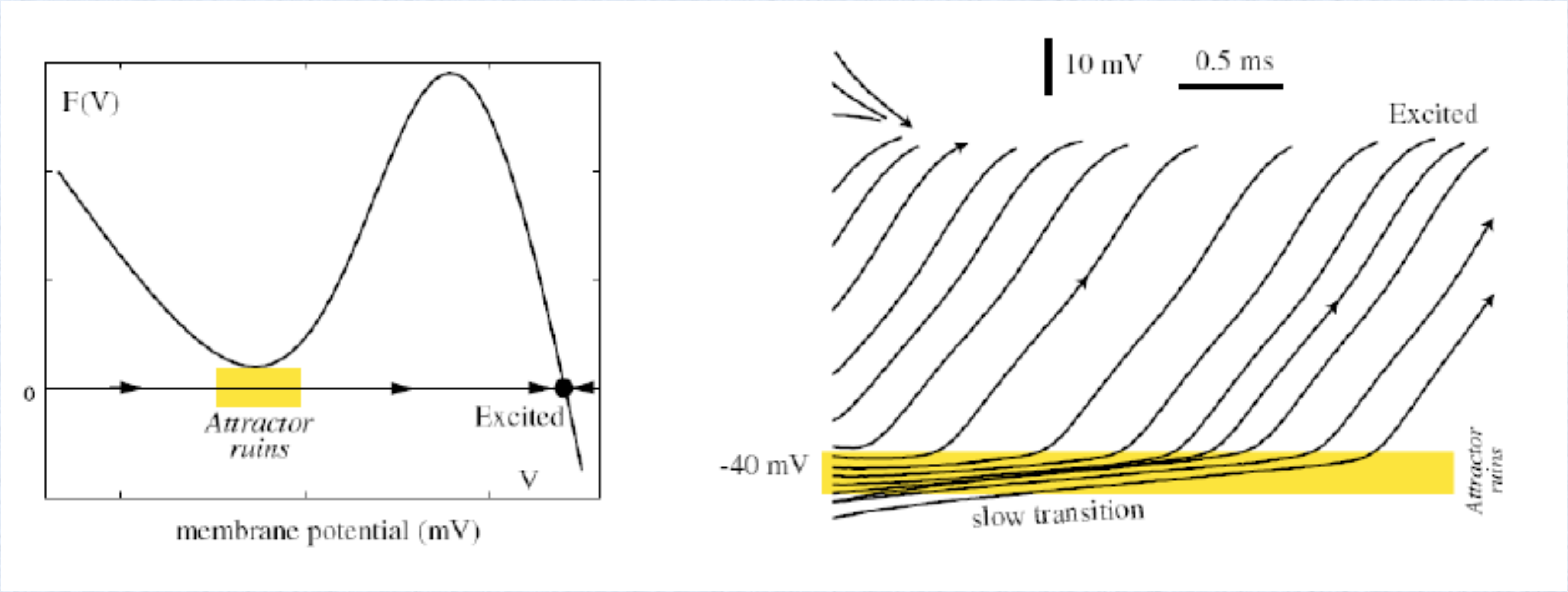
- Topological normal form for the saddle-node bifurcation

$$\frac{dV}{dt} = I + V^2$$

The system has a saddle-node bifurcation when V=0 and I=0

One-dimensional neural model

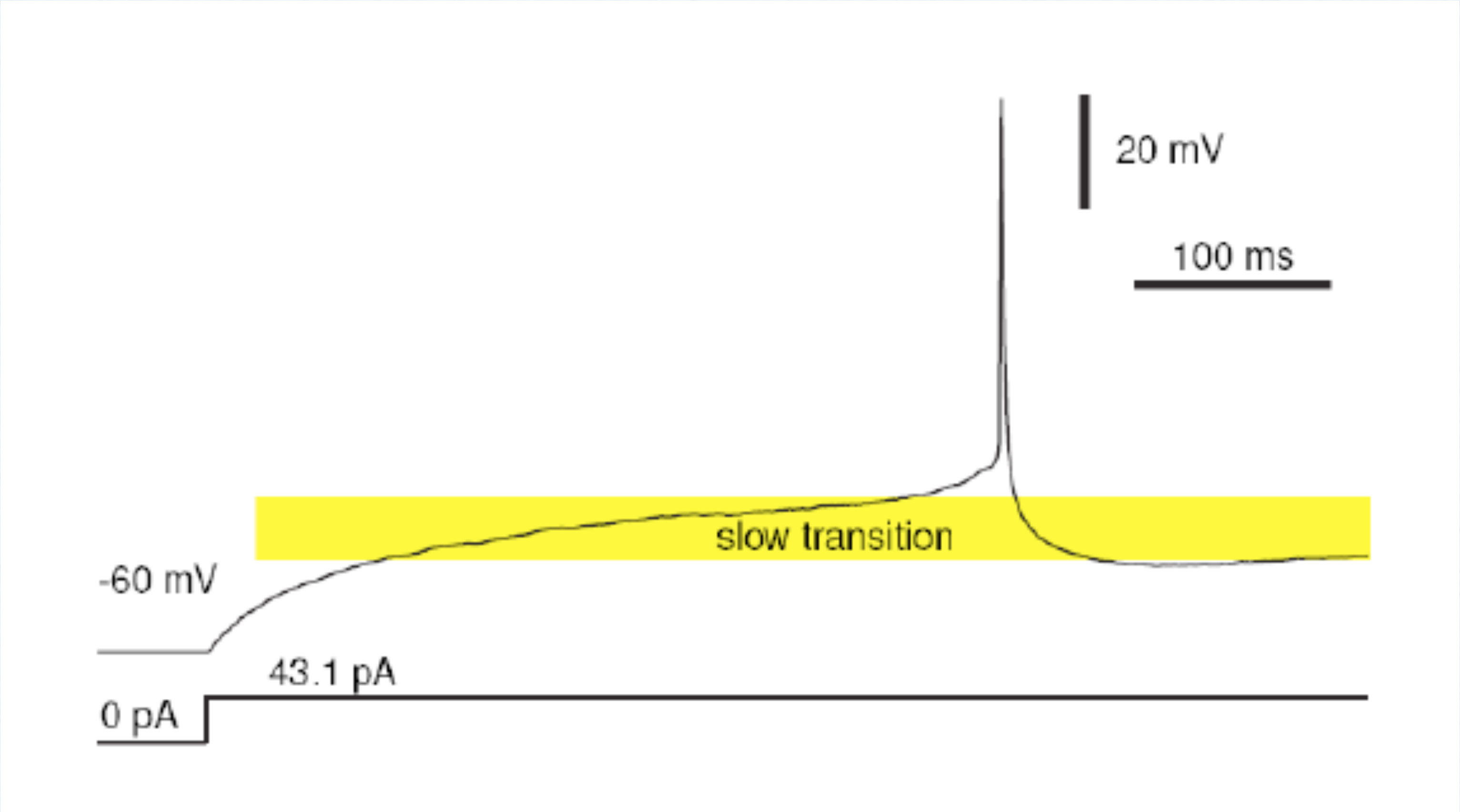
- Slow transitions



Slow transition through the ghost of the resting state attractor in a cortical pyramidal neuron with I=30 pA

One-dimensional neural model

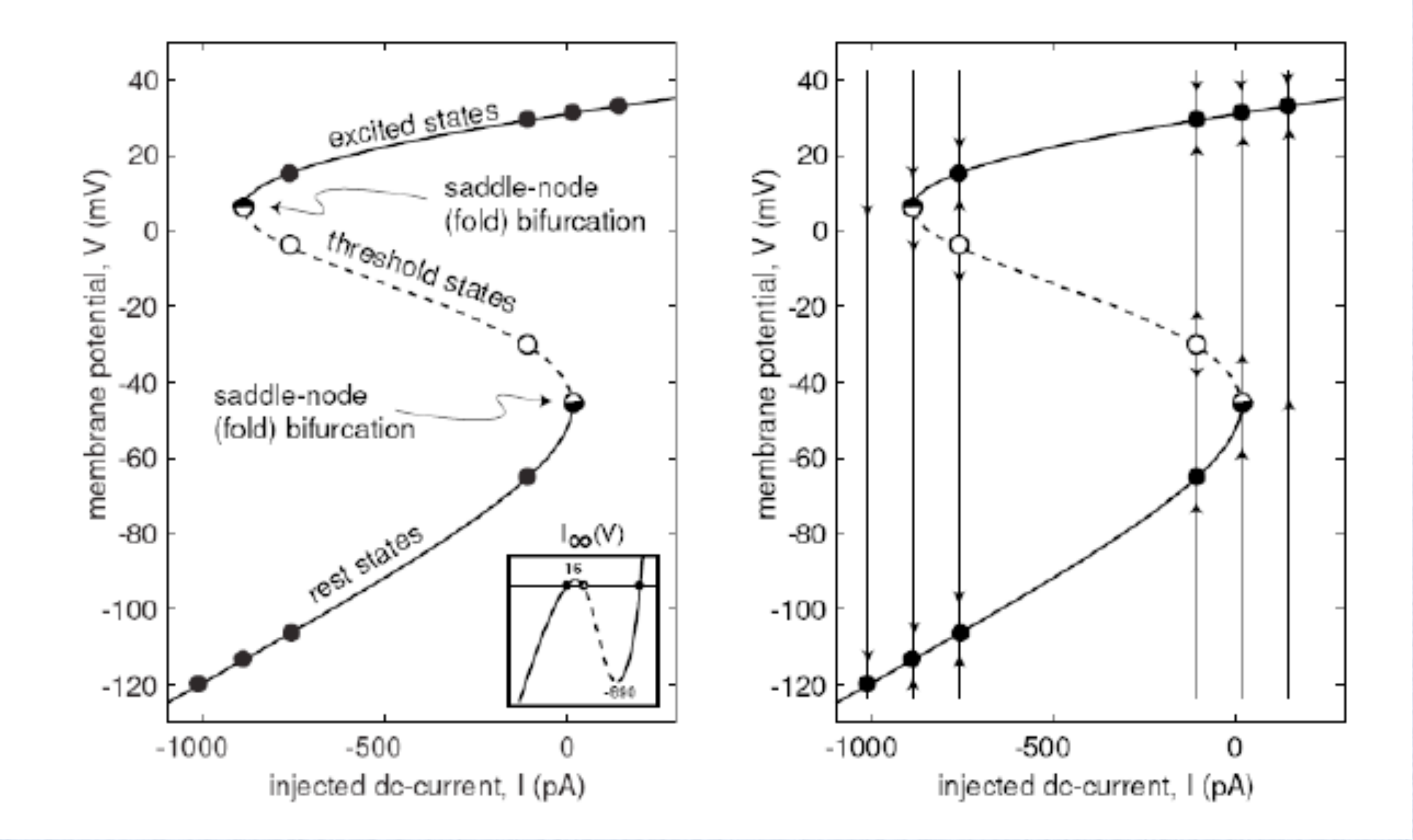
- Slow transitions



A 400 ms latency in a layer 5 pyramidal neuron of rat visual cortex

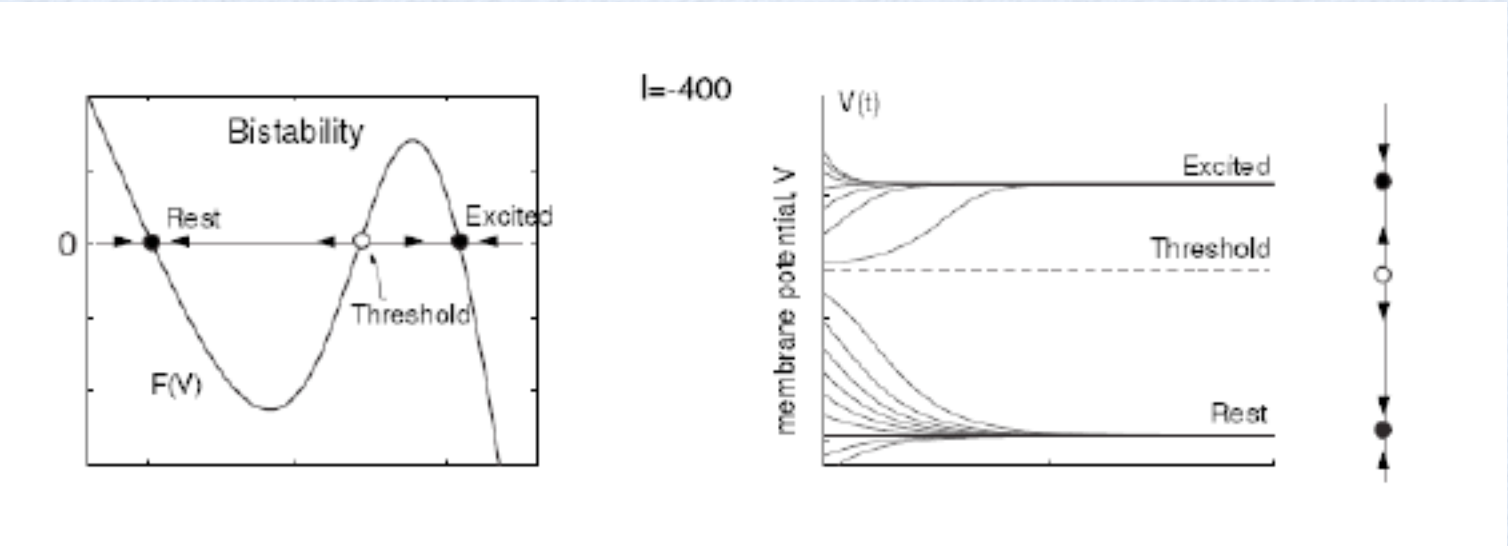
One-dimensional neural model

- Bifurcation diagram of the $I_{Na,p}$ model



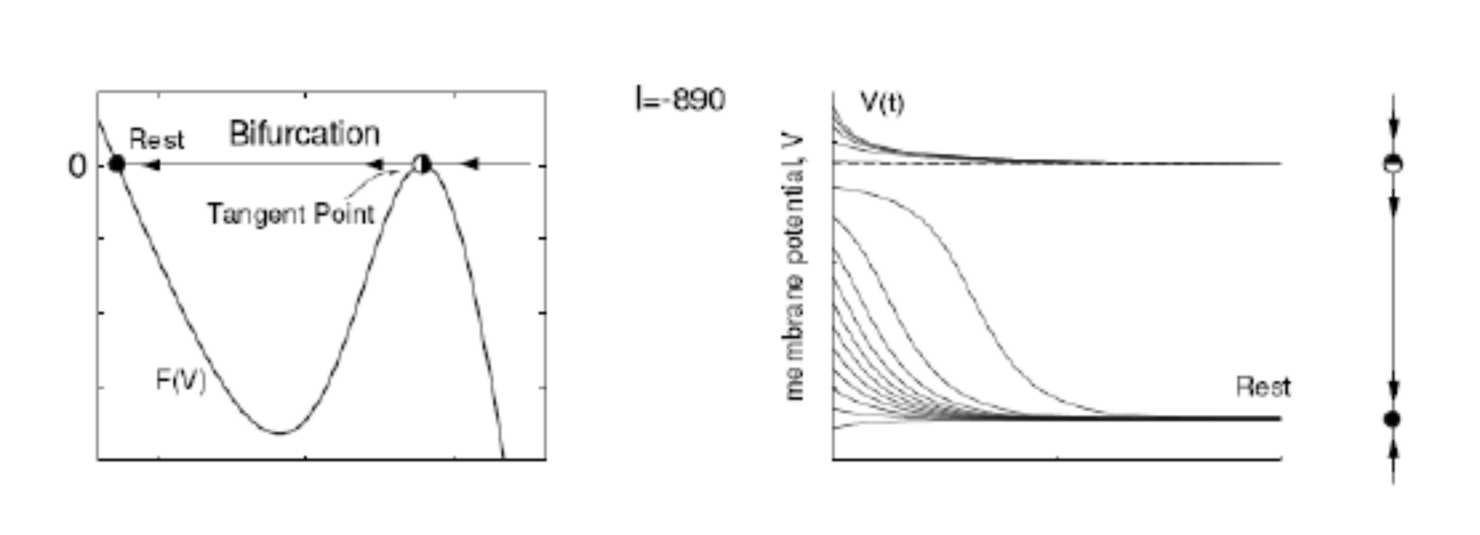
One-dimensional neural model

- Bifurcation diagram of the $I_{Na,p}$ model



One-dimensional neural model

- Bifurcation diagram of the $I_{Na,p}$ model



One-dimensional neural model

- Bifurcation diagram of the $I_{Na,p}$ model